TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC QUY NHƠN

Tìm nghiệm liouville của phương trình vi phân đại số cấp một bằng phép đổi biến

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TÓM TẮT

Chúng tôi trình bày một phương pháp tìm nghiệm liouville của phương trình vi phân đại số cấp một bằng phép đổi biến. Cụ thể, một phương trình vi phân đại số cấp một với hệ số thuộc vào một mở rộng liouville được biến đổi thành một phương trình vi phân với hệ số thuộc vào trường vi phân hữu tỷ bằng phép đổi biến trên trường cơ sở. Thêm nữa, sử dụng phép đổi biến giữa các hàm số, phương trình vi phân đại số cấp một với hệ số trên trường vi phân hữu tỷ có thể được biến đổi về dạng phương trình đơn giản hơn phù hợp với các thuật toán đã biết. Một số ví dụ được trình bày để minh họa phương pháp đã đưa ra.

Từ khóa: *Phương trình vi phân đại số, nghiệm liouville, phép đổi biến*.

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Finding liouvillian solutions of first-order algebraic ordinary differential equations by change of variables

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ABSTRACT

We present an approach for finding liouvillian solutions of first-order algebraic ordinary differential equations (AODEs) by means of change of variables. In particular, a first-order AODE with liouvillian coefficients can be transformed into an AODE over rational fields by the change of indeterminate over the ground fields. In addition, by the change of functions, the last AODE can be converted into the one which is suitable for known algorithms. Some examples are given to illustrate the method.

Keywords: *Algebraic ordinary differential equation, liouvillian solution, change of variables.*

1. INTRODUCTION

The ideas of using geometric properties which satisfy the differential constraint into the problem of solving differential equations are well-known. There are notable works for finding rational general solutions which are based on rational parametrizations of algebraic curves of genus zero.[1−3] Recently, by this technical method, we have presented an algorithm for determining liouvillian solutions of first-order AODEs of genus zero.

In this paper, we give an approach for solving first-order AODEs which is based on the change of variables. This continues the ideas considered in the previous works.[◇,4−5] In more details, we aim to transform certain first-order AODEs into sub-types with respect to the two cases of change of variables, that are the change of the functions and the change of the indeterminate over the ground fields. From these considerations, first-order AODEs with liouvillian coefficients can be converted into the AODEs over $\mathbb{C}(z)$ (Section 4). Moreover, such an AODE over $\mathbb{C}(z)$ can be transformed into an autonomous AODE or a rational one (Section 3) where known algorithms can be applied.[6−7]

2. PRELIMINARIES

We present some necessary definitions which are well known in literature.[8−10]

Definition 2.1. Let k be an algebraic field of characteristic zero. A *derivation* of the field k, denote by $'$, is an operation of k such that $\forall a, b \in k$, the followings hold.

$$(a+b)' = a' + b', \quad (ab)' = a'b + ab'.$$

A field k equipped with a derivation $'$ is called a *differential field*. An element $a \in k$ is called a *constant* if $a' = 0$. A field extension E of k is called a *differential field extension* of k if and only if the derivation of E restricted to k coincides with the derivation of k.

Definition 2.2. Let E be a differential field extension of k and let $'$ denote the derivation on E. $t \in E$ is called *primitive* over k if $t' \in k$. $t \in E \setminus 0$ is called *hyperexponential* over k if $t'/t \in k$. $t \in E$ is called *liouvillian* over k if t is either algebraic, or primitive or hyperexponential over k. E is a *liouvillian extension* of k if $E = k(t_1, t_2, \ldots, t_n)$, and there is a tower of differential fields $k = k_0 \subseteq k_1 \subseteq \cdots \subseteq k_n = E$ such that for each $i \in \{1, \ldots, n\}$, $k_i = k_{i-1}(t_i)$ and t_i is liouvillian over k_{i-1}.

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Definition 2.3. Let $F(y,w) \in k[y,w]$ be an irreducible polynomial in two variables and K be the algebraic closure of k. Then we define an *affine algebraic curve* over k by the set

$$\mathbb{L} := \left\{(a,b) \in \mathbb{A}^2(K) \mid F(a,b) = 0\right\}.$$

The polynomial $F(y,w)$ is called the *defining polynomial* of $\mathbb{L}$. We may write $F(y,w) = 0$ to indicate an algebraic curve $\mathbb{L}$.

Definition 2.4. Let k be a differential field with a derivation $'$ and let $F \in k[y,w]$. A first-order algebraic ordinary differential equation (AODE) is a differential equation of the form

$$F(Y, Y') = 0. \quad (1)$$

Then, the equation $F(y,w) = 0$ is called the *corresponding algebraic curve* of the first-order AODE (1).

By abuse of notations, when we refer to an AODE (1), we mean $k = \mathbb{C}(z)$ with $' = \dfrac{d}{dz}$ whose field of constants is $\mathbb{C}$ and $z' = 1$.

Definition 2.5. ξ is called a solution of the AODE (1) if $F(\xi, \xi') = 0$. If such ξ belongs to a liouvillian extension E of k then we call it a *liouvillian solution*. A solution ξ is called a *liouvillian general solution* if it does not vanish the separant $S_F = \dfrac{\partial F}{\partial Y'}$.

Remark 2.1. A general solution ξ defined by the way in Definition 2.5 first introduced in the work by Hubert,[11] and it is tantamount to the classical one defined in the book of Ritt.[8] Moreover, ξ is called a *singular solution* if it fails to annul S_F. It is well known that an AODE (1) has only finite singular solutions, hence, the paper's method is only applicable for finding liouvillian general solutions since it generates infinitely many solutions.

3. THE CHANGE OF VARIABLES $u = \psi(Y)$

We show how a geometric transformation induces a differential one. Let

$$G(u, u') = 0 \quad (2)$$

be a first-order AODE and $G(u,v) = 0$ be its algebraic corresponding curve over $\mathbb{C}(z)$. As above, let $F(y,w) = 0$ be the corresponding algebraic curve of the AODE (1). Assume that there is a transformation of the form

$$u = \psi(y,w), v = \gamma(y,w) \quad (3)$$

such that

$$G(u,v) = G(\psi(y,w), \gamma(y,w)) = F(y,w) = 0.$$

Then the transformation (3) induces a differential transformation between such two AODEs

$$u = \psi(Y, Y'), u' = \gamma(Y, Y') = \psi'(Y, Y'). \quad (4)$$

Lemma 3.1. *The transformation (4) must be of the form*

$$u = \psi(Y), u' = \psi'(Y). \quad (5)$$

Proof. In fact, if the first component of the transformation (4) contains the term Y' then the second component must include Y'' which is a contradiction if we compare with (3). □

Remark 3.1. The transformation (5) is based on the change $u = \psi(Y)$ and it can be started with any rational function $\psi(Y) \in \mathbb{C}(Y)$. However, just simple cases are considered in practical application. Recently, the change of variables $u = Y^n$ has been studied in the work by Dat and Chau. This consideration induces the one called a *power transformation*. Such a transformation may lead to a change of the genus of algebraic curves, by that, it can be applied for solving first-order AODEs whose genera are positive.

In the rest of this section, we consider a transformation induced by a rational function u of the form

$$u = M(Y) = \frac{\alpha Y + \beta}{\gamma Y + \delta},$$

where $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{C}(z)}$, $\alpha\delta - \beta\gamma \neq 0$. A *Möbius transformation* is a transformation of the form

$$u = \frac{\alpha Y + \beta}{\gamma Y + \delta}, u' = \left(\frac{\alpha Y + \beta}{\gamma Y + \delta}\right)'. \quad (6)$$

The inverse substitution of (6) is

$$Y = \frac{\delta u - \beta}{-\gamma u + \alpha}, Y' = \left(\frac{\delta u - \beta}{-\gamma u + \alpha}\right)'. \quad (7)$$

In this part, we follow the work by Ngo and Ha for the details of Möbius transformations.[4] They have been studied for finding algebraic and rational solutions.[4,12] Hence, there is no need to elaborate about them. Our contribution is to show that Möbius transformations are also applicable for determining liouvillian solutions. First, there is an expression for u'

$$
\begin{aligned}
\frac{\partial M(Y)}{\partial Y} =& \frac{\alpha\delta - \beta\gamma}{(\gamma Y + \delta)^2},\\
\frac{\partial M(Y)}{\partial z} =& \frac{(\alpha'\gamma - \gamma'\alpha)Y^2 + +\beta'\delta - \delta'\beta}{(\gamma Y + \delta)^2}\\
&+ \frac{(\alpha'\delta - \alpha\delta' + \beta'\gamma - \gamma'\beta)Y}{(\gamma Y + \delta)^2}, \quad (8)\\
u' = \frac{du}{dz} =& \frac{d(M(Y))}{dz}\\
=& \frac{\partial M(Y)}{\partial Y}Y' + \frac{\partial M(Y)}{\partial z}.
\end{aligned}
$$

Definition 3.1. Let

$$F(Y, Y') = \sum a_{ij} Y^i Y'^j$$

be an irreducible polynomial over $\mathbb{C}(z)$. Then we define the *differential total degree* of F by

$$\mu(F) = \max\{i + 2j \mid 0 \neq a_{ij} \in \mathbb{C}(z)\}.$$

Putting (6) into an AODE (2) and using (8), we obtain

$$
\begin{aligned}
G(u, u') =& G\left(\frac{\alpha Y + \beta}{\gamma Y + \delta}, \left(\frac{\alpha Y + \beta}{\gamma Y + \delta}\right)'\right)\\
=& \left(\frac{\alpha\delta - \beta\gamma}{\gamma Y + \delta}\right)^{\mu(G)} F(Y, Y') = 0.
\end{aligned} \quad (9)
$$

In the reverse, from the formulas (7) and (9), we have

$$
\begin{aligned}
&(\alpha - \gamma u)^{\mu(F)} F\left(\frac{\delta u - \beta}{-\gamma u + \alpha}, \left(\frac{\delta u - \beta}{-\gamma u + \alpha}\right)'\right)\\
&= G(u, u') = 0.
\end{aligned} \quad (10)
$$

Moreover, in (9) and (10), $\mu(G) = \mu(F)$.[5]

Definition 3.2. Let $F(Y, Y') = 0$ (1) and $G(u, u') = 0$ (2) be two first-order AODEs. We say F is *equivalent* to G if there is a Möbius transformation (6) such that the formula (10) is satisfied.

Möbius transformations preserve the genus among the corresponding algebraic curves since they are birational. Moreover, They give an equivalence relation among first-order AODEs and preserve the property of having algebraic solutions of the equivalence class.[4] Next, we will prove that Möbius transformations also preserve the property of having a liouvillian solution of the equivalence class.

Theorem 3.1. *Assume that F is equivalent to G. Then F has a liouvillian solution if and only if so does G. In the affirmative case, the correspondence of such solution is one to one.*

Proof. The case of having an algebraic general solution has been proved by Theorem 2.2.[4] From formula (10) and since

$$(-c\xi + a)^{\mu(F)} \neq 0,$$

we find that an AODE $G = 0$ has a liouvillian transcendental solution ξ if and only if

$$M^{-1}(\xi) = \frac{\delta\xi - \beta}{-\gamma\xi + \alpha}$$

is a transcendental solution $F = 0$. Finally, by formula (6), the correspondence of liouvillian solutions between F and G is one to one. □

Möbius transformation is used to check whether a first-order AODE is equivalent to an autonomous one.[4] If this is the case, then there is an algorithm for finding an algebraic general solution.[7] From that, an algebraic solution of the original AODE can be returned. Based on Theorem 3.1, we aim to apply Möbius transformations for determining liouvillian solutions. The following example, see Section 3, illustrates our idea.

Example 3.1. Consider first-order AODE

$$
\begin{aligned}
F(Y, Y') =& -z^3Y^3 + z^2Y'^2 - 2z^2Y^2\\
&+ 2zYY' - zY + Y^2 = 0.
\end{aligned} \quad (11)
$$

Putting $Y = \dfrac{u-1}{z}$ into the AODE (11) and using formula (10), we obtain

$$
\begin{aligned}
&z^4 F\left(\frac{u-1}{z}, (\frac{u-1}{z})'\right) =\\
&G(u, u') = u'^2 - u^3 + u^2 = 0.
\end{aligned} \quad (12)
$$

By Algorithm 4.1,[6] a liouvillian solution of the AODE (12) is

$$(\exp i(z+c)+1)^2 u - 2\exp i(z+c) = 0, \quad i^2 = -1.$$

Therefore, a liouvillian general solution of the AODE (11) is

$$(\exp i(z+c) + 1)^2 (zY + 1) - 2\exp i(z+c) = 0.$$

4. THE CHANGE OF VARIABLES $z = \varphi(x)$

This section studies some cases of differential transformations induced by change of variables over the ground fields. Let $k = \mathbb{C}(x)$ with $' = \frac{d}{dx}$ and let E be a liouvillian extension of k. Consider the differential equation

$$\tilde{F}(y, y') = 0, \tag{13}$$

where y is a function of x and $\tilde{F} \in E[y, w]$, i.e. a first-order AODE with the coefficients in a liouvillian extension E of $\mathbb{C}(x)$. For briefly, we call it an AODE with *liouvillian coefficients* (see Definition 2.4).

The issue of considering a differential equation with coefficients in an extension field is quite naturally. A general investigation for determining liouvillian solutions of a linear differential equation with liouvillian coefficients can be found in Singer.[13] In this section, our purpose is to consider some simple cases which convert an AODE (13) into an AODE (1) by means of change of variables $z = \varphi(x)$.

Definition 4.1. (Definition 2.7)[14] Let E be a liouvillian extension over $\mathbb{C}(x)$ and $z \in E \setminus \mathbb{C}$, then z is called a *rational liouvillian element* over $\mathbb{C}$ if $\frac{dz}{dx} \in \mathbb{C}(z)$.

Example 4.1. The element $z = \sqrt{x} + \sqrt{x+1}$ is a rational liouvillian element over $\mathbb{C}$ since z is algebraic over $\mathbb{C}(x)$ and

$$\frac{dz}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x+1}} \in \mathbb{C}(\sqrt{x} + \sqrt{x+1}).$$

Since Algorithm 1 is independent of the particular form of the indeterminate z, then z can be seen as a rational liouvillian element over $\mathbb{C}$. Hence, this algorithm can be extended to the case of solving first-order AODEs (13) by a change of variable. Assume that there is a change of variable

$$z = \varphi(x), \tag{14}$$

such that it turns an AODE (13) into (1), i.e.

$$\tilde{F}(y, y') = F(Y, Y') = 0.$$

If this occurs and $Y(z)$ is a liouvillian solution of the AODE (1), then

$$y(x) = Y \circ \varphi(x)$$

is a liouvillian solution of the AODE (13).

Remark 4.1. We remind the two differential fields $\left(\mathbb{C}(z), \frac{d}{dz}\right)$ and $\left(\mathbb{C}(x), \frac{d}{dx}\right)$ with their derivatives y' and Y' whose defined as follows

$$y' = \frac{dy}{dx}, \; Y' = \frac{dY}{dz}.$$

By the chain rule, a relation between y' and Y' is expressed as

$$y' = \frac{dy}{dx} = \frac{d(Y \circ \varphi)}{dx} = \frac{dY}{d\varphi}\frac{d\varphi}{dx} = \frac{dY}{dz}\frac{dz}{dx} = Y'\frac{dz}{dx}.$$

The above expression may be applied to detect a candidate change of variables (14).

4.1. The AODEs with transcendental coefficients

In the case of transcendental coefficients, we refer the readers to some standard works for reference.[9,13] Unfortunately, there is no a full algorithm to find the change of variable for this case, and we are going to deal with it in the future. Here, we just present some examples to illustrate the change of variables (14) in the affirmative cases.

Example 4.2. (I·463)[15] Consider first-order AODE

$$yy'^2 - \exp(2x) = 0. \tag{15}$$

The coefficients of the AODE (15) are in $\mathbb{C}(\exp x)$. By setting $z = \varphi(x) = \exp x$, then (15) is converted into an AODE (1)

$$z^2(YY'^2 - 1) = 0.$$

After dividing z^2, we obtain an autonomous AODE (I·462)[15]

$$YY'^2 - 1 = 0, \tag{16}$$

which has a liouvillian general solution

$$Y = \sqrt[3]{\frac{9}{4}(z+c)^2}.$$

Therefore, a liouvillian general solution of the AODE (15) is

$$y = Y \circ \varphi = \sqrt[3]{\frac{9}{4}(\exp x + c)^2}.$$

Example 4.3. (I·387)[15] Consider first-order AODE

$$y'^2 + (y' - y)\exp x = 0. \tag{17}$$

By setting $z = \varphi(x) = \exp x$, the AODE (17) is converted into an AODE

$$Y'^2 z^2 + Y' z^2 - Yz = 0 \tag{18}$$

which has a proper parametrization

$$\mathcal{P}(t) = (t^2 z + tz, t).$$

By using Algorithm 1, the associated ODE respect to $\mathcal{P}(t)$ is

$$t' = -\frac{t^2}{z(2t+1)}$$

which has only a general solution ensured by Risch,[16]

$$\ln(t^2 z) - \frac{1}{t} = c.$$

Due to the work by Rosenlicht,[17] this solution is not liouvillian. Therefore, the AODE (17) has no liouvillian general solution.

4.2. The AODEs with radical coefficients

In the case of solving an AODE with radical coefficients, Algorithm 3.5 by Caravantes et al. can be relied.[5] Assume that there is a change of variables

$$x = r(z) \in \mathbb{C}(z),$$

then it leads to the existence of the inverse substitution (14)

$$z = \varphi(x).$$

Since z is algebraic over $\mathbb{C}(x)$ and

$$\frac{dz}{dx} = (\frac{dr}{dz})^{-1} \in \mathbb{C}(z),$$

then z is a rational liouvillian element over $\mathbb{C}$. The following example illustrates the ideas.

Example 4.4. Consider the first-order AODE

$$\begin{aligned}\tilde{F}(y, y') = &- x\sqrt{x}y^3 + 4x^2 y'^2 - 2xy^2 \\ &+ 4xyy' - \sqrt{x}y + y^2 = 0\end{aligned} \tag{19}$$

By Algorithm 3.5,[5] there is a change of variables $z = \varphi(x) = \sqrt{x}$, which transforms the AODE (19) into (11)

$$\begin{aligned}F(Y, Y') = &- z^3 Y^3 + z^2 Y'^2 - 2z^2 Y^2 \\ &+ 2zYY' - zY + Y^2 = 0.\end{aligned}$$

From Example 3.1, then (19) has a liouvillian general solution

$$\begin{aligned}&(\exp i(\sqrt{x} + c) + 1)^2(\sqrt{x}y + 1) \\ &-2\exp i(\sqrt{x} + c) = 0.\end{aligned}$$

Remark 4.2. There are more examples of transforming first-order AODEs with radical coefficients into the AODEs (1).[5] Since all of the AODEs (1) obtained here are of genus zero, then they are suitable for Algorithm 1.

5. CONCLUSION

In this paper, we have investigated some ways to convert a first-order AODE into the one where known-algorithms exist. In details, first-order AODEs with liouvillian coefficients can be transformed into first-order AODEs (1) in Section 4. Moreover, an AODE (1) may be converted into an autonomous one by Möbius transformation in Section 3. In addition, if the AODEs (1) are of positive genera, the power transformations (respect to $u = Y^n$) may be considered. A full algorithm for determining liouvillian solutions of first-order AODEs will challenge us in the future.

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