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Ổn định trung bình mũ của hệ ngẫu nhiên rời rạc có độ trễ biến thiên theo thời gian, thông qua cách tiếp cận dựa trên IPR

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TÓM TẮT

Kỹ thuật IPR (Internally Positive Representation) giúp chuyển bài toán xét độ ổn định của hệ tuyến tính không dương thành xét độ ổn định của hệ tuyến tính dương mà ma trận hệ số được xây dựng từ việc trích xuất các ma trận của hệ thống cần xem xét. Trong bài báo này, chúng tôi trình bày sự phát triển của cách tiếp cận dựa trên IPR cho một lớp các hệ ngẫu nhiên rời rạc với trễ thời gian. Nghiên cứu của chúng tôi cố gắng tìm một ước lượng α mũ cho giá trị tuyệt đối của kỳ vọng của vectơ trạng thái. Để làm được điều này, đầu tiên, chúng tôi xem xét tính ổn định mũ của một hệ ngẫu nhiên rời rạc dương có độ trễ thay đổi theo thời gian. Ở đây cả tính dương và tính ổn định hàm mũ được xem xét theo nghĩa kỳ vọng. Tiếp theo, bằng cách sử dụng kỹ thuật IPR, chúng tôi phát triển kết quả thu được cho hệ ngẫu nhiên không dương. Cuối cùng, chúng tôi đưa ra một số ví dụ minh họa cho tính hiệu quả của phương pháp vừa phát triển.

Từ khóa: *Internally Positive Representation, ước lượng trạng thái trung bình mũ, hệ ngẫu nhiên rời rạc, trễ biến thiên theo thời gian bị chặn.*

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Exponential mean stability of stochastic discrete-time systems with time-varying delays via an IPR-based approach

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ABSTRACT

The internally positive representation (IPR) technique helps to reformulate the stability problem of non-positive linear systems into the stability problem of a class of positive linear systems, whose matrices are constructed from the extraction of matrices of the considered system. In this paper, we present a development of the IPR-based approach to a class of stochastic discrete-time systems with time-delays. Our study is devoted to the problem of finding an α-exponential estimate of the absolute of the expectation of the state vector. For this, firstly, we investigate the the exponential stability problem for a class of positive stochastic discrete-time systems with time-varying delays. Here, both the positivity and the exponential stability are considered in the sense of expectation. Next, by using the IPR technique, we develop the obtained result to a class of non-positive stochastic systems. Finally, a numerical example is given to illustrate the effectiveness of the developed approach.

Keywords: *Internally positive representation (IPR), exponential mean state estimate, stochastic discrete-time systems, bounded time-varying delay.*

1. INTRODUCTION

The IPR-based approach for analyzing the stability of linear dynamical systems has been proposed by the authors in.[1–3] It includes two main steps: (1) Constructing a positive linear system, whose matrices are designed from the extraction of matrices of the considered system, such that the stability of a considered system is followed from the stability of the constructed positive system; (2) Analyzing the stability of the constructed positive linear system. In five recent years, this approach has been developed to some classes of linear systems with time-delays, e.g., continuous-time linear systems with time-varying delays,[4,5] singular linear systems with time-varying delays,[6–8] difference equations with constant time-delays.[9] To the best of our knowledge, so far, there has not been any result which reported on the IPR-based approach to the stability problem of classes of stochastic systems with time-delays. This unsolved problem, therefore, will be investigated in this paper.

Because of a vast applicability in many areas such as finance, economics, biology, physics, communication,..., the topic on stability analysis of stochastic systems has been an attractive research issue for past decades, see, e.g.,[10–14] and the references therein. Rather than the asymptotic stability where guarantees only the convergence of the state vector, the exponential stability with a given rate and a known factor can provide more quantitative estimates, which is often required in practical applications. Most of existing results reported on two types of the exponential stability for classes of stochastic systems, including: (1) p-th moment exponential stability with $p \geq 2$ (in the case $p = 2$, it is referred as mean square exponential stability);[15–19] (2) Almost-sure exponential stability.[20–23] In 2014, Bolzern et al.,[13] proposed and investigated the 1-th moment exponential stability (i.e., the exponential mean stability) for a class of Markov jump systems. They have also shown that the estimate of the expectation of the state vector obtained from the exponential mean stability is more accurate than the one obtained from the exponential mean square stability. Recently, the problem of exponential mean stability has also been developed to some classes of positive Markov jump systems with/without time-delays.[24–26] However, it seems that, so far, there

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has not been any result reported on the exponential mean stability problem for classes of non-Markov jump stochastic systems.

Within ten recent years, a considerable number of research attention has been paid to the problem on stability analysis of classes of positive linear systems with time-varying delays. There are two popular approaches, including: (1) the approach based on the solution comparisons between a positive system with time-varying delays and the one with constant time-delays,[27–37] and (2) the approach based on the direct comparison between the system solution and a constructed exponential function.[37–42] With regarding to the topic on the stability analysis of positive stochastic systems, almost existing works have reported only on classes of positive stochastic systems of the Markov jump type. Because coefficient matrices of a positive Markov jump system belong to a predefined set of Metzler or nonnegative matrices, under a positive initial condition, its state vector is always positive.[13,43–47] Very recently, Liang and Jin,[52] have considered two other classes of positive stochastic (non-Markov) systems, these are more general notions of positivity for classes of stochastic (non-Markov) systems: the positivity in the sense of probability and the positivity in the sense of expectation. The former notion means that, for a predefined threshold, m, between 0 and 1, from any positive initial condition, there exists a time point such that from this time point, the probability that state vectors of the system are non-negative is not less than the threshold m. The latter notion is used for stochastic systems in which the expectation of state vectors is always non-negative for any positive initial condition. These two notions of positive stochastic systems have not yet been developed to any discrete-time or continuous-time stochastic systems with time delays.

Motivated by these discussions, in this paper, we will consider a class of linear stochastic discrete-time systems with time-varying delays which can be seen as a discrete-time version of stochastic continuous-time systems with time-varying delays and random uncertainties. An IPR is firstly constructed to deal with the α-exponential mean estimate for an arbitrary (not necessarily positive) system. We then develop results about the exponential mean stability for positive stochastic systems. Under the effect of stochastic factors, the system is not positive in the normal sense as in deterministic systems. However, we will show that under some conditions on coefficient matrices and stochastic process, the system is still positive in the sense of expectation (see Definition 4.1 below). From this property, we will study the monotonicity of the expectation of the state vector of stochastic systems. As a result, for the first time, a solution comparison principle for linear stochastic discrete-time systems with time-varying delays will be introduced. By using this solution comparison and a state transformation, we will derive an α-exponential mean estimate for state vector of positive stochastic systems. A sufficient condition for the α-exponential mean boundedness of positive stochastic systems (in the sense of expectation) which is based on the spectral property of the coefficient matrices is then introduced. This new approach will give us an estimation with time-varying coefficients. For the sake of demonstrating the effectiveness of the IPR approach, we also introduce another result about the α-exponential mean boundedness of linear stochastic systems which is based on positive "upper bound" systems. Together with theoretical results, a numerical example is also conducted to show that the approach based on the IPR will give us i) a less conservative condition for the exponential mean stability of the stochastic discrete-time system with time-varying delays than the approach based on an "upper bound" system, and ii) a more accurate α-exponential mean estimates of state vector of the system. Our contributions in this paper can be summarized as below:

- For the first time, an IPR is applied for a class of stochastic discrete-time systems with time-varying delays to derive an α-exponential mean estimate for state vectors.
- The notion of the positivity in the sense of expectation is introduced for a class of linear stochastic discrete-time systems with time-varying delays. We then prove some sufficient conditions for stochastic systems to be positive in the sense of expectation. A solution comparison between positive stochastic systems with time-varying delays is established. This comparison is then combined with a state transformation to derive an α-exponential mean estimate for positive stochastic discrete-time systems with time-varying delays. This is the key tool for the IPR method.

The paper is organized as below. The next section introduces notations, definitions and some preliminary results. The IPR approach is presented in Section 3 to derive an α-exponential mean estimate of non-positive linear stochastic discrete-timeys stems. Section 4 is devoted to positivity and solution comparison principle of linear stochastic systems. In addition, an α-exponential mean estimate for positive stochastic systems is also established in this section. Section 5 introduces another approach to obtain an α-exponential mean estimate for stochastic systems which is based on "upper bound" systems. An illustrative example will be presented in Section 6 to verify theoretical results. Section 7 concludes the paper.

2. SYSTEM, NOTATIONS AND DEFINITIONS

Notations: $\mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{R}^n_{0,+}$ are respectively the set of nonnegative integers, the n-dimensional vector space and the nonnegative orthant in $\mathbb{R}^n$; $e = [1\ 1\ \cdots\ 1]^\top \in \mathbb{R}^n$; for two vectors $x = [x_1\ x_2\ \cdots\ x_n]^\top$, $y = [y_1\ y_2\ \cdots\ y_n]^\top$ in $\mathbb{R}^n$, two $n \times n$-matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $x \prec y$ ($x \preceq y$) means that $x_i < y_i$ ($x_i \le y_i$), $\forall i = 1, \cdots, n$ and $A \prec B$ ($A \preceq B$) means that $a_{ij} < b_{ij}$ ($a_{ij} \le b_{ij}$), $\forall i, j = 1, \cdots, n$; A is a nonnegative matrix if $0 \preceq A$; $x \succeq y$ ($A \succeq B$) means that $y \preceq x$ ($B \preceq A$); $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is the spectral radius of A; I_n is the identity matrix of size n. The maximum, minimum of a finite set of vectors (of matrices) are understood componentwise. Similarly, the absolute value of a matrix A (or a vector x) is also understood componentwise.

Let (Ω, σ, P) be a basic probability space. The notation $\mathbb{E}\zeta$ denotes the componentwise expectation of a random variable $\zeta = [\zeta_1\ \zeta_2\ \cdots \zeta_n]^\top$ in $\mathbb{R}^n$, i.e. $\mathbb{E}\zeta = [\mathbb{E}\zeta_1\ \mathbb{E}\zeta_2\ \cdots \mathbb{E}\zeta_n]^\top$. For a positive integer $h \in \mathbb{N}$, let $\mathcal{C}([-h, 0], \mathbb{R}^n)$ be the set of all functions $\phi : \{-h, -h+1, \cdots, 0\} \to \mathbb{R}^n$. Let $\mathcal{C}_\sigma([-h, 0], \mathbb{R}^n)$ be the family of $\mathcal{C}([-h, 0], \mathbb{R}^n)$-valued random variables on (Ω, σ, P) such that $\max\limits_{s \in \{-h, -h+1, \cdots, 0\}} |\mathbb{E}\phi(s)| < \infty$, for any $\phi(\cdot) \in \mathcal{C}_\sigma([-h, 0], \mathbb{R}^n)$.

The result in this paper can be easily developed for linear stochastic discrete-time systems with multiple time-varying delays. Hence, for the sake of simplicity, in this paper, we consider the following linear stochastic discrete-time system with a time-varying delay

$$\begin{aligned} x(t+1) =& [A_0 + \xi(t)B_0]x(t) \qquad (1)\\ &+ [A_1 + \xi(t)B_1]x(t - h_1(t)),\ t \in \mathbb{N},\\ x(s) =& \phi(s),\ s \in \{-h, -h+1, \cdots, 0\}, \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $\xi(t)$ is a scalar random process satisfying $\mathbb{E}\xi(t) = \hat{\xi}$ $\forall t \in \{-h, -h+1, \cdots, 0\}$, for some $\hat{\xi} \in \mathbb{R}$ and $\xi(0), \xi(1), \cdots$ are assumed to be mutually independent; A_0, A_1, B_0 and B_1 are four matrices in $\mathbb{R}^{n\times n}$; an unknown time-varying delay $h_1(t) \in [0, h]$, where $h > 0$ is a known integer and $\phi \in \mathcal{C}_\sigma([-h, 0], \mathbb{R}^n)$ is an unknown random function satisfying

$$|\mathbb{E}\phi(s)| \preceq \overline{\phi}(s),\ s \in \{-h, -h+1, \cdots, 0\}, \quad (2)$$

where $\overline{\phi} \in \mathcal{C}([-h, 0], \mathbb{R}^n_{0,+})$ is a known function. Let us denote by $x(t, \phi)$ the unique solution, under the initial value function $\phi(\cdot)$, of system (1). The process $\xi(t)$ whose the expectation equals to 0 is considered very often in discrete-time stochastic systems, e.g..[17,48–50] In this paper, we will study the system (1) under a more general stochastic process $\xi(t)$.

Inspired by the notion of exponential mean stability in,[13] we introduce the following definition of α-exponentially mean boundedness for the stochastic system (1).

Definition 2.1. Let $\alpha > 1$ be a positive real number. System (1) is said to be α-exponentially mean bounded if there exist a vector-valued function $\eta(t, A_0, B_0, A_1, B_1, h, \bar{\phi}, \hat{\xi}) \in \mathbb{R}^n_{0,+}$, $t \in \mathbb{N}$ such that

$$|\mathbb{E}x(t, \phi)| \preceq \eta(t, \cdot)\alpha^{-t},\ \ t \in \mathbb{N}. \quad (3)$$

The function $\eta(\cdot)$ is called the factor function. In the case where system (1) is α-exponentially mean bounded, for some $\alpha > 1$, an estimation in the form (3) is called an α-exponential mean estimate of this system.

Before introducing main results of this paper in next sections, we recall a well-known result related to properties of Schur matrices.

Lemma 2.2 ([51])**.** Let M be a nonnegative matrix in $\mathbb{R}^{n\times n}$. Then M is Schur matrix, i.e. $\rho(M) < 1$, if and only if one of the following conditions holds: i) there exits an vector $z \in \mathbb{R}^n_+$ such that $(M - I_n)z \prec 0$; ii) $(I_n - M)^{-1} \succeq 0$.

3. AN IPR APPROACH FOR α-EXPONENTIAL MEAN BOUNDEDNESS OF NON-POSITIVE STOCHASTIC SYSTEMS

In this section, by using an IPR approach for the system (1), we will establish an exponential mean estimate for this system. For any $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, let us introduce the following min-positive representation,[9]

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, \quad (4)$$

where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}$,

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and

$$\Pi(M) = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}, \quad (5)$$

where $M^+ = \max\{M, 0\}, M^- = \max\{-M, 0\}$.

Let us consider the following stochastic system

$$\begin{aligned} \hat{x}(t+1) =& \Pi(A_0 + \xi(t)B_0)\,\hat{x}(t) \quad (6) \\ &+ \Pi(A_1 + \xi(t)B_1)\,\hat{x}(t - h_1(t)),\ t \in \mathbb{N}, \\ \hat{x}(s) =& \pi(\phi(s)),\ s \in \{-h, -h+1, \cdots, 0\}. \end{aligned}$$

The next lemma gives us some properties of the min-positive representation.

Lemma 3.1. Let $x(t, \phi)$ and

$$\hat{x}(t, \phi) = [\hat{x}^1(t, \pi(\phi))\ \hat{x}^2(t, \pi(\phi))] \in \mathbb{R}^{2n}$$

be respectively the solution of (1) and (6). For all $t \in \mathbb{N}$, we then have

(a) $x(t, \phi) = x(t, \phi)^+ - x(t, \phi)^-$ and $|x(t, \phi)| = x(t, \phi)^+ + x(t, \phi)^-$;

(b) $M = M^+ - M^-$ and $|M| = M^+ + M^-$;

(c) $x(t, \phi) = \hat{x}^1(t, \pi(\phi)) - \hat{x}^2(t, \pi(\phi))$.

Proof. The first two properties can be verified easily. The proof of (c) can be found in,[9] Theorem 6. □

For each $t \in \mathbb{N}$, Lemma 3.1-(c) implies that

$$\begin{aligned} |\mathbb{E}x(t, \phi)| &= |\mathbb{E}\hat{x}^1(t, \pi(\phi)) - \mathbb{E}\hat{x}^2(t, \pi(\phi))| \quad (7) \\ &\leq |\mathbb{E}\hat{x}^1(t, \pi(\phi))| + |\mathbb{E}\hat{x}^2(t, \pi(\phi))|. \end{aligned}$$

We will prove, in the next section (see Lemma 4.2-*i*)), that the system (6) is positive in the sense of expectation. In particular, we have

$$|\mathbb{E}\hat{x}(t, \pi(\phi))| = \mathbb{E}\hat{x}(t, \pi(\phi)) \succeq 0,\ \forall t \in \mathbb{N}.$$

Moreover, we will prove (see Theorem 4.4 below) that for any $\alpha > 1$ which satisfies

$$\rho\left(\alpha\Pi\left(A_0 + \hat{\xi}B_0\right) + \alpha^{h+1}\Pi\left(A_1 + \hat{\xi}B_1\right)\right) < 1,$$

we have

$$\mathbb{E}\hat{x}(t, \pi(\phi)) \preceq \hat{z}(t, \hat{\phi})\alpha^{-t},\ t \in \mathbb{N}, \quad (8)$$

where $\hat{\phi} \in \mathcal{C}([-h, 0], \mathbb{R}^n)$ will be defined later and $\hat{z}(t, \hat{\phi})$ is the solution of the following system

$$\begin{aligned} \hat{z}(t+1) =& \alpha\Pi(A_0 + \xi(t)B_0)\,\hat{z}(t) \quad (9) \\ &+ \alpha^h\Pi(A_1 + \xi(t)B_1)\,\hat{z}(t - h_1(t)), \\ & t \in \mathbb{N}, \\ \hat{z}(s) =& \hat{\phi}(s),\ s \in \{-h, -h+1, \cdots, 0\}. \end{aligned}$$

In addition, $\hat{z}(t, \cdot)$ is a non-increasing function. This means that the system (6) is α-exponentially mean bounded.

By combining the previous facts, we obtain the following theorem which gives us a sufficient condition for the α-exponential mean boundedness of system (1).

Theorem 3.2. Let $\alpha > 1$ be such that

$$\rho\left(\alpha\Pi\left(A_0 + \hat{\xi}B_0\right) + \alpha^{h+1}\Pi\left(A_1 + \hat{\xi}B_1\right)\right) < 1. \quad (10)$$

Then, system (1) is α-exponentially mean bounded and the α-exponential mean estimate is given by

$$|\mathbb{E}x(t, \phi)| \preceq \bar{z}(t, \hat{\phi})\alpha^{-t}, \quad (11)$$

where $\bar{z}(t, \hat{\phi}) := \mathbb{E}\hat{z}^1(t, \hat{\phi}) + \mathbb{E}\hat{z}^2(t, \hat{\phi})$ and $\hat{z}(t, \hat{\phi}) = [\hat{z}^1(t, \hat{\phi})\ \hat{z}^2(t, \hat{\phi})] \in \mathbb{R}^{2n}$ is the solution of (9). In addition, the factor function $\bar{z}(\cdot, \hat{\phi})$ is non-increasing.

In the next section, we will establish an α-exponential mean estimate for positive stochastic discrete-time systems in the sense of expectation from which the result in Theorem 3.2 is followed.

4. AN α-EXPONENTIAL MEAN ESTIMATE FOR POSITIVE STOCHASTIC DISCRETE-TIME SYSTEMS AND ITS APPLICATIONS

Different from positive Markov jump linear systems, under the influence of stochastic uncertainty $\xi(t)$, the system (1) might not be positive even if the initial value function belongs to the nonnegative orthant. From this fact and the notion of positive systems, see, e.g.,[52,53] we introduce the following definition of positive stochastic systems in the sense of expectation.

Definition 4.1. System (1) is said to be positive in the sense of expectation if for any random initial value function $\phi \in \mathcal{C}_\sigma([-h, 0], \mathbb{R}^n)$ such that $\mathbb{E}\phi(\cdot) \succeq 0$, we then have $\mathbb{E}x(t, \phi) \succeq 0$.

From now on, for simplicity, sometimes we say that a stochastic system is positive to mean that this system is positive in the sense of expectation. We will prove that under some assumptions related to the coefficient matrices, the random process and the initial value function, system (1) is positive. From the positivity of this system, we will apply solution comparison-based methods to establish relations between positive systems, see, e.g..[34,54] More specifically, this approach will make a comparison between state vectors of the stochastic discrete-time system with time-varying delay (1) with state vectors of the following “similar” constant time delay system

$$\begin{aligned} y(t+1) =&[A_0+\xi(t)B_0]y(t) \\ &+[A_1+\xi(t)B_1]y(t-h),\ t\in\mathbb{N}, \\ y(s) =&\psi(s),\ s\in\{-h,-h+1,\cdots,0\}, \end{aligned} \tag{12}$$

where $\psi \in \mathcal{C}_\sigma([-h,0],\mathbb{R}^n)$ is a random function with known expectation. In addition, the system (1) is also compared with the following “upper” system

$$\begin{aligned} \bar{x}(t+1) =&[\bar{A}_0+\xi(t)\bar{B}_0]\bar{x}(t) \\ &+[\bar{A}_1+\xi(t)\bar{B}_1]\bar{x}(t-h_1(t)),\ t\in\mathbb{N}, \\ \bar{x}(s) =&\phi(s),\ s\in\{-h,-h+1,\cdots,0\}, \end{aligned} \tag{13}$$

where $\bar{A}_0,\bar{A}_1,\bar{B}_0$ and $\bar{B}_1$ are matrices in $\mathbb{R}^{n\times n}$ such that $0 \preceq A_0+\hat{\xi}B_0 \preceq \bar{A}_0+\hat{\xi}\bar{B}_0$ and $0 \preceq A_1+\hat{\xi}B_1 \preceq \bar{A}_1+\hat{\xi}\bar{B}_1$. The positivity and the monotonic property of these systems will be applied to derive an α-exponential mean estimate for the positive stochastic system (1).

4.1. Positivity and solution comparisons

In this section, we will prove the positivity of systems (1), (12) and (13). From this property, we then derive some solution comparisons between these systems. The main results of this section are presented in the next lemma.

Lemma 4.2. Let us assume that the matrices A_0, A_1, B_0, B_1 and the random process $\xi(t)$ satisfy $A_0+\mathbb{E}\xi(t)B_0 \succeq 0$ and $A_1+\mathbb{E}\xi(t)B_1 \succeq 0$ for all $t\in\mathbb{N}$. Then, the following assertions hold true.

i) Assume that $\mathbb{E}\phi(\cdot)\succeq 0$ and $\mathbb{E}\psi(\cdot)\succeq 0$. Then, systems (1), (12) and (13) are positive in the sense of expectation.

ii) For all $\phi_1,\phi_2\in\mathcal{C}_\sigma([-h,0],\mathbb{R}^n)$ such that $0\preceq\mathbb{E}\phi_1(s)\preceq\mathbb{E}\phi_2(s)$ $\forall s\in\{-h,-h+1,\cdots,0\}$, then

$$\mathbb{E}x(t,\phi_1)\preceq\mathbb{E}x(t,\phi_2)\ \forall t\in\mathbb{N}. \tag{14}$$

iii) For all $\phi\in\mathcal{C}_\sigma([-h,0],\mathbb{R}^n)$ such that $\mathbb{E}\phi(\cdot)\succeq 0$, we have

$$\mathbb{E}x(t,\phi)\preceq\mathbb{E}\bar{x}(t,\phi)\ \forall t\in\mathbb{N}. \tag{15}$$

iv) For every random function $\delta\in\mathcal{C}_\sigma([-h,0],\mathbb{R}^n)$ such that $\mathbb{E}\delta(t+1)\preceq\mathbb{E}\delta(t)$ for all $t\in\{-h,-h+1,\cdots,-1\}$ and $(A_1+\hat{\xi}B_1)\mathbb{E}\delta(-h)+(A_0+\hat{\xi}B_0)\mathbb{E}\delta(0)\preceq\mathbb{E}\delta(0)$, we then have

iv)-1) $\mathbb{E}y(t+1,\delta)\preceq\mathbb{E}y(t,\delta),\ \forall t\in\mathbb{N}$,

iv)-2) $\mathbb{E}x(t,\delta)\preceq\mathbb{E}y(t,\delta),\ \forall t\in\mathbb{N}$,

where $x(t,\delta)$ and $y(t,\delta)$ are respectively solutions of systems (1) and (12) under the initial conditions $\phi(s)=\psi(s)=\delta(s)$ for all $s\in\{-h,-h+1,\cdots,0\}$.

Proof. *i)* We just need to prove the positivity in the sense of expectation of the system (1). The positivity of two systems (12) and (13) follows as a particular case of (1). From the initial condition that $\mathbb{E}\phi(t)\succeq 0$, for all $t\in\{-h,-h+1,\cdots,0\}$ and noting that $\mathbb{E}\xi(0)=\hat{\xi}$, $A_0+\hat{\xi}B_0\succeq 0$ and $A_1+\hat{\xi}B_1\succeq 0$, we then have

$$\begin{aligned} &\mathbb{E}x(1,\phi) \\ =&\ \mathbb{E}\{[A_0+\xi(0)B_0]x(0) \\ &+[A_1+\xi(0)B_1]x(-h_1(0))\} \\ =&\ (A_0+\hat{\xi}B_0)\mathbb{E}\phi(0)+(A_1+\hat{\xi}B_1)\mathbb{E}\phi(-h_1(0)) \\ \succeq&\ 0. \end{aligned}$$

Suppose that $\mathbb{E}x(t,\phi)\succeq 0$ for all $t\in\{-h,-h+1,\cdots,t_0\}$ for some $t_0\in\mathbb{N}_0$. We now prove that $\mathbb{E}x(t_0+1,\phi)\succeq 0$. Indeed, from the assumption that $\mathbb{E}\xi(t_0)=\hat{\xi}$, $A_0+\hat{\xi}B_0\succeq 0$, $A_1+\hat{\xi}B_1\succeq 0$, and noting that $0\le h_1(t_0)\le h$, we then get from the induction hypothesis that

$$\begin{aligned} \mathbb{E}x(t_0+1,\phi) =&\mathbb{E}\{[A_0+\xi(t_0)B_0]x(t_0) \\ &+[A_1+\xi(t_0)B_1]x(t_0-h_1(t_0))\} \\ =&(A_0+\hat{\xi}B_0)\mathbb{E}x(t_0) \\ &+(A_1+\hat{\xi}B_1)\mathbb{E}x(t_0-h_1(t_0)) \\ \succeq&0. \end{aligned}$$

By induction argument, we obtain the conclusion.

ii) Let us denote

$$e(t,\hat{\phi})=x(t,\phi_2)-x(t,\phi_1),$$

where

$$\hat{\phi}(t)=\phi_2(t)-\phi_1(t),\forall t\in\{-h,-h+1,\cdots,0\}.$$

It can be verified that $e(t,\hat{\phi})$ is the solution of the following linear stochastic discrete-time system

$$\begin{aligned} e(t+1) =&[A_0+\xi(t)B_0]e(t) \\ &+[A_1+\xi(t)B_1]e(t-h_1(t)),\ t\in\mathbb{N}, \\ e(s) =&\hat{\phi}(s),\ s\in\{-h,-h+1,\cdots,0\}, \end{aligned}$$

By using assertion i) with noting that

$$\mathbb{E}\hat{\phi}(t)=\mathbb{E}\phi_2(t)-\mathbb{E}\phi_1(t)\succeq 0 \quad \forall t\in\{-h,-h+1,\cdots,0\},$$

the above system is positive in the sense of expectation. This means that

$$\mathbb{E}e(t,\hat{\phi})=\mathbb{E}x(t,\phi_2)-\mathbb{E}x(t,\phi_1)\succeq 0$$

from which the conclusion ii) follows.
iii) Let $t\in\mathbb{N}$. By using the positivity of systems (1), (13), and the assumption that

$$0\preceq A_0+\hat{\xi}B_0\preceq \bar{A}_0+\hat{\xi}\bar{B}_0,$$
$$0\preceq A_1+\hat{\xi}B_1\preceq \bar{A}_1+\hat{\xi}\bar{B}_1,$$

taking the expectation on both sides of (1) with noting that $\mathbb{E}\xi(t)=\hat{\xi}$, $\mathbb{E}x(t)\succeq 0$ and $\mathbb{E}x(t-h_1(t))\succeq 0$, one gets

$$\begin{aligned}\mathbb{E}x(t+1,\phi)=&\mathbb{E}\{[A_0+\xi(t)B_0]x(t)\\&+[A_1+\xi(t)B_1]x(t-h_1(t))\}\\=&(A_0+\hat{\xi}B_0)\mathbb{E}x(t)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}x(t-h_1(t))\\\preceq&(\bar{A}_0+\hat{\xi}\bar{B}_0)\mathbb{E}x(t)\\&+(\bar{A}_1+\hat{\xi}\bar{B}_1)\mathbb{E}x(t-h_1(t))\\=&(\bar{A}_0+\hat{\xi}\bar{B}_0)\mathbb{E}\bar{x}(t)\\&+(\bar{A}_1+\hat{\xi}\bar{B}_1)\mathbb{E}\bar{x}(t-h_1(t))\\=&\mathbb{E}\{[\bar{A}_0+\xi(t)\bar{B}_0]\bar{x}(t)\\&+[\bar{A}_1+\xi(t)\bar{B}_1]\bar{x}(t-h_1(t))\}\\=&\mathbb{E}\bar{x}(t+1,\phi),\end{aligned}$$

from which completes the proof of iii).
iv) Firstly, we will prove iv)-1) by induction. For $t=0$, from assumption on the initial value function δ and $\mathbb{E}\xi(0)=\hat{\xi}$, we then have from system (12) that

$$\begin{aligned}\mathbb{E}y(1,\delta)=&\mathbb{E}\{[A_0+\xi(0)B_0]y(0)\\&+[A_1+\xi(0)B_1]y(-h)\}\\=&(A_0+\hat{\xi}B_0)\mathbb{E}y(0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}y(-h)\\=&(A_0+\hat{\xi}B_0)\mathbb{E}\delta(0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}\delta(-h)\\\preceq&\mathbb{E}\delta(0)=\mathbb{E}y(0,\delta).\end{aligned}$$

Suppose that $\mathbb{E}y(t+1,\delta)\preceq\mathbb{E}y(t,\delta)$ for all $t\in\{-h,-h+1,\cdots,t_0\}$ for some $t_0\in\mathbb{N}$.
We will prove that $\mathbb{E}y(t_0+2,\delta)\preceq\mathbb{E}y(t_0+1,\delta)$. Indeed, from system (12) and the assumption that $\mathbb{E}\xi(t_0+1)=\mathbb{E}\xi(t_0)=\hat{\xi}$, one gets

$$\begin{aligned}\mathbb{E}y(t_0+2,\delta)=&\mathbb{E}\{[A_0+B_0\xi(t_0+1)]y(t_0+1)\\&+[A_1+B_1\xi(t_0+1)]y((t_0+1)-h)\}\\=&(A_0+\hat{\xi}B_0)\mathbb{E}y(t_0+1)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}y((t_0+1)-h)\\\preceq&(A_0+\hat{\xi}B_0)\mathbb{E}y(t_0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}y(t_0-h)\\=&\mathbb{E}\{[A_0+\xi(t_0)B_0]y(t_0)\\&+[A_1+\xi(t_0)B_1]y(t_0-h)\}\\=&\mathbb{E}y(t_0+1,\delta).\end{aligned}$$

By induction argument, the proof of iv)-1) is then completed.

Finally, we use again induction reasoning to prove that iv)-2) is true. For the case $t=1$, by using the initial condition and the fact that $\mathbb{E}\xi(0)=\hat{\xi}$ and $0\le h_1(0)\le h$, we then have

$$\begin{aligned}\mathbb{E}x(1,\delta)=&\mathbb{E}\{[A_0+\xi(0)B_0]x(0)\\&+[A_1+\xi(0)B_1]x(-h_1(0))\}\\=&(A_0+\hat{\xi}B_0)\mathbb{E}x(0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}x(-h_1(0))\\=&(A_0+\hat{\xi}B_0)\mathbb{E}\delta(0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}\delta(-h_1(0))\\\preceq&(A_0+\hat{\xi}B_0)\mathbb{E}\delta(0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}\delta(-h)\\=&(A_0+\hat{\xi}B_0)\mathbb{E}y(0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}y(-h)\\=&\mathbb{E}\{[A_0+\xi(0)B_0]y(0)\\&+[A_1+\xi(0)B_1]y(-h)\}\\=&\mathbb{E}y(1,\delta).\end{aligned}$$

Suppose that $\mathbb{E}x(t,\delta)\preceq\mathbb{E}y(t,\delta)$ for all $t\in\{-h,-h+1,\cdots,t_0\}$ for some $t_0\in\mathbb{N}$. We will prove that $\mathbb{E}y(t_0+1,\delta)\preceq\mathbb{E}x(t_0+1,\delta)$. Indeed, by using induction hypothesis, $\mathbb{E}\xi(t_0)=0$ and

iv)-1), systems (1) and (12) give us

$$\begin{aligned}\mathbb{E}x(t_0+1,\delta) =&\mathbb{E}\{[A_0+\xi(t_0)B_0]x(t_0)\\&+[A_1+\xi(t_0)B_1]x(t_0-h_1(t_0))\}\\=&(A_0+\hat{\xi}B_0)\mathbb{E}x(t_0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}x(t_0-h_1(t_0))\\\preceq&(A_0+\hat{\xi}B_0)\mathbb{E}y(t_0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}y(t_0-h_1(t_0))\\\preceq&(A_0+\hat{\xi}B_0)\mathbb{E}y(t_0)\\&+(A_1+\hat{\xi}B_1)\mathbb{E}y(t_0-h)\\=&\mathbb{E}\{[A_0+\xi(t_0)B_0]y(t_0)\\&+[A_1+\xi(t_0)B_1]y(t_0-h)\}\\=&\mathbb{E}y(t_0+1,\delta).\end{aligned}$$

From induction argument, we have thus proved the conclusion iv)-2). □

4.2. An α-exponential mean estimates for positive stochastic systems

Let us assume throughout this section that the coefficient matrices A_0, A_1, B_0, B_1 and the random process $\xi(t)$ of system (1) satisfy conditions in Lemma 4.2 for which this system is positive. In this section, we will provide a sufficient condition for the α-exponential mean boundedness of the positive system (1). By using results developed in Section 4.1, we will derive an α-exponential mean estimate for state vectors of system (1) with time-varying factor function. Our approach is based on an exponential state transformation and solution comparisons.

Step 1: An exponential state transformation
Let $\alpha > 1$ and let us consider the following α-exponential state transformation

$$p(t) = \alpha^t x(t). \tag{16}$$

We introduce the following notations

$$\varphi(s) := \alpha^s \phi(s) \text{ and } \overline{\varphi}(s) := \alpha^s \bar{\phi}(s),\quad s \in \{-h, -h+1, \cdots, 0\}.$$

System (1) then becomes the following system

$$\begin{aligned}p(t+1) =&\alpha[A_0+\xi(t)B_0]p(t)\\&+\alpha^{(h_1(t)+1)}[A_1+\xi(t)B_1]p(t-h_1(t)),\\p(s) =&\varphi(s),\ s\in\{-h,-h+1,\cdots,0\},\end{aligned} \tag{17}$$

and from (2), one has

$$0 \preceq \mathbb{E}\varphi(s) \preceq \overline{\varphi}(s)\quad \forall s \in \{-h, -h+1, \cdots, 0\}. \tag{18}$$

Step 2: Solution comparisons
We consider the following "upper" system of (17)

$$\begin{aligned}\hat{p}(t+1) =&\alpha[A_0+\xi(t)B_0]\hat{p}(t)\\&+\alpha^{h+1}[A_1+\xi(t)B_1]\hat{p}(t-h_1(t)),\\\hat{p}(s) =&\varphi(s),\ s\in\{-h,-h+1,\cdots,0\}.\end{aligned} \tag{19}$$

It follows from $0 \le h_1(t) \le h\ \forall t \in \mathbb{N}$, $\alpha > 1$ and $A_1 + \hat{\xi}B_1 \in \mathbb{R}^n_{0,+}$ that

$$0 \preceq \alpha^{(h_1(t)+1)}(A_1+\hat{\xi}B_1) \preceq \alpha^{h+1}(A_1+\hat{\xi}B_1)\ \forall t \in \mathbb{N}.$$

From this, Lemma 4.2 – ii), iii) and (18), we then obtain

$$\mathbb{E}p(t,\varphi) \preceq \mathbb{E}\hat{p}(t,\varphi) \preceq \mathbb{E}\hat{p}(t,\bar{\varphi}),\ \ \forall t \in \mathbb{N}. \tag{20}$$

Finally, we consider the following "similar" system of (19)

$$\begin{aligned}z(t+1) =&\alpha[A_0+\xi(t)B_0]z(t)\\&+\alpha^{h+1}[A_1+\xi(t)B_1]z(t-h),\ t\in\mathbb{N},\\z(s) =&\eta(s),\ s\in\{-h,-h+1,\cdots,0\},\end{aligned} \tag{21}$$

where the function $\eta(\cdot) \in \mathcal{C}_\sigma([-h, 0], \mathbb{R}^n)$ will be defined such that the expectation of the solution $z(t, \eta)$ is a non-increasing function and is an upper bound of the expectation of the solution $\hat{p}(t, \bar{\varphi})$. The next lemma will give us a condition for the existence of such initial value function.

Lemma 4.3. Assume that

$$\rho\left(\alpha\left(A_0+\hat{\xi}B_0\right)+\alpha^{h+1}\left(A_1+\hat{\xi}B_1\right)\right) < 1.$$

Then, there exists an initial value function $\eta(\cdot) \in \mathcal{C}_\sigma([-h, 0], \mathbb{R}^n)$ such that $\mathbb{E}z(\cdot, \eta)$ is a non-increasing function and

$$\mathbb{E}\hat{p}(t,\bar{\varphi}) \preceq \mathbb{E}z(t,\eta),\ \forall t \in \mathbb{N}. \tag{22}$$

Proof. Let us consider the following linear programming problem

$$\begin{bmatrix}-I_n & I_n & 0 & \cdots & 0 & 0\\0 & -I_n & I_n & \cdots & 0 & 0\\0 & 0 & -I_n & \cdots & 0 & 0\\\vdots & \vdots & \vdots & \ddots & I_n & 0\\0 & 0 & 0 & \cdots & -I_n & I_n\end{bmatrix} u \preceq 0, \tag{23}$$

$$v - u \preceq 0, \tag{24}$$

$$\left[\alpha^{h+1}(A_1+\hat{\xi}B_1)\ 0\cdots0\ \alpha(A_0+\hat{\xi}B_0)-I_n\right]u \preceq 0, \tag{25}$$

where $v := [\overline{\varphi}^\top(-h)\ \ \overline{\varphi}^\top(-h+1)\ \cdots\ \overline{\varphi}^\top(0)]^\top \in \mathbb{R}^{(h+1)n}$ and $u := [u^\top_{-h}\ \ u^\top_{-h+1}\ \cdots\ u^\top_0]^{\in}\mathbb{R}^{(h+1)n}$ are unknown.

By using the same argument as in,[36] Step 2 we conclude the proof of this lemma. □

Noting that since both h, $\eta(\cdot)$ and $\mathbb{E}\xi(\cdot)$ are known, we can use system (21) to calculate the expectation of the state vector $z(t,\eta)$ for all $t \in \mathbb{N}$. By combining (16), (20) and (22), we obtain an α-exponential mean estimate with a decreasing factor function for the solution $x(t,\phi)$ of system (1) which is stated in the following theorem.

Theorem 4.4. Let $\alpha > 1$ be such that

$$\rho(\alpha(A_0 + \hat{\xi}B_0) + \alpha^{h+1}(A_1 + \hat{\xi}B_1)) < 1. \quad (26)$$

Then, there exists an initial value function $\eta(\cdot) \in \mathcal{C}([-h,0],\mathbb{R}^n)$ such that the system (1) is α-exponentially mean bounded and its α-exponential mean estimate is given by

$$\mathbb{E}x(t,\phi) \preceq \mathbb{E}z(t,\eta)\alpha^{-t}, \ \forall t \in \mathbb{N}, \quad (27)$$

where $z(t,\eta)$ is the solution of the system (21). In addition, $\mathbb{E}z(\cdot,\eta)$ is a non-increasing function.

Remark 4.5. In Lemma 4.3, instead of considering the linear programming (23)–(25), let us consider the following one

$$\min_{u\in\mathbb{R}^{(h+1)n}} f(u) := c^\top(u - v) \ \text{ s.t. } (23),(24),(25), \quad \text{(LP)}$$

where $c := [1 \ \ 1 \ \cdots \ 1]^\top \in \mathbb{R}^{(h+1)n}$. Let u^* be an optimal solution of (LP). By setting

$$\eta(s) = u^*_s, \ \ \forall s \in \{-h, -h+1, \cdots, 0\}, \quad (28)$$

the function $\eta(\cdot)$ not only satisfies conditions of Lemma 4.3 but also makes $\sum_{s=-h}^{0} ||\mathbb{E}(\eta(s) - \bar{\varphi}(s))||_1$ smallest. The latter condition means that in general (in the sense of expectation) the function $\eta(\cdot)$ is not too far from $\bar{\varphi}(\cdot)$. This will provide an α-exponential mean estimate of the state vector $x(t,\phi)$ as accurate as possible.

Remark 4.6. By applying the result in Theorem 4.4 to system (6) and noting that

$$\Pi\left(A_0 + \hat{\xi}B_0\right) \succeq 0, \Pi\left(A_1 + \hat{\xi}B_1\right) \succeq 0$$
$$\text{and } \mathbb{E}\pi(\phi(s)) \succeq 0,$$

we then obtain the α-exponentially mean estimate for the non-positive system (1) (given by Theorem 3.2).

5. AN α-EXPONENTIAL MEAN ESTIMATE VIA AN "UPPER BOUND" SYSTEM

For the sake of demonstrating the effectiveness of the IPR approach when applying to the non-positive stochastic discrete-time system (1), we introduce another α-exponential mean estimate for this system via an "upper bound" system, see, e.g..[17,35] In this approach, we will bound from above the system (1) by the following positive stochastic discrete-time system (1)

$$\begin{aligned} u(t+1) =& [|A_0| + |\xi(t)||B_0|]u(t) \quad (29)\\ &+ [|A_1| + |\xi(t)||B_1|]u(t - h_1(t)), \ t \in \mathbb{N},\\ u(s) =& |\phi(s)|, \ s \in \{-h, -h+1, \cdots, 0\}. \end{aligned}$$

Let us denote by $u(t,|\phi|)$ the solution of this system. By virtue of Lemma 4.2 – i), this system is positive in the sense of expectation, i.e. $\mathbb{E}u(t,|\phi|) \succeq 0$, for all $t \in \mathbb{N}$. The following result gives us an α-exponential mean estimate of system (1) based on this "upper bound" system.

Theorem 5.1. Let us assume that the stochastic process $\xi(t)$ satisfies $\mathbb{E}|\xi(t)| \leq \bar{\xi}$ for all $t \in \mathbb{E}$, for some $\bar{\xi} \geq 0$. Assume that there exists $\alpha > 1$ such that

$$\rho(\alpha(|A_0| + \bar{\xi}|B_0|) + \alpha^{h+1}(|A_1| + \bar{\xi}|B_1|)) < 1. \quad (30)$$

Then, there exists a vector-valued function $\lambda(t) \in \mathbb{R}^n_{0,+}, \ t \in \mathbb{N}$ such that

$$\mathbb{E}x(t,\phi) \preceq \lambda(t)\alpha^{-t}, \ \forall t \in \mathbb{N}, \quad (31)$$

In addition, $\lambda(\cdot)$ is a non-increasing function.

Proof. To complete the proof of Theorem 5.1, we just need to prove that system (29) is an upper bound of system (1) in the sense of expectation, i.e.,

$$|\mathbb{E}x(t,\phi)| \preceq \mathbb{E}u(t,|\phi|), \ \forall t \in \mathbb{N}. \quad (32)$$

The above inequality will be proved by induction. For all $s \in \{-h, -h+1, \cdots, 0\}$, by applying the Jensen's inequality, see, e.g.,[55] for the convex function $f(x) = |x|$ and using the initial conditions for systems (1) and (29), one gets

$$|\mathbb{E}x(s,\phi)| = |\mathbb{E}\phi(s)| \preceq \mathbb{E}|\phi(s)| = \mathbb{E}u(s,|\phi|).$$

Suppose that (32) is valid for all $t \in \{-h, -h+1, \cdots, t_0\}$, for some $t_0 \in \mathbb{N}$. Let us prove that this inequality also holds at $t = t_0 + 1$. Indeed, from the definition of systems (1) and (29) and remembering that $0 \leq h_1(t_0) \leq h$, we then have

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$$
\begin{aligned}
|\mathbb{E}x(t_0,\phi)| &= |\mathbb{E}\{[A_0+\xi(t_0)B_0]x(t_0)\\
&\quad + [A_1+\xi(t_0)B_1]x(t_0-h_1(t_0))\}|\\
&= |\{[A_0+\mathbb{E}\xi(t_0)B_0]\mathbb{E}x(t_0)\\
&\quad + [A_1+\mathbb{E}\xi(t_0)B_1]\mathbb{E}x(t_0-h_1(t_0))\}|\\
&\preceq [|A_0|+|\mathbb{E}\xi(t_0)||B_0|]|\mathbb{E}x(t_0)|\\
&\quad + [|A_1|+|\mathbb{E}\xi(t_0)||B_1|]|\mathbb{E}x(t_0-h_1(t_0))\ |\\
&\preceq [|A_0|+\mathbb{E}|\xi(t_0)||B_0|]\mathbb{E}u(t_0)\\
&\quad + [|A_1|+\mathbb{E}|\xi(t_0)||B_1|]\mathbb{E}u(t_0-h_1(t_0))\\
&= \mathbb{E}\{[|A_0|+|\xi(t_0)||B_0|]u(t_0)\\
&\quad + [|A_1|+|\xi(t_0)||B_1|]u(t_0-h_1(t_0))\}\\
&= \mathbb{E}u(t_0+1,|\phi|),
\end{aligned}
$$

where the second inequality is obtained from the induction hypothesis and the Jensen's inequality. This concludes the proof. □

Remark 5.2.

1. The result in Theorem 5.1 seems to be more natural than the one in Theorem 3.2. However, we should note that system (29) is an overestimate of system (1). As a consequence, the range of α to get an α-exponential mean estimate (condition (30)) will be narrowed. In addition, the α-exponential mean estimate given in this theorem is also looser than the one obtained by Theorem 3.2. These advantages of the IPR approach will be demonstrated in Section 6

2. By using a direct evaluation on the expectation of the square norm of the state vector, Xu and Ge,[17] proved the mean square exponential stability of system (1) under the usual condition on the stochastic process $\xi(t)$, i.e., $\mathbb{E}\xi(t)=0$ and $\mathbb{E}\xi(t)^2=1$. From this result and the Jensen's inequality, we can derive an α-exponential mean estimate for the system (1). It worth noting that we can apply a direct evaluation on the expectation of state vector (not the square norm of state vector) to obtain a more accurate α-exponential mean estimate for this system. However, these two estimates will be not as accurate as the ones obtained in Theorems 3.2 and 5.1.

6. ILLUSTRATIVE EXAMPLE

This section is devoted to verify the effectiveness of the IPR approach for deriving the α-exponential mean estimate of the stochastic discrete-time system with time-varying delay (1). Let us consider system (1) with the following coefficient matrices

$$
A_0 = \begin{bmatrix} 0.51 & 0.10 & -0.12\\ 0.04 & -0.12 & 0.04\\ 0.12 & 0.05 & -0.06 \end{bmatrix}
$$

$$
A_1 = \begin{bmatrix} 0.12 & 0.01 & -0.08\\ -0.03 & -0.15 & 0.02\\ 0.04 & -0.01 & 0.10 \end{bmatrix}
$$

$$
B_0 = \begin{bmatrix} -0.21 & -0.03 & 0.11\\ -0.05 & 0.02 & 0.01\\ 0.02 & 0.01 & -0.02 \end{bmatrix}
$$

$$
B_1 = \begin{bmatrix} 0.01 & -0.02 & 0.02\\ 0.05 & -0.03 & -0.07\\ -0.05 & 0.01 & -0.01 \end{bmatrix}
$$

The stochastic process $\xi(t)$ will be chosen such that $\mathbb{E}\xi(t) = 0.2$ and $\mathbb{E}|\xi(t)| \le 0.8$, for all $t \in \mathbb{N}$. We assume that the expectation of the initial value function is bounded by $\bar{\phi}(-2) = [3.3,\ 3.8,\ 1.7], \bar{\phi}(-1) = [2.4,\ 2.8,\ 0.7]$ and $\bar{\phi}(0) = [-3.0,\ -3.3,\ 1.7]$. The bound of the time-varying delays h is set by 2.

By using one dimensional search, we can find the largest value of the decay rate α such that Theorems 4.4 and 5.1 can be applied. The results are given in Table 1. From this table, we can see that the sufficient condition derived by the IPR method the (condition (10)) gives us a larger range of decay rate α than the condition obtained in Theorem 5.1 (condition (30)). This means that the α-exponential mean estimate of state vectors obtained by Theorem 4.4 has a broader applicability.

Table 1. Ranges of decay rate α.

Methods	Range of α
Theorem 3.2 (condition (10))	[1, 1.294]
Theorem 5.1 (condition (30))	[1, 1.046]

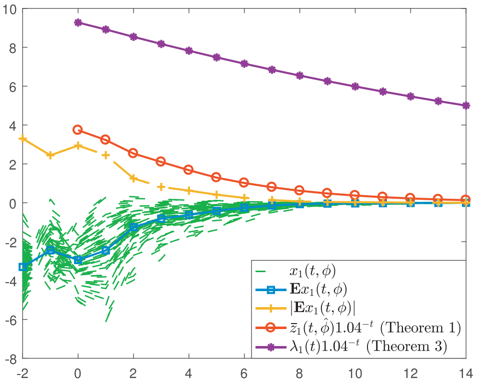

Figure 1. Trajectories of $x_1(t,\phi)$, the mean value $\mathbb{E}x_1(t,\phi)$ and its α-exponential mean estimates.

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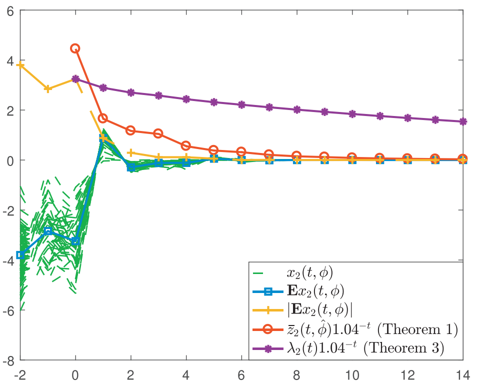

Figure 2. Trajectories of $x_2(t,\phi)$, the mean value $\mathbb{E}x_2\,(t,\phi)$ and its α-exponential mean estimates.

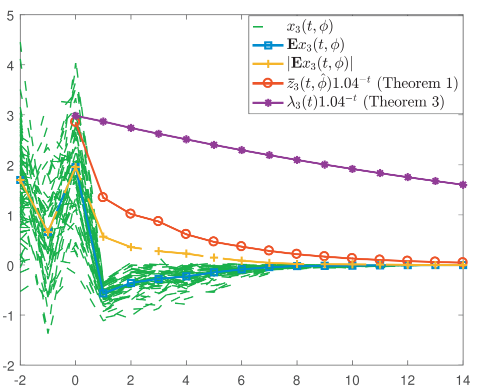

Figure 3. Trajectories of $x_3(t,\phi)$, the mean value $\mathbb{E}x_3(t,\phi)$ and its α-exponential mean estimates.

To illustrate the behavior of system (1) and α-exponential mean estimates of this system, for each $t \in \mathbb{N}$ and $s \in \{-2,-1,0\}$, we generate 100 random variables $\xi(t)$ and $\zeta(s)$ such that $\mathbb{E}\xi(t) = 0.2, |\mathbb{E}\xi(t)| \leq 0.8$ and $\mathbb{E}\zeta(s) = 0$ to create 100 stochastic systems under the form (1). The initial condition is chosen by $\phi(s) = \bar{\phi}(s) + \zeta(s)$ for $s = -2,-1,0$. Let us choose the decay rate $\alpha = 1.04$ for which both results in Theorem 3.2 and 5.1 can be applied. Figures 1–3 show us the behavior of 100 realizations of the system (1) together with its mean values $\mathbb{E}x(t,\phi)$ and its α-exponential mean estimates $\mathbb{E}\bar{z}(t,\eta)$. As we can see, the expectation of state vectors (the line) is not positive. The behavior of the line means that these systems are α-exponentially mean bounded. Moreover, these figures also show us that the α-exponential mean estimate obtained by the IPR approach (the line) is more accurate than the one given by the "upper bound" approach (the line). This verifies the effectiveness of the IPR approach on the stochastic discrete-time system with time-varying delays (1).

7. CONCLUSION AND PERSPECTIVES

In this paper, we consider a class of linear stochastic discrete-time systems with time-varying delays which can be seen as the usual linear discrete-time system with time-varying delays and with random uncertainty. Under the assumption that the random process does not lose the positivity of coefficient matrices, we prove that the system is still positive in the sense of expectation. In addition, a new solution comparison for stochastic systems is derived and then is applied to obtain an α-exponential mean estimate of positive stochastic systems. This result is used together with the IPR approach to obtain an α-exponential mean estimate for non-positive stochastic discrete-time systems. Some numerical examples are performed to demonstrate the effectiveness of the IPR approach on stochastic systems. The approach in this paper can be also applied to some other problems, e.g., interval observer,[30,56,57] ℓ_∞-gain analysis,[58,59] etc. Finally, we list here some other open problems which can be studied for stochastic systems: the positivity in the sense of probability for stochastic systems, the exponential estimate in the sense of probability for stochastic systems, etc.

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