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SEGRE'S UPPER BOUND FOR THE REGULARITY INDEX OF $2n+1$ DOUBLE POINTS IN P^n

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Abstract: We prove the Segre's upper bound for the regularity index of $2n+1$ double points that do not exist n+1 points of them lying on an $(n-2)$-plane in P^n.

1 Introduction

Let $P_1,\dots,P_s$ be a set of distinct points in a projective space with n-dimension $P^n := P_k^n$, with k as an algebraically closed field. Let $\wp_1,\dots,\wp_s$ be the homogeneous prime ideals of the polynomial ring $R := k[x_0,\dots,x_n]$ corresponding to the points $P_1,\dots,P_s$. Let $m_1,\dots,m_s$ be positive integers. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_1^{m_1}$, denote $Z = m_1P_1 + \cdots + m_sP_s$ the zero-scheme defined by I. We call Z to be a set of fat points.

A set of s fat points in P^n is said to be equimultiple if $m_1 = \cdots = m_s = m.$. In case $m_1 = \cdots = m_s = 2,$ a set of fat points

$$Z = 2P_1 + \cdots + 2P_s$$

is said to be a set of s double points in P^n.

The homogeneous coordinate ring of Z is

$$A = R/(\wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}).$$

The ring $A = \oplus_{t\ge 0} A_t$ is a one-dimension k-graded Cohen-Macaulay algebra whose multiplicity is $e(A) = \sum_{i=1}^{s} \binom{m_i+n-1}{n}$. The Hilbert function $H_A(t) = dim_k A_t$ strictly increases until it reaches the multiplicity $e(A)$, at which it stabilizes. The regularity index of Z is defined to be the least integer t such that $H_A(t) = e(A)$, and we denote it by $reg(Z)$ or $reg(A)$. It is not easy to count the regularity index of a set of fat points. Thus, one usually finds a sharp upper bound for $reg(Z)$.

In 1961, Segre [6] showed the upper bound for regularity index of generic fat points $Z = m_1P_1 + \cdots + m_sP_s$ in P^2:

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$$reg(Z) \le max\{m_1 + m_2 - 1, [\frac{m_1 + \cdots + m_s}{2}]\}$$

with $m_1 \ge \cdots \ge m_s$. This bound was later extended for fat points in general position in P^n by Catalisano [3]

$$reg(Z) \le max\{m_1 + m_2 - 1, [\frac{m_1 + \cdots + m_s + n - 2}{n}]\}.$$

In 1996, Trung gave the following conjecture on a sharp upper bound for regularity index of arbitrary fat points in P^n :

$$reg(Z) \le max\{T_j \,/\, j = 1,...,n\},$$

where

$$T_j = max\{[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j}] \,/\, P_{i_1},...,P_{i_q} \; lieonanj - plane\},$$

this bound is said to be Segre's upper bound.

The conjecture of Trung is more general and better than the previous results. However, it is only proved right in projective spaces with dimension $n \le 3$ [5], [7], [8], for the case of a set of double points $Z = 2P_1 + \cdots + 2P_s$ in P^n with dimension $n = 4$ [9] and for a set of $s+2$ fat points which is not on an $(s-1)$-space [10]. Recently, Ballico et al have proved the case $n+3$ arbitrary fat points in P^n [2]. Up to now, there have not been any results of Trung's conjecture published yet.

In this article, we prove the conjecture of Trung right in the case $2n+1$ double points that does not exist $n+1$ points lying on an $(n-2)$-plane in P^n. The case $s \le n+3$ in P^n was proved right by Thien [10], Ballico et al [2]. That Trung's conjecture, $n+3 < s < 2n+1$ is proved right remains an open question.

2 Segre's upper bound for the regularity index of $2n+1$ double points in P^n

From now on, we consider a hyperplane and its identical defining linear form. The following propositions are the important results for the proof of Segre's upper bound.

Proposition 2.1. *Let $X = \{P_1,...,P_{2n+1}\}$ be a set of $2n+1$ distinct points and there do not exist $n+1$ points of X lying on an $(n-2)$-plane in P^n. Let $\wp_i$ be the homogeneous prime ideal corresponding $P_i, i = 1,...,2n+1$. Let*

$$Z = 2P_1 + \cdots + 2P_{2n+1}.$$

Put

$$T_j = max\{[\frac{1}{j}(2q+j-2)] \,/\, P_{i_1},...,P_{i_q} \ lieonanj-plane\},$$

$$T_Z = max\{T_j \,/\, j=1,...,n\}.$$

Then, there exists a point $P_{i_0} \in X$ such that

$$reg(R/(J+\wp_{i_0}^2)) \le T_Z,$$

where

$$J = \bigcap_{k \ne i_0} \wp_k^2.$$

Proof: We denote $/H/$ by the number points of X lying on an j-plane H. The proposition was proved in projective spaces with dimension $n \le 4$ [5], [7], [8], [9]. Thus, we will prove the case with dimension $n \ge 5$.

We can see that there are $(n-1)$-planes $H_1,...,H_d$ in P^n with d as the least integer such that the following two conditions are satisfied:

$(i)\ X \subset \cup_{i=1}^{d} H_i,$

$(ii)\ /H_i \cap (X) \setminus \bigcup_{j=1}^{i-1} H_j / = max\{/ H \cap (X \setminus \bigcup_{j=1}^{i-1} H_j) / \,/\, H \ isan\ \text{(n-1)} - plane\}.$

Since $n+1$ points do not lie on a $(n-2)$-plane, $1 \le d \le 3$. We consider the following cases:

Case 1. $d=3$. Since a hyperplane always passes through at least n points of X and $d=3$, we have $/H_1/=/H_2/=n, /H_3/=1$. We may assume that $P_{2n+1} \notin H_1 \cup H_2$. Choose $P_{2n+1} = P_{i_0} = (1,0,...,0)$, then $\wp_{i_0} = (x_1,...,x_n)$. Clearly, H_1, H_2 avoid P_{i_0}. We have $H_1H_1H_2H_2 \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$ where $c_1 + \cdots + c_n = i (i=0,1)$. By [4, Lemma 3] we have

$$reg(R/(J+\wp_{i_0}^2)) \le 4+i \le 5 \le T_Z.$$

Case 2. $d=2$. We have $X \subset H_1 \cup H_2$. Therefore, $/H_1/ \ge n+1$. We call q the number points of X lying on H_2. Put $Y=\{P_1,...,P_q\}$. Since $n+1$ points of X do not lie on a $(n-2)$-plane, Y does not lie on a $(q-3)$-plane. We consider the following cases:

Case 2.1. Y lies on a $(q-1)$-plane, and Y does not lie on a $(q-2)$-plane.

Choose $P_q = P_{i_0} = (1,0,...,0)$, $P_1 = (0,\underset{2}{1},...,0),...,$ $P_{q-1} = (0,...,\underset{q}{1},...,0)$, then $\wp_{i_0} = (x_1,...,x_2)$. Since we always have a $(q-2)$-plane, say K, passing through $P_1,...,P_{q-1}$ and avoiding P_{i_0}; therefore, we always have a hyperplane, say L, containing K and avoiding P_{i_0}. We have $H_1H_1LL \in J$. Thus $H_1H_1LLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i (i=0,1)$. By [4, Lemma 3] we have

$$reg(R/(J+\wp_{i_0}^2)) \le 4+i \le 5 \le T_Z.$$

Case 2.2. Y lies on a $(q-2)$-plane $\alpha, q \geq 3,$ then H_1 contains $2n+1-q$ points of $X.$ Consider the set $W=\{P_{q+1},...,P_{2n+1}\} \subset H_1 \cap X,$ then there are $(n-2)$-planes $Q_1,...,Q_d$ in P^n such that the following two conditions are satisfied:

$(i)\ W \subset \cup_{i=1}^{d} Q_i,$

$(ii)\ |\, Q_i \cap W \setminus \bigcup_{j=1}^{i-1} Q_j \,| = max\{|\, Q \cap (W \setminus \bigcup_{j=1}^{i-1} Q_j) \,|\ /\ Q\ is an\ \text{(n-2)}-plane\}.$

Since $q \geq 3,$ then we have $2n+1-q \leq 2n-2.$ Moreover, every $(n-2)$-plane always passes through $n-1$ points, so we have $d=2.$ We consider the following cases:

Case 2.2.1. If Q_1 contains n points, then there are the points, that is, $P_{i_1},...,$ $P_{i_{n+1-q}}$ of W lying on $Q_2.$ Choose $P_{i_1}=P_{i_0}=(1,0,...,0),$ then $\wp_{i_0}=(x_1,...,x_n).$ There always exists a $(n-2)$-plane, say $K,$ containing $P_{i_2},...,P_{i_{n+1-q}}$ and Y and avoiding P_{i_0} (if not, then there exists $n+1$ points lying on a $(n-2)$-plane). Therefore, we can choose a hyperplane L_1 containing Q_1 and a hyperplane L_2 containing K avoiding $P_{i_0}.$

We have $L_1L_1L_2L_2 \in J,$ thus $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i (i=0,1).$ By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

Case 2.2.2. If Q_1 contains $n-1$ points then we call $T=\{P_1,...,P_{n+2}\} \subset X \setminus Q_1.$ We consider the two following cases of T :

a) T lies on a $(n-1)$-plane, say $L,$ then $P_{n+3},...,P_{2n+1}$ lie on $Q_1.$ Since X does not lie on a $(n-1)$-plane, there exists a point in $Q_1 \setminus L.$ Assume that $P_{2n+1} \notin L.$ Moreover, since points on Q_1 are in the general position, there exists a $(n-3)$-plane passing through $n-2$ points of Q_1 and avoiding $P_{2n+1}.$ We call π the $(n-3)$-plane passing through $n-2$ points $P_{n+3},...,P_{2n}$ of Q_1 and $P_{2n+1} \notin \pi,$ choose $P_{2n+1}=P_{i_0}=(1,0,...,0),$ then $\wp_{i_0}=(x_1,...,x_n).$ We always have a hyperplane L_1 containing π and avoiding $P_{i_0}.$ We have $LLL_1L_1 \in J,$ thus $LLL_1L_1M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i (i=0,1).$ By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

b) T does not lie on a $(n-1)$-plane, thus T lies on $P^n.$ We have the two following cases:

- If T has $n+1$ points lying on a $(n-1)$-plane, say $L,$ then there exists a point in T without in $L.$ Assume that $P_{n+2} \notin L.$ Choose $P_{n+2}=P_{i_0}=(1,0,...,0),$ then $\wp_{i_0}=(x_1,...,x_n).$ We always have a hyperplane L_1 passing through Q_1 and avoiding $P_{i_0}.$ We have $LLL_1L_1 \in J,$ thus $LLL_1L_1M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i (i=0,1).$ By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

• If T does not have $n+1$ points lying on a $(n-1)$-plane, then T is in the general position. Choose $P_{n+2} = P_{i_0} = (\underset{1}{1},0,...,0)$, $P_1 = (0,\underset{2}{1},...,0)$, ..., $P_n = (0,0,...,0,\underset{n+1}{1})$, then $\wp_{i_0} = (x_1,...,x_n)$. For every monomial $M = x_1^{c_1}\cdots x_n^{c_n}, c_1+\cdots+c_n = i (i=0,1)$, we have $M \in \wp_1^{i-c_1} \cap\cdots\cap \wp_1^{i-c_n}$. Put $m_l = 2-i+c_l (l=1,....,n), m_{n+1}=2$ and

$$t = max\{2,[(\sum_{l=1}^{n+1} m_l + n-1)/n]\}.$$

We have

$$t+i = max\{2,[(\sum_{l=1}^{n+1} m_l + n-1)/n]\}+i$$

$$\leq max\{2+i,[(\sum_{i=1}^{n+1} m_l + ni + n-1)/n]\}$$

$$\leq max\{2+i,[(3n+2)/n] \leq 3,$$

therefore,

$$t \leq 3-i.$$

By [4, Lemma 4], we can find t $(n-1)$-planes, say $L_1,...,L_t$, avoiding P_{i_0} such that for every point $P_l (l=1,...,n+1)$, there are m_l $(n-1)$-planes of $\{L_1,...,L_t\}$ passing through P_l. Then

$$L_1\cdots L_t \in \wp_1^{m_1} \cap\cdots\cap \wp_n^{m_n} \cap \wp_{n+1}^2.$$

On the other hand, since $M \in \wp_1^{2-m_1} \cap\cdots\cap \wp_n^{2-m_n}$, we have $L_1\cdots L_t M \in \wp_1^2 \cap\cdots\cap \wp_{n+1}^2$.

Moreover, we always have an hyperplane L containing Q_1 and avoiding P_{i_0}. We have $LL \in \wp_{n+3}^2 \cap\cdots\cap \wp_{2n+1}^2$, thus $LLL_1\cdots L_t M \in J$. By [4, Lemma 3] we have

$$reg(R/(J+\wp_{i_0}^2)) \leq (5-i)+i = 5 \leq T_Z.$$

Case 3. $d=1$. We have $X \subset H_1$. Then there are $(n-2)$-planes $Q_1,...,Q_s$ in P^n, with s be the smallest number integer such that the following two conditions are satisfied:

(i) $X \subset \cup_{i=1}^s Q_i$,

(ii) $|Q_i \cap (X \setminus \bigcup_{j=1}^{i-1} Q_j)| = max\{|Q \cap (X \setminus \bigcup_{j=1}^{i-1} Q_j)| \mid Q \text{ is an } (n\text{-}2)-plane\}$.

Since the $(n-2)$-planes contain the most n points of X and they always pass through $n-1$ points, therefore $s=3$, we have the following cases:

(1) $|Q_1| = |Q_2| = n, |Q_3| = 1.$

(2) $|Q_1| = n, |Q_2 \setminus Q_1| = n-1, |Q_3 \setminus (Q_1 \cup Q_2)| = 2.$

(3) $|Q_1| = n-1, |Q_2| = n-1, |Q_3| = 3.$

Case 3.1. $|Q_1|=|Q_2|=n, |Q_3|=1$.

Assume that $P_1 \in Q_3$, therefore, $P_1 \notin Q_1 \cup Q_2$. Choose $P_1 = P_{i_0} = (1,0,...,0)$, thus $\wp_{i_0} = (x_1,...,x_n)$. We always have a hyperplane L_1 containing Q_1 and a hyperplane L_2 containing Q_2 and avoiding P_{i_0}. Therefore, $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1}\cdots x_n^{c_n}, c_1+\cdots+c_n = i (i=0,1)$. By [4, Lemma 3] we have

$$reg(R/(J+\wp_{i_0}^2)) \le 4+i \le 5 \le T_Z.$$

Case 3.2. $|Q_1|=n, |Q_2 \setminus Q_1| = n-1, |Q_3 \setminus (Q_1 \cup Q_2)| = 2$.

Assume that Q_1 contains $P_{n+2},...,P_{2n+1}$. Put $Y=\{P_1,...,P_{n+1}\}$, therefore, there is a set of points of Y lying on a $(n-1)$-plane such that there do not exist n points of Y lying on a $(n-2)$-plane. Choose $P_{n+1} = P_{i_0} = (1,0,...,0)$, $P_1 = (0,\underset{2}{1},...,0)$, ..., $P_{n-1} = (0,...,0,\underset{n}{1},...,0)$, thus $\wp_{i_0} = (x_1,...,x_n)$. For every monomial $M = x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n = i (i=0,1)$, we have $M \in \wp_1^{i-c_1} \cap \cdots \cap \wp_{n-1}^{i-c_{n-1}}$. Put $m_l = 2-i+c_l (l=1,...,n-1), m_n = 2$ and

$$t = max\{2, [(\sum_{i=1}^{n} m_l + n-2)/(n-1)]\}.$$

We have

$$t+i = max\{2, [(\sum_{i=1}^{n} m_l + n-2)/(n-1)]\} + i$$

$$\le max\{2+i, [(\sum_{i=1}^{n} m_l + (n-1)i + n-2)/(n-1)]\}$$

$$\le max\{2+i, [(3n + \sum_{j=1}^{n} c_j - 4)/(n-1)]\}$$

$$\le max\{2+i, [3n-3/(n-1)]\} \le 3.$$

Thus,

$$t \le 3-i.$$

By [4, Lemma 4] we can find t $(n-2)$-planes, say $G_1,...,G_t$ avoiding P_{i_0} such that for every point $P_l (l=1,...,n+1)$ there are m_l $(n-2)$-planes of $\{G_1,...,G_t\}$ passing through P_l. With $j=1,...,t$ we find a hyperplane L_j containing G_j and avoiding P_{i_0}. Therefore

$$L_1\cdots L_t \in \wp_1^{m_1} \cap \cdots \cap \wp_{n-1}^{m_{n-1}} \cap \wp_n^2.$$

On the other hand, since $M \in \wp_1^{2-m_1} \cap \cdots \cap \wp_{n-1}^{2-m_{n-1}}$, we have $L_1\cdots L_t M \in \wp_1^2 \cap \cdots \cap \wp_{n-1}^2 \cap \wp_n^2$. Moreover, we may choose a hyperplane L containing Q_1 and avoiding P_{i_0}, thus $LLL_1\cdots L_t M \in J$ for every $M = x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n = i (i=0,1)$. By [4, Lemma 3], we have

$$reg(J+\wp_{i_0}^2)\le(5-i)+i\le 5\le T_Z.$$

Case 3.3. $|Q_1|=n-1, |Q_2|=n-1, |Q_3|=3$. Then, there are not n points of X lying on a $(n-2)$-plane. Therefore, we consider a set of points of X lying on P^{n-1}, thus X is in the general position. Choose $P_{2n+1}=P_{i_0}=(1,0,...,0)$, $P_1=(0,\underset{2}{1},...,0),...,P_{n-1}=(0,...,\underset{n}{1},0)$, then $\wp_{i_0}=(x_1,...,x_n)$. For every monomial $M=x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n=i(i=0,1)$, we have $M\in\wp_1^{i-c_1}\cap\cdots\cap\wp_{n-1}^{i-c_{n-1}}$. Put $m_l=2-i+c_l,(l=1,...,n-1)$, $m_l=2,(l=n,...,2n)$ and

$$t=max\{2,[(\sum_{i=1}^{2n}m_l+n-2)/(n-1)]\}.$$

We have

$$t+i=max\{2,[(\sum_{i=1}^{2n}m_l+n-2)/(n-1)]\}+i$$

$$\le max\{2+i,[(\sum_{i=1}^{2n}m_l+(n-1)i+n-2)/(n-1)]\}$$

$$\le max\{2+i,[(4n+\sum_{j=1}^{n-1}c_j+n-2)/(n-1)]\}$$

$$\le max\{2+i,[2(2n+1)+(n-1)-2)/(n-1)]\}=max\{T_1,T_{n-1}\}\le T_Z.$$

By [4, Lemma 4] we can find t $(n-2)$-planes, say $G_1,...,G_t$, avoiding P_{i_0} such that for every point $P_l(l=1,...,n+1)$, then there exist m_l $(n-2)$-planes of $\{G_1,...,G_t\}$ passing through P_l, with $j=1,...,t$, we always have a hyperplane L_j containing G_j and avoiding P_{i_0}. Therefore,

$$L_1\cdots L_t\in\wp_1^{m_1}\cap\cdots\cap\wp_{n-1}^{m_{n-1}}\cap\wp_n^2\cap\cdots\cap\wp_{2n}^2.$$

On the other hand, since $M\in\wp_1^{2-m_1}\cap\cdots\cap\wp_{n-1}^{2-m_{n-1}}$, then $L_1\cdots L_tM\in\wp_1^2\cap\cdots\cap\wp_{2n}^2$. By [4, Lemma 3] we have

$$reg(J+\wp_{i_0}^2)\le t+i\le T_Z.$$

The proof of proposition 2.1 is completed.

Proposition 2.2. *Let $X=\{P_1,...,P_{2n+1}\}$ be a set of $2n+1$ distinct points and there do not exist $n+1$ points of X lying on an $(n-2)$-plane in P^n. Let $Y=\{P_{i_1},...,P_{i_s}\},2\le s\le 2n$, be a subset of X. Let $\wp_i$ be the homogeneous prime ideal corresponding $P_i,i=1,...,2n+1$. Let*

$$Z=2P_1+\cdots+2P_{2n+1}.$$

Put

$$T_j=max\{[\frac{1}{j}(2q+j-2)]/P_{i_1},...,P_{i_q}\ lieonanj-plane\},$$

$$T_Z = max\{T_j \ / \ j = 1,...,n\}.$$

Then, there exists a point $P_{i_0} \in Y$ such that

$$reg(R / (J + \wp_{i_0}^2)) \leq T_Z,$$

where

$$J = \bigcap_{P_k \in Y \setminus \{P_{i_0}\}} \wp_k^2.$$

Proof: We denote $/ H /$ by the number points of X lying on a j-plane H. The proposition was proved in projective spaces with dimension $n \leq 4$ [5], [7], [8], [9]. Thus, we will prove the case with dimension $n \geq 5$. Firstly, we can see that $T_Z \geq 5$. Without loss of generality, assume that $Y = \{P_1,...,P_s\}$. We consider the following cases:

Case 1. $2 \leq s \leq n+1$. Since there are not $n+1$ points lying on an $(n-2)$-plane, then we have the two following cases of Y:

Case 1.1. Y does not lie on a $(s-2)$-plane. Therefore, there exists a $(s-2)$-plane, say α, passing through $s-1$ points $P_1,...,P_{s-1}$ and avoiding P_s. Choose $P_s = P_{i_0} = (1,0,...,0)$; therefore, $\wp_{i_0} = (x_1,...,x_n)$. We always have a hyperplane L containing α and avoiding P_{i_0}. We have $LL \in J$; therefore, $LLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \leq 2 + i \leq 5 \leq T_Z.$$

Case 1.2. Y lies on a $(s-2)$-plane. Then, there exists a $(s-3)$-plane, say β, passing through l points of $Y, s-2 \leq l \leq s-1$. We consider the following two cases:

- $l = s-1$. Assume that $P_1,...,P_{s-1} \in \beta$. Choose $P_s = P_{i_0} = (1,0,...,0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. We always have a hyperplane L containing β and avoiding P_{i_0}. We have $LL \in J$; therefore, $LLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \leq 2 + i \leq 5 \leq T_Z.$$

- $l = s-2$. Assume that $P_1,...,P_{s-2} \in \beta$. Choose $P_s = P_{i_0} = (1,0,...,0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. We always have a hyperplane L_1 containing β; L_2 passing through P_{s-1} and avoiding P_{i_0}. We have $L_1L_1L_2L_2 \in J$; therefore, $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

Case 2. $n+2 \leq s \leq 2n$. Assume that $Y = \{P_1,...,P_s\}$ is a set of s double points on P^n. Then there are $(n-1)$-planes $H_1,...,H_d$ in P^n with d as the least integer such that the following two conditions are satisfied:

$(i) Y \subset \cup_{i=1}^{d} H_i,$

$(ii) \mid H_i \cap (Y \setminus \bigcup_{j=1}^{i-1} H_j) \mid = max\{\mid H \cap (Y \setminus \bigcup_{j=1}^{i-1} H_j) \mid \mid H \text{ is a } \text{(n-1)} - plane\}.$

Since every hyperplane always passes through at least n points of X, we have $1 \le d \le 2$. We consider the following cases of d:

Case 2.1. $d = 2$. We have $Y \subset H_1 \cup H_2, \mid H_1 \mid \ge \mid H_2 \mid, \mid H_1 \mid \ge n$. We call q the number points of Y lying on $H_2, 1 \le q \le n$. Without loss of generality, assume that $P_1,...,P_q$ lying on H_2. Put $V = \{P_1,...,P_q\}$. Since there are not $n+1$ points of X lying on a $(n-2)$-plane, we see that V does not lie on a $(q-3)$-plane. We consider the following cases:

Case 2.1.1. V lies on a $(q-1)$-plane and does not lie on a $(q-2)$-plane. Choose $P_q = P_{i_0} = (1,0,...,0)$, $P_1 = (0,\underset{2}{1},...,0),...,$ $P_{q-1} = (0,...,\underset{q}{1},...,0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. There always exists a $(q-2)$-plane, say K, passing through $P_1,...,P_{q-1}$ and avoiding P_{i_0}, therefore we always have a hyperplane L containing K and avoiding P_{i_0}. We have $H_1H_1LL \in J$; therefore, $H_1H_1LLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \le 4 + i \le 5 \le T_Z.$$

Case 2.1.2. V lies on a $(q-2)$-plane α, we have $3 \le q \le n-1$. Then $\mid H_1 \mid$ contains $s-q$ points of Y. Assume that $W = \{P_{q+1},...,P_s\} \subset Y \cap H_1$. Then, there are two $(n-2)$-planes Q_1, Q_2 such that the following two conditions are satisfied:

$(i) W \subset Q_1 \cup Q_2,$

$(ii) \mid Q_i \cap W \setminus \bigcup_{j=1}^{i-1} Q_j \mid = max\{\mid Q \cap (W \setminus \bigcup_{j=1}^{i-1} Q_j) \mid \mid Q \text{ is a } \text{(n-2)} - plane\}.$

We consider the following two cases of Q_1:

Case 2.1.2.1. Q_1 contains n points. Then, $H_1 \ge n+1$; so, $s \ge n+q+1$. From the conditions of (i) and (ii), there are $s-n-q$ points of Y lying on Q_2, assume that $P_{i_1},...,P_{s-n-q} \in Q_2$. Choose $P_{i_1} = P_{i_0} = (1,0,...,0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. Since $s-n-3 \le n-3$, there is a $(s-n-3)$-plane, say K, containing V and $P_{i_2},...,P_{s-n-q}$ and avoiding P_{i_0} (if not, then there are $n+1$ points of X lying on a $(n-2)$-plane). Therefore, we always have an hyperplane L_1 containing Q_1, and a hyperplane L_2 containing K and avoiding P_{i_0}. We have $L_1L_1L_2L_2 \in J$; therefore, $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R / (J + \wp_{i_0}^2)) \le 4 + i \le 5 \le T_Z.$$

Case 2.1.2.2. Q_1 contains $n-1$ points. Assume that $T = \{P_1,...,P_{s-n+1}\}$ is a subset of $s-n+1$ points of Y which does not lie on Q_1. Since $\alpha \cap Y \subseteq T$, a set of points of T lie on an $(s-n-1)$-plane. We have $s-n-1 \le n-1$. We call β a $(s-n-1)$-plane containing T. Since a set of points of Y

does not lie on a $(n-1)$-plane, there exists a point in $Q_1 \setminus \beta$, we may assume that it is P_s. Choose $P_s = P_{i_0} = (1,0,...,0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. We consider on a $(n-2)$-plane Q_1. Since there exists a $(n-3)$-plane, say π, passing through $n-2$ points of $Y \cap Q_1$ and avoiding P_{i_0}, we always have a hyperplane L_1 containing β, and an hyperplane L_2 containing π and avoiding P_{i_0}. We have $L_1L_1L_2L_2 \in J$; therefore, $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R/(J + \wp_{i_0}^2)) \le 4 + i \le 5 \le T_Z.$$

Case 2.2. $d = 1$. We have $Y \subset H_1$. Then, there are $(n-2)$-planes $Q_1,...,Q_r$ in P^n such that the following two conditions are satisfied:

$(i) Y \subset \cup_{i=1}^r Q_i$,

$(ii) |Q_i \cap (Y \setminus \bigcup_{j=1}^{i-1} Q_j)| = max\{|Q \cap (Y \setminus \bigcup_{j=1}^{i-1} Q_j)| \mid Q \text{ is an (n-2)} - plane\}$.

Since $(n-2)$-planes contain the most n points of Y and they always pass through $n-1$ points, we have $2 \le r \le 3$. We consider the following cases of r.

Case 2.2.1. $r = 3$. Then $2n \ge s \ge 2n-1$. Since every $Q_i (i = 1,2)$ contains the most $n-1$ points, we have $|Q_1 \cup Q_2| \ge n-2$. So $1 \le |Q_3 \setminus (Q_1 \cup Q_2)| \le 2$. We consider the following cases of $|Q_3|$.

a) $|Q_3 \setminus (Q_1 \cup Q_2)| = 1$. Assume that $P_s \in Q_3, P \notin Q_1 \cup Q_2$. Choose $P_s = P_{i_0} = (1,0,...,0)$ then $\wp_{i_0} = (x_1,...,x_n)$. We always have a hyperplane L_1 containing Q_1, and an hyperplane L_2 containing Q_2 and avoiding P_{i_0}. We have $L_1L_1L_2L_2 \in J$; therefore, $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$. By [4, Lemma 3] we have

$$reg(R/(J + \wp_{i_0}^2)) \le 4 + i \le 5 \le T_Z.$$

b) $|Q_3 \setminus (Q_1 \cup Q_2)| = 2$. Then $s = 2n, |Q_1| = |Q_2| = n-1$. Thus there are not any n points of Y lying on an $(n-2)$-plane. Choose $P_{2n} = P_{i_0} = (\underset{1}{1},0,...,0)$, $P_1 = (0,\underset{2}{1},...,0),...,P_{n-1} = (0,...,\underset{n}{1},0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. For every monomial $M = x_1^{c_1} \cdots x_n^{c_n}$, $c_1 + \cdots + c_n = i (i = 0,1)$, we have $M \in \wp_1^{i-c_1} \cap \cdots \cap \wp_{n-1}^{i-c_{n-1}}$. Put $m_l = 2 - i + c_l (l = 1,...,n-1)$, $m_l = 2 (l = n,...,2n-1)$ and

$$t = max\{2, [(\sum_{i=1}^{2n-1} m_l + n - 2)/(n-1)]\}$$

We have

$$t + i = max\{2, [(\sum_{i=1}^{2n-1} m_l + n - 2)/(n-1)]\} + i$$

$$\le max\{2 + i, [(\sum_{i=1}^{2n-1} m_l + (n-1)i + n - 2)/(n-1)]\}$$

$$\le max\{2+i,[(4n+\sum_{j=1}^{n-1}c_j+n-4)/(n-1)]\}$$

$$\le max\{2+i,[2(2n+1)+(n-1)-2)/(n-1)]\}=max\{T_1,T_{n-1}\}\le T_Z.$$

By [4, Lemma 4], we can find t $(n-2)$-plane, say $G_1,...,G_t$ avoiding P_{i_0} such that for every point $P_l(l=1,...,n),$ there exists an m_l $(n-2)$-plane of $\{G_1,...,G_t\}$ passing through P_l. With $j=1,...,t,$ we always have an hyperplane L_j containing G_j and avoiding P_{i_0}. Then

$$L_1\cdots L_t\in\wp_1^{m_1}\cap\cdots\cap\wp_{n-1}^{m_{n-1}}\cap\wp_n^2\cap\cdots\cap\wp_{2n}^2.$$

On the other hand, since $M\in\wp_1^{2-m_1}\cap\cdots\cap\wp_{n-1}^{2-m_{n-1}}$, then we have $L_1\cdots L_tM\ \in\wp_1^2\cap\cdots\cap\wp_{2n}^2$. By [4, Lemma 3], we have

$$reg(J+\wp_{i_0}^2)\le t+i\le T_Z.$$

Case 2.2.2. $r=2$. We have $Y\subset Q_1\cup Q_2$. We consider the following subcases of Q_1:

Subcase 2.2.2.1. $|Q_1|=n-1$. Assume that $P_1,...,P_{n-1}$ are $n-1$ points of Y lying on Q_1. Since Y does not lie on a $(n-2)$-plane, there exists a point in $Q_1\setminus Q_2$, assume that $P_1\in Q_1\setminus Q_2$. Moreover, since points on Q_1 are in the general position, there exists a $(n-3)$-plane, say α, passing through $n-2$ points $P_2,...,P_{n-1}$ and avoiding P_1. Choose $P_1=P_{i_0}=(1,0,...,0)$, then we have $\wp_{i_0}=(x_1,...,x_n)$. We always have an hyperplane L_1 containing α, and an hyperplane L_2 containing Q_2 and avoiding P_{i_0}. We have $L_1L_1L_2L_2\in J;$ therefore, $L_1L_1L_2L_2M\in J$ for every monomial $M=x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n=i(i=0,1)$. By [4, Lemma 3], we have

$$reg(R/(J+\wp_{i_0}^2))\le 4+i\le 5\le T_Z.$$

Subcase 2.2.2.2. $|Q_1|=n$. Assume that $P_1,...,P_n\in Q_1\cap Y$ and $P_{n+1},...,P_s\in Y\setminus Q_1$. Put $U=\{P_{n+1},...,P_s\}$. Since $n+2\le s\le 2n$, then $2\le s-n\le n$. We consider the two following cases:

a) U lies on a $(s-n-1)$-plane. Then a set of points of U is in the general position, there exists a $(s-n-2)$-plane, say γ_1, passing through $s-n-1$ points $P_{n+1},...,P_{s-1}$ and avoding P_s. Choose $P_s=P_{i_0}=(1,0,...,0)$, then we have $\wp_{i_0}=(x_1,...,x_n)$. We always have a hyperplane L_1 containing γ_1, and an hyperplane L_2 containing Q_1 and avoiding P_{i_0}. We have $L_1L_1L_2L_2\in J;$ therefore, $L_1L_1L_2L_2M\in J$ for every monomial $M=x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n=i(i=0,1)$. By [4, Lemma 3], we have

$$reg(R/(J+\wp_{i_0}^2))\le 4+i\le 5\le T_Z.$$

b) U lies on a $(s-n-2)$-plane, say γ_2. Then $3\le s-n$ and $\gamma_2\cap Q_1\cap Y=\varnothing$ (if not, then there are $n+1$ points of X lying on an $(n-2)$-plane). We consider the following cases:

- If there are $n-1$ points of $Y\cap Q_1$ lying on a $(n-3)$-plane, say γ, then there exists a point, assume that $P_1\in Q_1\setminus\gamma$. Choose $P_1=P_{i_0}=(1,0,...,0)$, then we have $\wp_{i_0}=(x_1,...,x_n)$. We always have a hyperplane L_1 containing γ, and an hyperplane L_2 containing γ_2 and avoiding P_{i_0}. We have $L_1L_1L_2L_2 \in J$, therefore, $L_1L_1L_2L_2M \in J$ for every monomial $M = x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n = i(i=0,1)$. By [4, Lemma 3] we have

$$reg(R/(J+\wp_{i_0}^2)) \le 4+i \le 5 \le T_Z.$$

• If there are not any $n-1$ points of $Y\cap Q_1$ lying on a $(n-3)$-plane, then every $(n-3)$-plane only passes through $n-2$ points of $Y\cap Q_1$. Choose $P_n = P_{i_0} = (\underset{1}{1},0,...,0)$, $P_1 = (0,\underset{2}{1},...,0),...,P_{n-2} = (0,...,0,\underset{n-1}{1},0,0)$, then we have $\wp_{i_0} = (x_1,...,x_n)$. For every monomial $M = x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n = i(i=0,1)$, we have $M \in \wp_1^{i-c_1}\cap\cdots\cap\wp_{n-2}^{i-c_{n-2}}$. Put $m_l = 2-i+c_l(l=1,...,n-2)$, $m_{n-1}=2$ and

$$t = max\{2,[(\sum_{i=1}^{n-1} m_l + n-3)/(n-2)]\}.$$

$$\text{we have } t+i = max\{2,[(\sum_{i=1}^{n-1} m_l + n-3)/(n-2)]\}+i$$

$$\le max\{2+i,[(\sum_{i=1}^{n-1} m_l + (n-2)i+n-3)/(n-2)]\}$$

$$\le max\{2+i,[(3n+\sum_{j=1}^{n-1} c_j - 5)/(n-2)]\}$$

$$\le max\{2+i,[3n-4/(n-2)]\} = 3.$$

Therefore

$$t \le 3-i.$$

By [4, Lemma 4], we can find t $(n-3)$-planes, say $G_1,...,G_t$, avoiding P_{i_0} such that for every point $P_l(l=1,...,n)$, there exist m_l $(n-3)$-planes of $\{G_1,...,G_t\}$ passing through P_l. With $j=1,...,t$, we always have a hyperplane L_j containing G_j and avoiding P_{i_0}. Then,

$$L_1\cdots L_t \in \wp_1^{m_1}\cap\cdots\cap\wp_{n-2}^{m_{n-2}}\cap\wp_{n-1}^2.$$

On the other hand, since $M \in \wp_1^{2-m_1}\cap\cdots\cap\wp_{n-2}^{2-m_{n-2}}$, we have $L_1\cdots L_tM \in \wp_1^2\cap\cdots\cap\wp_{n-1}^2$ for every monomial $M = x_1^{c_1}\cdots x_n^{c_n}$, $c_1+\cdots+c_n = i(i=0,1)$. Moreover, we always have an hyperplane L containing γ_2 and avoiding P_{i_0}. We have $LL \in \wp_{n+1}^2\cap\cdots\cap\wp_s^2$, therefore $LLL_1\cdots L_tM \in \wp_1^2\cap\cdots\cap\wp_{n-1}^2\cap\wp_{n+1}^2\cap\cdots\cap\wp_s^2 = J$. By [4, Lemma 3], we have

$$reg(J+\wp_{i_0}^2) \le (5-i)+i = 5 \le T_Z.$$

The proof of Proposition 2.2 is completed.

The following theorem is the main result of this article.

Theorem 2.3. *Let* $X=\{P_1,...,P_{2n+1}\}$ *be a set of* $2n+1$ *distinct points in* P^n *such that there are not* $n+1$ *points of* X *lying on an* $(n-2)$ *-plane. Let*

$$Z=2P_1+\cdots+2P_{2n+1}.$$

Then

$$reg(Z)\le max\{T_j\ /\ j=1,...,n\}=T_Z,$$

where

$$T_j=\{[\frac{2q+j-2}{j}]\ /\ P_{i_1},...,P_{i_q}\ lieonanj-plane\}.$$

Proof: Firstly, we have the following claim:

Let $X=\{P_1,...,P_{2n+1}\}$ *in* P^n, $Y=\{P_{i_1},...,P_{i_s}\}$ *be a subset of* $X, 1\le s\le 2n.$ *Then*

$$reg(R/J_s)\le T_Z,$$

where

$$J_s=\bigcap_{P_i\in Y}\wp_i^2.$$

We will prove this claim by induction on the number points of Y.

If $s=1$. Let $\wp_1$ be the defining homogeneous prime ideal of P_1. Put $J_1=\wp_1^2, A=R/J_1$. Then,

$$reg(R/J_1)=1\le T_Z.$$

Assume that the claim is right for all subsets Y of X whose number points are smaller or equal to $s-1$. Let $Y=\{P_{i_1},...,P_{i_s}\}$. By Proposition 2.2, there exists a point $P_{i_0}\in Y$ such that

$$reg(R/(J_{s-1}+\wp_{i_0}^2))\le T_Z, \qquad (1)$$

where $J_{s-1}=\bigcap_{P_i\in Y\backslash\{P_{i_0}\}}\wp_i^2$. Note that, J_{s-1} is the intersection of ideals containing $s-1$ double points of Y. By conjecture of induction, we have

$$reg(R/J_{s-1})\le T_Z. \qquad (2)$$

By [4, Lemma 1], we have

$$reg(R/J_s)=\{1, reg(R/(J_{s-1}), reg(R/(J_{s-1}+\wp_{i_0}^2))\}. \qquad (3)$$

From (1), (2) and (3) we have

$$reg(R/J_s)\le T_Z.$$

The proof of the above claim is completed.

Now, we prove Theorem 2.3. Let $X=\{P_1,...,P_{2n+1}\}$ in P^n. By Proposition 2.1, there exists a point $P_{i_0}\in X$ such that

$$reg(R/(J+\wp_{i_0}^2)) \le T_Z. \qquad (4)$$

where $J = \bigcap_{P_i \in X \setminus \{P_{i_0}\}} \wp_i^2$. Note that, J is the intersection of ideals containing $2n$ double points of X.

Therefore, by the above claim with $s = 2n,$ we have :

$$reg(R/J) \le T_Z. \qquad (5)$$

By [4, Lemma 1], we have

$$regR/I = \{1, reg(R/J), reg(R/(J+\wp_{i_0}^2))\} \qquad (6)$$

where $I = J \cap \wp_{i_0}^2$.

From (4), (5) and (6) we have

$$reg(Z) \le T_Z.$$

The proof of Theorem 2.3 is completed.

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