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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR 3D VISCOUS CAMASSA-HOLM EQUATIONS WITH THE DAMPING TERM IN THE WHOLE SPACE

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Abstract. We consider the 3D viscous Camassa-Holm equations with damping term in the whole space. We first prove the existence, uniqueness of global weak solutions to the equations. Then, the Sobolev regularity of the solutions is studied.
Keywords: the viscous Camassa-Holm equations; existence; Sobolev regularity.

1. Introduction

The viscous Camassa-Holm equations with the damping term are written by

$$\begin{cases} \partial_t v - \nu\Delta v + u \cdot \nabla v + v \cdot \nabla u^{\mathrm{T}} + \gamma u + \nabla \pi = g, \\ u - \alpha^2 \Delta u = v, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where u is the velocity vector, π is the pressure function and g is the external force that will be assumed to belong to suitable spaces. Here, we have used the notation $v \cdot \nabla u^{\mathrm{T}} = \sum v_j \nabla u_j$.

The viscous Camassa-Holm equations (VCHE for short) or the Navier-Stokes-α equations arose from the work on shallow water introduced in [1]. The existence and properties of solutions to the VCHE have attracted the attention of many mathematicians in the last few years. In bounded domains with Dirichlet boundary conditions or in a box with periodic boundary conditions, there are many results on the existence, regularity of solutions and the existence of global attractors for Navier-Stokes-α equations, see e.g. [2-4] and references therein. In the whole space, the existence and time decay rates of global solutions were investigated by many authors in [5-9]

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In this paper, we study the existence and uniqueness of weak solutions to the 3D VCHE with the damping term in the whole space $\mathbb{R}^3$. In the previous work [7], the authors studied the existence, uniqueness and regularity of solutions to VCHE (1.1) in the case $g \equiv 0$ and $\gamma = 0$. It is noticed that the definition of weak solutions in [7] is stronger than that in our paper, so our result on the existence and uniqueness of global weak solutions cannot be deduced from the corresponding results in [7] although these proofs are quite similar. In particular, in our paper the initial datum u_0 only belongs to $H^1_\sigma(\mathbb{R}^3)$ compared with $u_0 \in H^2_\sigma(\mathbb{R}^3)$ in that paper. In addition, we also prove that the weak solutions will belong to $C([0,T]; H^{n+1}_\sigma(\mathbb{R}^3)) \cap L^2(0,T; H^{n+2}_\sigma(\mathbb{R}^3))$ for any $T > 0$ whenever the initial datum u_0 and the external force g belong to smoother spaces, namely $H^{n+1}_\sigma(\mathbb{R}^3)$ and $H^{n-1}_\sigma(\mathbb{R}^3)$, respectively. Thus, in some sense, this result is an extension and continuation of the previous corresponding result in [7].

The rest of this paper is organized as follows. In Section 2, we introduce some notations, results and inequalities related to function spaces, which are necessary in our paper. In the last section, we prove the existence, uniqueness and regularity of global weak solutions to equations (1.1) in the whole space $\mathbb{R}^3$.

2. Preliminaries

Let Ω be a smooth bounded domain in $\mathbb{R}^3$ or be the whole space. We denote $\mathcal{V} = \left\{\varphi \in (C_0^\infty(\Omega))^3 \mid \nabla \cdot \varphi = 0\right\}$, and let $L^2_\sigma(\Omega) = H^0_\sigma(\Omega)$ and $H^n_\sigma(\Omega)$ be the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ and in $(H^n(\Omega))^3$ respectively. We also use notation $H^{-n}_\sigma(\Omega)$ as the dual space of $H^n_\sigma(\Omega)$.

For simplicity, we denote by $\langle u, v\rangle$ the scalar product in $H^0_\sigma(\Omega)$ and also the dual pairing of $H^n_\sigma(\Omega) - H^{-n}_\sigma(\Omega)$. In addition, we use notation $\|u\| = (\int_\Omega |u|^2 dx)^{1/2}$ to denote the norm in the space $H^0_\sigma(\Omega)$.

We summarize some properties of the nonlinear terms.

Lemma 2.1 ([7]). *Let $u, v, w \in \mathcal{V}$, it holds that*

(i) $\langle v \cdot \nabla u^T, w\rangle = \langle w \cdot \nabla u, v\rangle$,

(ii) $\langle u \cdot \nabla v, v\rangle = 0$,

(iii) $\langle u \cdot \nabla v, w\rangle = -\langle u \cdot \nabla w, v\rangle$.

We recall some inequalities needed to deal with the nonlinear terms.

Lemma 2.2 ([10]). *Let Ω be a smooth bounded domain or $\mathbb{R}^3$. Then we have the following inequalities:*

(i) (Gagliardo-Nirenberg inequality)

$$\|u\|_{L^3(\Omega)} \le C\|u\|^{1/2}\|\nabla u\|^{1/2}, \quad \forall u \in H^1_\sigma(\Omega).$$

(ii) (Sobolev inequality)

$$\|u\|_{L^6(\Omega)} \leq C\|u\|_{H^1_\sigma(\Omega)} \quad \forall u \in H^1_\sigma(\Omega).$$

(iii) (Agmon's inequality)

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|^{1/2}_{H^1_\sigma(\Omega)}\|u\|^{1/2}_{H^2_\sigma(\Omega)} \quad \forall u \in H^2_\sigma(\Omega).$$

We have the following definition of weak solutions to (1.1).

Definition 2.1 (Weak solution). *Let $u_0 \in H^1_\sigma(\mathbb{R}^3)$ and $g \in L^2(0,T;H^{-2}_\sigma(\mathbb{R}^3))$. A weak solution to the system* (1.1) *on the interval $(0,T)$ is a function u such that*

$$u \in C([0,T];H^1_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^2_\sigma(\mathbb{R}^3)) \text{ with } \frac{du}{dt} \in L^2(0,T;H^0_\sigma(\mathbb{R}^3)),$$

as well as $u(0) = u_0$ and for any $\varphi \in H^2_\sigma(\mathbb{R}^3)$, the following equality holds

$$\begin{aligned}\langle\frac{dv}{dt},\varphi\rangle &+ \nu\langle\nabla v,\nabla\varphi\rangle + \gamma\langle u,\varphi\rangle + \langle u\cdot\nabla v,\varphi\rangle \\ &+ \langle v\cdot\nabla u^{\mathsf{T}},\varphi\rangle = \langle v(0),\varphi\rangle + \langle g,\varphi\rangle, \quad \textit{for a.e. } t \in [0,T],\end{aligned}$$

and for almost everywhere $t \in [0,T]$,

$$\langle u,\varphi\rangle + \alpha^2\left\langle\nabla u,\nabla\varphi\right\rangle = \left\langle v,\varphi\right\rangle.$$

3. Existence, uniqueness and regularity of weak solutions

To prove the global well-posedness of problem (1.1) in the whole space $\mathbb{R}^3$, we follow general lines of the approach in [7]. More precisely, the proof is divided into three steps. First, we prove the existence of global weak solutions in smooth bounded domains. Then we extend the obtained weak solutions to the whole space and pass to the limits to get the solution to the given problem. Finally, we prove the uniqueness of the solution.

Theorem 3.1. *Let $g \in L^2(0,T;H^{-2}_\sigma(\mathbb{R}^3))$ and $u_0 \in H^1_\sigma(\mathbb{R}^3)$. Then for any $T > 0$, we have a unique weak solution u to equations* (1.1) *on $[0,T]$.*

Proof. Step 1: Existence in smooth bounded domains: We consider $\Omega \subset \mathbb{R}^3$ be any bounded open set with smooth boundary.

Let P_m be the orthogonal projection onto the first m eigenfunctions of the Stokes operator on $(H^2(\Omega))^3 \cap H^1_\sigma(\Omega)$. The Galerkin approximation of (1.1) is given by

$$\begin{cases}\dfrac{dv_m}{dt} + P_m(u_m\cdot\nabla v_m) + P_m(v_m\cdot\nabla u_m^{\mathsf{T}}) - \nu\Delta v_m + \gamma u_m = P_m g, \\ u_m - \alpha^2\Delta u_m = v_m, \\ u_m(0) = P_m u_0.\end{cases} \tag{3.1}$$

Taking the inner product of the both sides of the first (3.1) with u_m and using the fact that $\langle u_m \cdot \nabla v_m, u_m\rangle + \langle v_m \cdot \nabla u_m^{\mathrm{T}}, u_m\rangle = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_m\|^2 + \alpha^2\|\nabla u_m\|^2) + \nu(\|\nabla u_m\|^2 + \alpha^2\|\Delta u_m\|^2) + \gamma\|u_m\|^2 \\
&= \langle g, u_m\rangle \le \|g\|_{H_\sigma^{-2}(\mathbb{R}^3)}\|u_m\|_{H_\sigma^2(\mathbb{R}^3)} \\
&\le \frac{1}{2\min\{\gamma,\nu,\nu\alpha^2\}}\|g\|^2_{H_\sigma^{-2}(\mathbb{R}^3)} + \frac{\min\{\gamma,\nu,\nu\alpha^2\}}{2}\|u_m\|^2_{H_\sigma^2(\mathbb{R}^3)} \\
&\le \frac{1}{2\min\{\gamma,\nu,\nu\alpha^2\}}\|g\|^2_{H_\sigma^{-2}(\mathbb{R}^3)} + \frac{1}{2}\left(\nu(\|\nabla u_m\|^2 + \alpha^2\|\Delta u_m\|^2) + \gamma\|u_m\|^2\right).
\end{aligned}$$

Here we have used the fact that

$$\nu(\|\nabla u_m\|^2 + \alpha^2\|\Delta u_m\|^2) + \gamma\|u_m\|^2 \ge \min\{\gamma,\nu,\nu\alpha\}\|u_m\|^2_{H_\sigma^2(\mathbb{R}^3)}.$$

Then integrating both sides from 0 to T, we have for all $t \in [0,T]$

$$\begin{aligned}
&\|u_m(t)\|^2 + \alpha^2\|\nabla u_m(t)\|^2 + \int_0^T \left(\gamma\|u_m(s)\|^2 + \nu(\|\nabla u_m(s)\|^2 + \alpha^2\|\Delta u_m(s)\|^2)\right) ds \\
&\le \frac{1}{\min\{\gamma,\nu,\nu\alpha^2\}}\|g\|^2_{H_\sigma^{-2}(\mathbb{R}^3)} + \|u_0\|^2 + \alpha^2\|\nabla u_0\|^2.
\end{aligned}$$

Therefore, we can conclude that $\{u_m\}$ is bounded in $L^\infty(0,T;H_\sigma^1(\Omega)) \cap L^2(0,T;H_\sigma^2(\Omega))$. Therefore, by the Banach-Alaoglu theorem, we can ensure that there is a subsequence, which we still denote by u_m, satisfying

$$u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_\sigma^1(\Omega)),$$

$$u_m \rightharpoonup u \text{ in } L^2(0,T;H_\sigma^2(\Omega)).$$

Since the boundedness of $\{u_m\}$ in $L^2(0,T;H_\sigma^2(\Omega))$ implies the boundedness of $\{v_m\}$ in $L^2(0,T;H_\sigma^0(\Omega))$, then it can be seen that $\{\frac{dv_m}{dt}\}$ is bounded in $L^2(0,T;H_\sigma^{-2}(\Omega))$ so that $\{\frac{du_m}{dt}\}$ belong to $L^2(0,T;H_\sigma^0(\Omega))$. Hence, there is a subsequence (which we still denote as $\frac{du_m}{dt}$) such that $\frac{du_m}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2(0,T;H_\sigma^0(\Omega))$. Moreover, there is a subsequence $v_m \rightharpoonup v$ in $L^2(0,T;H_\sigma^0(\Omega))$ and by Aubin-Lions lemma, we can extract a subsequence $u_m \to u$ in $L^2(0,T;H_\sigma^1(\Omega))$ strongly.

From this convergence and the above boundedness of u_m, one can prove that

$$u_m \cdot \nabla v_m \rightharpoonup u \cdot \nabla v \text{ in } L^2(0,T;H_\sigma^{-2}(\Omega)),$$

and

$$v_m \cdot \nabla u_m^{\mathrm{T}} \rightharpoonup v \cdot \nabla u^{\mathrm{T}} \text{ in } L^2(0,T;H_\sigma^{-2}(\Omega)).$$

As we have proved above that $u \in L^2(0,T;H_\sigma^2(\Omega))$ and $\frac{du}{dt} \in L^2(0,T;H_\sigma^0(\Omega))$. Therefore, $u \in C([0,T];H_\sigma^1(\Omega))$. The fact that $u(0) = u_0$ is proved by a standard argument.

Step 2: Existence in the whole space. We have proved the existence of the weak solution of (1.1) for any bounded open set with a smooth boundary. To prove the existence of weak solution on the whole space $\mathbb{R}^3$, let R_m be a sequence tending to infinity w.r.t m and χ_{R_m} a smooth cutoff function which is equal to 1 inside the ball of radius $R_m - \epsilon$ for $\epsilon > 0$ and zero on the boundary of the ball of radius R_m. Each R_m, there is a weak solution $\widetilde{u}^{R_m}$ on the ball B_m with initial conditions $u_0\chi_{R_m}$ (which has $|u_0\chi_{R_m}| \leq |u_0|$), we denote the extended function u^{R_m} which is equal to $\widetilde{u}^{R_m}$ (the weak solution corresponding to $u_0\chi_{R_m}$) inside the open ball B_m and equal to 0 outside the open ball B_m. Moreover, all the boundedness above is independent of the size Ω. Using the Banach-Alaoglu theorem, there exists a function

$$u \in L^\infty(0,T; H^1_\sigma(\mathbb{R}^3)) \cap L^2(0,T; H^2_\sigma(\mathbb{R}^3))$$

such that

$$\begin{aligned} u^{R_m} &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T; H^1_\sigma(\mathbb{R}^3)), \\ u^{R_m} &\rightharpoonup u \text{ in } L^2(0,T; H^2_\sigma(\mathbb{R}^3)). \end{aligned}$$

Now we denote B_m a ball radius R_m centered at zero and for each m we have the bound

$$\|du^{R_m}/dt\|_{L^2(0,T;H^0_\sigma(B_m))} \leq C$$

Remark that we have $H^0_\sigma(B_m) \subset H^0_\sigma(B_M)$ for every $m \geq M$ with $\|\cdot\|_{H^0_\sigma(B_M)} \leq \|\cdot\|_{H^0_\sigma(B_m)}$ then

$$\|du^{R_m}/dt\|_{L^2(0,T;H^0_\sigma(B_M))} \leq C.$$

So by the Aubin-Lions lemma, we get a subsequence $u^{R_m} \to u$ in $L^2(0,T; H^1_\sigma(B_M))$ strongly for $M = 1, 2, 3, \ldots$ To prove that u is a weak solution in the whole space, given any test function $\varphi \in \mathcal{V}$, let M be large enough so that the compact supports of φ is contained in B_M, then for all $m \geq M$ and $t > 0$, we have

$$\begin{aligned} \langle v^{R_m}(t), \varphi\rangle + \nu \int_0^t \langle \nabla v^{R_m}, \nabla\varphi) ds + \gamma \int_0^t \langle u^{R_m}, \varphi\rangle ds + \int_0^t \langle u^{R_m} \cdot \nabla v^{R_m}, \varphi\rangle ds \\ + \int_0^t \langle v^{R_m} \cdot \nabla (u^{R_m})^{\mathtt{T}}, \varphi\rangle ds = \langle v_0\chi_{R_m}, \varphi\rangle + \langle g, \varphi\rangle \end{aligned}$$

since u^{R_m} is a weak solution in B_m and $\varphi \in C^\infty_c(B_m)$ for $m > M$. We pass to the limit $m \to \infty$ by using weak convergence on the whole space and the strong convergence of u^{R_m} in $L^2(0,T; H^1_\sigma(B_M))$ so it has proved that u is a weak solution for $\Omega = \mathbb{R}^3$ with the initial datum u_0.

Step 3: Uniqueness of solutions. Let u_1 and u_2 be two solutions to (1.1) corresponding to the initial data u_{01} and u_{02}. Put $u = u_1 - u_2$ and $u_0 = u_{01} - u_{02}$ and $v = u - \alpha^2\Delta u = v_1 - v_2$, we have

$$\begin{cases} \dfrac{dv}{dt} + u_1 \cdot \nabla v_1 + v_1 \cdot \nabla u_1^{\mathtt{T}} - u_2 \cdot \nabla v_2 - v_2 \cdot \nabla u_2^{\mathtt{T}} - \nu\Delta v + \gamma u = 0, \\ u(0) = u_{01} - u_{02}. \end{cases} \tag{3.2}$$

Taking the inner product both sides of the first (3.2) with u, using Lemma 2.1 then we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|^2+\alpha^2\|\nabla u\|^2)+\nu(\|\nabla u\|^2+\alpha^2\|\Delta u\|^2)+\gamma\|u\|^2\\
&=-\langle u_1\cdot\nabla v_1,u\rangle-\langle u\cdot\nabla u_1,v_1\rangle+\langle u_2\cdot\nabla v_2,u\rangle+\langle u\cdot\nabla u_2,v_2\rangle\\
&\le|\langle u_2\cdot\nabla v,u\rangle|+|\,\langle u\cdot\nabla u_2,v\rangle\,|\\
&\le\alpha^2|\langle u_2\cdot\nabla\Delta u,u\rangle|+|\langle u\cdot\nabla u_2,u\rangle|+\alpha^2|\langle u\cdot\nabla u_2,\Delta u\rangle|. \qquad (3.3)
\end{aligned}$$

Using the Hölder inequality, Gagliardo-Nirenberg inequality, Sobolev inequality and the Young inequality, we first have

$$\begin{aligned}
\alpha^2|\langle u_2\cdot\nabla\Delta u,u\rangle|&\le\alpha^2\int_{\mathbb{R}^3}(|\nabla u_2||\Delta u||u|+|u_2||\Delta u||\nabla u|)\,dx\\
&\le\alpha^2\|\nabla u_2\|_{L^6(\mathbb{R}^3)}\|\Delta u\|\|u\|_{L^3(\mathbb{R}^3)}+\alpha^2\|u_2\|_{L^6(\mathbb{R}^3)}\|\Delta u\|\|\nabla u\|_{L^3(\mathbb{R}^3)}\\
&\le C\alpha^2\|u_2\|_{H^2_\sigma(\mathbb{R}^3)}\|\Delta u\|\|u\|^{1/2}\|\nabla u\|^{1/2}\\
&\quad+C\alpha^2\|u_2\|_{H^1_\sigma(\mathbb{R}^3)}\|\Delta u\|^{3/2}\|\nabla u\|^{1/2}\\
&\le\frac{\nu}{2}(\|\nabla u\|^2+\alpha^2\|\Delta u\|^2)\\
&\quad+C\left(\alpha\nu^{-1}\|u_2\|^2_{H^2_\sigma(\mathbb{R}^3)}+\nu^{-3}\|u_2\|^4_{H^1_\sigma(\mathbb{R}^3)}\right)(\|u\|^2+\alpha^2\|\nabla u\|^2).
\end{aligned}$$

Using the Hölder inequality, Gagliardo-Nirenberg inequality, Sobolev inequality and the Cauchy inequality, then

$$\begin{aligned}
|\langle u\cdot\nabla u_2,u\rangle|&\le\|u\|_{L^3(\mathbb{R}^3)}\|\nabla u_2\|_{L^6(\mathbb{R}^3)}\|u\|\\
&\le C\|u_2\|_{H^2_\sigma(\mathbb{R}^3)}\|u\|^{3/2}\|\nabla u\|^{1/2}\\
&\le C\alpha^{-1}\gamma^{-1}\|u_2\|^2_{H^2_\sigma(\mathbb{R}^3)}(\|u\|^2+\alpha^2\|\nabla u\|^2)+\gamma(\|u\|^2+\alpha^2\|\nabla u\|^2),
\end{aligned}$$

and

$$\begin{aligned}
\alpha^2|\langle u\cdot\nabla u_2,\Delta u\rangle|&\le\alpha^2\|u\|_{L^3(\mathbb{R}^3)}\|\nabla u_2\|_{L^6(\mathbb{R}^3)}\|\Delta u\|\\
&\le C\|u\|^{1/2}\|\nabla u\|^{1/2}\|u_2\|_{H^2_\sigma(\mathbb{R}^3)}\|\Delta u\|\\
&\le\frac{\nu}{2}(\|\nabla u\|^2+\alpha^2\|\Delta u\|^2)+C\alpha\nu^{-1}\|u_2\|^2_{H^2_\sigma(\mathbb{R}^3)}(\|u\|^2+\alpha^2\|\nabla u\|^2).
\end{aligned}$$

Substituting three above estimates into (3.3) to deduce that

$$\begin{aligned}
&\frac{d}{dt}(\|u\|^2+\alpha^2\|\nabla u\|^2)\\
&\le C\left((\alpha\nu^{-1}+\alpha^{-1}\gamma^{-1})\|u_2\|^2_{H^2_\sigma(\mathbb{R}^3)}+\nu^{-3}\|u_2\|^4_{H^1_\sigma(\mathbb{R}^3)}\right)(\|u\|^2+\alpha^2\|\nabla u\|^2). \qquad (3.4)
\end{aligned}$$

Using the Gronwall inequality to (3.4) with notice that $u_2\in L^\infty(0,T;H^1_\sigma(\mathbb{R}^3))\cap L^2(0,T;H^2_\sigma(\mathbb{R}^3))$, we obtain the uniqueness of weak solution. □

We now study the Sobolev regularity of the weak solution u when the initial data u_0 and external force g are more regular.

Theorem 3.2. *For any integer $n \geq 0$, assume that $g \in L^2(0,T;H_\sigma^{n-2}(\mathbb{R}^3))$ and $u_0 \in H_\sigma^{n+1}(\mathbb{R}^3)$, then the solution constructed in Theorem 3.1 with initial data u_0 satisfying $u \in C([0,T];H_\sigma^{n+1}(\mathbb{R}^3)) \cap L^2(0,T;H_\sigma^{n+2}(\mathbb{R}^3))$ with $\frac{du}{dt} \in L^2(0,T;H_\sigma^{n}(\mathbb{R}^3))$.*

Proof. For each $n \in \mathbb{N}$, we will prove that there exists a positive constant ρ_n depending on g, u_0 such that for all $t \in [0,T]$

$$\begin{aligned}&\|u(t)\|^2_{H_\sigma^n(\mathbb{R}^3)} + \alpha^2\|u(t)\|^2_{H_\sigma^{n+1}(\mathbb{R}^3)}\\&+\int_0^T \left\{\gamma\|u(s)\|^2_{H_\sigma^n(\mathbb{R}^3)} + \nu\left(\|u(s)\|^2_{H_\sigma^{n+1}(\mathbb{R}^3)} + \alpha^2\|u(s)\|^2_{H_\sigma^{n+2}(\mathbb{R}^3)}\right)\right\} ds \leq \rho_n. \qquad (3.5)\end{aligned}$$

For $n=0$, we easily see that (3.5) holds. Assuming that (3.5) is also true for n, we will prove that it holds for $n+1$. Let $g \in H_\sigma^{n-1}(\mathbb{R}^3)$ and $u_0 \in H_\sigma^{n+2}(\mathbb{R}^3)$. By taking inner product both sides of (1.1) with $-\nabla^{2n+2}u$ and integrating by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla^{n+1}u\|^2 + \alpha^2\|\nabla^{n+2}u\|^2) + \gamma\|\nabla^{n+1}u\|^2 + \nu(\|\nabla^{n+2}u\|^2 + \alpha^2\|\nabla^{n+3}u\|^2)\\&\leq |\langle g, \nabla^{2n+2}u\rangle| + |\langle u\cdot\nabla v, \nabla^{2n+2}u\rangle| + |\langle \nabla^{2n+2}u\cdot\nabla u, v\rangle|. \qquad (3.6)\end{aligned}$$

Here, we have denoted $\nabla^s = (-\Delta)^{s/2}$ and $\|\nabla^s u\|^2 = \int_{\mathbb{R}^3}|\nabla^s u|^2 dx$ for any $s \in \mathbb{N}$.
Firstly, using the Cauchy inequality we have

$$\begin{aligned}|\langle g, \nabla^{2n+2}u\rangle| &= |\langle \nabla^{n-1}g, \nabla^{n+3}u\rangle|\\&\leq \|\nabla^{n-1}g\|\|\nabla^{n+3}u\| \leq \frac{2}{\nu\alpha^2}\|\nabla^{n-1}g\|^2 + \frac{\nu\alpha^2}{8}\|\nabla^{n+3}u\|^2. \qquad (3.7)\end{aligned}$$

We now have

$$|\langle u\cdot\nabla v, \nabla^{2n+2}u\rangle| \leq |\langle u\cdot\nabla u, \nabla^{2n+2}u\rangle| + \alpha^2|\langle u\cdot\nabla(\nabla^2 u), \nabla^{2n+2}u\rangle|.$$

Using the Gagliardo-Nirenberg inequality, the Sobolev inequality and the Cauchy inequality then

$$\begin{aligned}&|\langle u\cdot\nabla u, \nabla^{2n+2}u\rangle| = |\langle \nabla^{n+1}(u\cdot\nabla u), \nabla^{n+1}u\rangle|\\&\leq \sum_{i=0}^{n+1}\binom{n+1}{i}|\langle \nabla^{n+1-i}u\cdot\nabla(\nabla^i u), \nabla^{n+1}u\rangle|\\&= \sum_{i=0}^{n+1}\binom{n+1}{i}|\langle \nabla^{n+1-i}u\cdot\nabla(\nabla^{n+1}u), \nabla^i u\rangle|\\&\leq C\sum_{i=0}^{n}\|\nabla^{n+2}u\|\|\nabla^{n+1-i}u\|_{L^3(\mathbb{R}^3)}\|\nabla^i u\|_{L^6(\mathbb{R}^3)}\end{aligned}$$

$$
\begin{aligned}
&+ C\|\nabla^{n+2}u\|\|\nabla^{n+1}u\|_{L^3(\mathbb{R}^3)}\|u\|_{L^6(\mathbb{R}^3)}\\
&\le C\sum_{i=0}^{n}\|\nabla^{n+2}u\|\|\nabla^{n+1-i}u\|^{1/2}\|\nabla^{n+2-i}u\|^{1/2}\|\nabla^i u\|_{H^1_\sigma(\mathbb{R}^3)}\\
&+ C\|\nabla^{n+2}u\|\|\nabla^{n+1}u\|^{1/2}\|\nabla^{n+2}u\|^{1/2}\|u\|_{H^1_\sigma(\mathbb{R}^3)}\\
&\le \frac{C}{\alpha\nu}\|u\|^2_{H^{n+1}_\sigma(\mathbb{R}^3)}\sum_{i=0}^{n}(\|\nabla^{n+1-i}u\|^2+\alpha^2\|\nabla^{n+2-i}u\|^2)+\frac{\nu}{8}\|\nabla^{n+2}u\|^2\\
&\le \frac{C}{\alpha\nu}\|u\|^2_{H^{n+1}_\sigma(\mathbb{R}^3)}\left(\|u\|^2_{H^{n+1}_\sigma(\mathbb{R}^3)}+\alpha^2\|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)}\right)+\frac{\nu}{8}\|\nabla^{n+2}u\|^2. \qquad (3.8)
\end{aligned}
$$

Now, using the Gagliardo-Nirenberg inequality, the Sobolev inequality, the Argmon inequality and the Cauchy inequality, we have

$$
\begin{aligned}
&\alpha^2|\langle u\cdot\nabla(\nabla^2 u),\nabla^{2n+2}u\rangle| = \alpha^2|\langle\nabla^{n+1}(u\cdot\nabla(\nabla^2u)),\nabla^{n+1}u\rangle|\\
&\le \alpha^2\sum_{i=0}^{n}\binom{n+1}{i}|\langle\nabla^{n+1-i}u\cdot\nabla(\nabla^{i+2}u),\nabla^{n+1}u\rangle| + C\alpha^2|\langle u\cdot\nabla(\nabla^{n+1}u),\nabla^{n+3}u\rangle|\\
&= \alpha^2\sum_{i=0}^{n}\binom{n+1}{i}|\langle\nabla^{n+1-i}u\cdot\nabla(\nabla^{n+1}u),\nabla^{i+2}u\rangle| + C\alpha^2|\langle u\cdot\nabla(\nabla^{n+1}u),\nabla^{n+3}u\rangle|\\
&\le C\alpha^2\sum_{i=0}^{n}\|\nabla^{n+1-i}u\|_{L^6(\mathbb{R}^3)}\|\nabla^{n+2}u\|_{L^3(\mathbb{R}^3)}\|\nabla^{i+2}u\|\\
&+ C\alpha^2\|u\|_{L^\infty(\mathbb{R}^3)}\|\nabla^{n+2}u\|\|\nabla^{n+3}u\|\\
&\le C\alpha^2\sum_{i=0}^{n}\|\nabla^{n+1-i}u\|_{H^1_\sigma(\mathbb{R}^3)}\|\nabla^{n+2}u\|^{1/2}\|\nabla^{n+3}u\|^{1/2}\|\nabla^{i+2}u\|\\
&+ C\alpha^2\|u\|^{1/2}_{H^1_\sigma(\mathbb{R}^3)}\|u\|^{1/2}_{H^2_\sigma(\mathbb{R}^3)}\|\nabla^{n+2}u\|\|\nabla^{n+3}u\|\\
&\le \frac{C}{\nu\alpha}\|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)}\sum_{i=0}^{n}(\|\nabla^{i+1}u\|^2+\alpha^2\|\nabla^{i+2}u\|^2)+\frac{\nu}{8}\left(\|\nabla^{n+2}u\|^2+\alpha^2\|\nabla^{n+3}u\|^2\right)\\
&\le \frac{C}{\nu\alpha}\|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)}\left(\|u\|^2_{H^{n+1}_\sigma(\mathbb{R}^3)}+\alpha^2\|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)}\right)+\frac{\nu}{8}\left(\|\nabla^{n+2}u\|^2+\alpha^2\|\nabla^{n+3}u\|^2\right). \qquad (3.9)
\end{aligned}
$$

Finally, for the last term, we have

$$
\begin{aligned}
&|\langle\nabla^{2n+2}u\cdot\nabla u, v\rangle| = \alpha^2|\langle\nabla^{2n+2}u\cdot\nabla u,\nabla^2u\rangle|\\
&= \alpha^2\sum_{i=1}^{n}\binom{n+1}{i}|\langle\nabla^{n+1}u\cdot\nabla(\nabla^{n+1-i}u),\nabla^{i+2}u\rangle|\\
&+ C\alpha^2|\langle\nabla^{n+1}u\cdot\nabla u,\nabla^{n+3}u\rangle| + C\alpha^2|\langle\nabla^{n+1}u\cdot\nabla(\nabla^{n+1}u),\nabla^2u\rangle|\\
&\le C\alpha^2\sum_{i=1}^{n}\|\nabla^{n+1}u\|_{L^3(\mathbb{R}^3)}\|\nabla^{n+2-i}u\|_{L^6(\mathbb{R}^3)}\|\nabla^{i+2}u\|
\end{aligned}
$$

$$
\begin{aligned}
&+ C\alpha^2 \|\nabla^{n+1}u\|_{L^3(\mathbb{R}^3)} \|\nabla u\|_{L^6(\mathbb{R}^3)} \|\nabla^{n+3}u\| \\
&+ C\alpha^2 \|\nabla^{n+1}u\| \|\nabla^{n+2}u\|_{L^3(\mathbb{R}^3)} \|\nabla^2 u\|_{L^6(\mathbb{R}^3)} \\
\le\ & C\alpha^2 \sum_{i=1}^{n} \|\nabla^{n+1}u\|^{1/2} \|\nabla^{n+2}u\|^{1/2} \|\nabla^{n+2-i}u\|_{H^1_\sigma(\mathbb{R}^3)} \|\nabla^{i+2}u\| \\
&+ C\alpha^2 \|\nabla^{n+1}u\|^{1/2} \|\nabla^{n+2}u\|^{1/2} \|\nabla u\|_{H^1_\sigma(\mathbb{R}^3)} \|\nabla^{n+3}u\| \\
&+ C\alpha^2 \|\nabla^{n+1}u\| \|\nabla^{n+2}u\|^{1/2} \|\nabla^{n+3}u\|^{1/2} \|\nabla^2 u\|_{H^1_\sigma(\mathbb{R}^3)} \\
\le\ & \frac{C}{\nu\alpha} \|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)} \left(\|u\|^2_{H^{n+1}_\sigma(\mathbb{R}^3)} + \alpha^2 \|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)} \right) + \frac{\nu}{8} \left(\|\nabla^{n+2}u\|^2 + \alpha^2 \|\nabla^{n+3}u\|^2 \right).
\end{aligned}
\tag{3.10}
$$

From (3.6), (3.7), (3.8), (3.9), (3.10), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla^{n+1}u\|^2 + \alpha^2\|\nabla^{n+2}u\|^2) + 2\gamma\|\nabla^{n+1}u\|^2 + \nu(\|\nabla^{n+2}u\|^2 + \alpha^2\|\nabla^{n+3}u\|^2) \\
\le\ & \frac{C}{\nu\alpha} \|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)} \left(\|u\|^2_{H^{n+1}_\sigma(\mathbb{R}^3)} + \alpha^2 \|u\|^2_{H^{n+2}_\sigma(\mathbb{R}^3)} \right) + \frac{8}{\nu\alpha^2} \|g\|^2_{H^{n-1}_\sigma(\mathbb{R}^3)}.
\end{aligned}
$$

Using the inductive assumption, then we can use the Gronwall inequality to conclude that (3.5) holds for $n+1$. Hence, u is bounded in $L^\infty(0,T;H^{n+1}_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^{n+2}_\sigma(\mathbb{R}^3))$.

We now prove the boundedness of $\frac{du}{dt}$ in $L^2(0,T;H^n_\sigma(\mathbb{R}^3))$. Since u is bounded in $L^\infty(0,T;H^{n+1}_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^{n+2}_\sigma(\mathbb{R}^3))$, we can conclude that the nonlinear terms are bounded in $L^2(0,T;H^{n-2}_\sigma(\mathbb{R}^3))$. Hence, $\frac{du}{dt}$ in $L^2(0,T;H^n_\sigma(\mathbb{R}^3))$. □

Remark 3.1. One can see that if $g \equiv 0$ and $\gamma = 0$, we recover the regularity result in [7, Theorem A.7] for $n \ge 1$.

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