

AN ESTIMATE FOR THE GAUSSIAN CURVATURE OF MINIMAL SURFACES IN EUCLIDEAN FOUR-SPACE WITH RAMIFICATION

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Abstract. In this article, we will establish an estimate for the Gaussian curvature of minimal surfaces in \mathbb{R}^4 whose classical Gauss map is ramified over the set of distinct points. This is a continuation of previous work of Fujimoto (1993, *Aspect of Math., Vieweg, Wiesbaden*, Vol. E21); Osserman-Ru (1997, *J. Differ. Geom.*, Vol. 46, pp. 578-593) and Ha (2014, *Differ. Geom. Appl.*, Vol. 32, pp. 130-138.)

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1. Introduction

Value distribution theory of the Gauss map of complete regular minimal surfaces has a long history, in particular, much attention has been given to this theory from the viewpoint of the Nevanlinna theory. The relations of the omitted properties or ramifications of the Gauss map and the Gaussian curvature of minimal surfaces have been studied more, especially in recent years (see [1-13]). For instance, Fujimoto [3] proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in \mathbb{R}^3 then its classical Gauss map can omit at most 4 points, and the bound is sharp. The value distribution of classical Gauss map of complete non-flat minimal surfaces in \mathbb{R}^4 was also given by Chen [1], Fujimoto [3] and some other authors (see [7, 14]...). After that, Fujimoto [15] extended those results for complete minimal surfaces in \mathbb{R}^m in the case their generalized Gauss map is assumed nondegenerate. In which, the Gauss map can omit at most $m(m+1)/2$ hyperplanes in general position in the complex projective space $\mathbb{P}^{m-1}(\mathbb{C})$. Next, Ru [13] refined these results by studying generalized Gauss map of minimal surfaces in \mathbb{R}^m with ramification.

By estimating the Gaussian curvature of minimal surfaces, we can get the "value distribution" properties of the Gauss map (see [4, 12] for the case of classical Gauss of minimal surfaces in \mathbb{R}^3 , see [5, 10] for the case of generalized Gauss map of minimal surfaces in \mathbb{R}^m). For example, Fujimoto [4] obtained an estimate for the Gaussian

curvature of minimal surfaces in \mathbb{R}^3 that if their classical Gauss map is ramified over q values a^j with multiplicity at least m_j for each j such that

$$\sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) > 4,$$

then there exists a constant C depending on the set of ramified points but not the surface such that

$$|K(p)|^{1/2}d(p) \leq C \tag{1.1}$$

where $K(p)$ is the Gaussian curvature of the surface at p and $d(p)$ is the geodesic distance from p to the boundary of M .

Osserman-Ru [10] generalized Ros's proof [12] to minimal surfaces in \mathbb{R}^m with $m \geq 3$. They proved that if a minimal surface M whose generalized Gauss map omits more than $m(m + 1)/2$ hyperplanes in general position, then there exists a constant C depending on the set of omitted hyperplanes but not the surface such that the inequality of the form (1.1) holds. Ha [5] extended Osserman-Ru's result to the case of ramification. Liu-Pang [9] used normal family theory to study Gaussian curvature estimate of minimal surfaces in \mathbb{R}^m and obtained a similar result. In particular, Kawakami [16] obtained an estimate similar to the inequality of the form (1.1) for the Gaussian curvature of various classes of the surfaces in three-space. However, an estimate for the Gaussian curvature of minimal surfaces in Euclidean four-space has not been obtained yet. Here, we would like to emphasize the following.

- i) The image of the generalized Gauss map $G : M \rightarrow \mathbb{P}^{m-1}(\mathbb{C})$ is contained in the hyper quadric $Q_{m-2}(\mathbb{C}) \subset \mathbb{P}^{m-1}$, and that $Q_1(\mathbb{C})$ is biholomorphic to $\mathbb{P}_1(\mathbb{C})$ and that $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. So the results in the cases $m = 3, m = 4$ are better than a result which holds for any $m \geq 3$ can be if restricted to the special cases $m = 3, m = 4$.
- ii) H. Fujimoto's method is only valid in the case $m = 3$ and it does not seem to be able to extend to higher dimension.

By using the methods due to Ros [12], Osserman-Ru [10], Ha [5] and Liu-Pang [9] as well as Campana's orbifold theory [17, 18], we will give an estimate for the Gaussian curvature of minimal surfaces in \mathbb{R}^4 whose classical Gauss map is ramified over the set of distinct points.

For this article, we first recall some definitions in [15, 19].

Let $x = (x_1, x_2, x_3, x_4) : M \rightarrow \mathbb{R}^4$ be a non-flat regular minimal surface in \mathbb{R}^4 . As is well-known, the set of all oriented 2-planes in \mathbb{R}^4 is canonically identified with the quadric

$$Q_2(\mathbb{C}) := \{(w_1 : \dots : w_4) | w_1^2 + \dots + w_4^2 = 0\}$$

in $\mathbb{P}^3(\mathbb{C})$. By the definition, the generalized Gauss map $G : M \rightarrow Q_2(\mathbb{C})$ is the map which maps each point p of M to the point of $Q_2(\mathbb{C})$ corresponding to the oriented tangent plane of M at p . The quadric $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. By using suitable identifications, we may regard G as a pair of meromorphic functions $g = (g_1, g_2)$ on M to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and call it the classical Gauss map.

We note that every meromorphic function can be regarded as holomorphic map with values in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. We may thus regard g as a holomorphic map from M into $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

Let z be a local holomorphic coordinate. Set $\phi_i := \partial x_i / dz$ for $i = 1, \dots, 4$. We cover x by the real Ablel-Jacobi map

$$x(z) = \mathcal{R} \int (\phi_1, \phi_2, \phi_3, \phi_4) dz.$$

We get

$$g_1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}, \quad g_2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}$$

and the metric on M induced from \mathbb{R}^4 is given by

$$ds^2 = |\phi|^2 (1 + |g_1|^2)(1 + |g_2|^2) |dz|^2, \quad (1.2)$$

where $\phi dz := \phi_1 - \sqrt{-1}\phi_2$. We remark that although the ϕ_i , ($i = 1, 2, 3, 4$) and ϕ depend on z , $g = (g_1, g_2)$ and ds^2 do not. We observe that by the assumption M is not flat, g is not constant. We call $(\phi dz, g_1, g_2)$ the Weierstrass data. This is related to the ϕ_j in one to one way

$$\begin{cases} \phi_1 = (1 + g_1 g_2) \phi dz \\ \phi_2 = \sqrt{-1}(1 - g_1 g_2) \phi dz \\ \phi_3 = (g_1 - g_2) \phi dz \\ \phi_4 = -\sqrt{-1}(g_1 + g_2) \phi dz. \end{cases} \quad (1.3)$$

We know that the zeros of ϕdz of order k coincide exactly with the poles of g_1 or g_2 of order k . If we are given a holomorphic differential ϕdz and two g_1 and g_2 are meromorphic functions on M , we get the ϕ_j by this formula and the zeros of ϕdz of order k coincide exactly with the poles of g_1 or g_2 of order k . Now the Gaussian curvature K of M is given by

$$K = -\frac{2}{|\phi|^2 (1 + |g_1|^2)(1 + |g_2|^2)} \left(\frac{|g_1'|^2}{(1 + |g_1|^2)^2} + \frac{|g_2'|^2}{(1 + |g_2|^2)^2} \right). \quad (1.4)$$

Surface M is flat if and only if $K \equiv 0$ or equivalently g is constant.

A minimal surface is complete if all divergent paths have an infinite length concerning the metric (1.2).

We say that the meromorphic function $g : M \rightarrow \mathbb{P}^1(\mathbb{C})$ is ramified over a point $a \in \mathbb{P}^1(\mathbb{C})$ with multiplicity as least m if g does not take the value a with multiplicity less than m . If the image of g omits a , we will say that g is ramified over a with multiplicity ∞ .

The main purpose of this article is to prove the following.

Main Theorem. *Suppose that M is a minimal surface in \mathbb{R}^4 and $g = (g_1, g_2)$ is the classical Gauss map of M . Let $a^{l_1}, \dots, a^{l_{q_l}}$ ($l = 1, 2$) be q_l distinct points in $\mathbb{P}^1(\mathbb{C})$ with $q_l \geq 2$. Suppose that g_l ($l = 1, 2$) is ramified over a^{l_j} with multiplicity at least m_{l_j} for each j such that*

$$\gamma_1 = \sum_{j=1}^{q_1} \left(1 - \frac{1}{m_{1j}}\right) > 2, \quad (1.5)$$

$$\gamma_2 = \sum_{j=1}^{q_2} \left(1 - \frac{1}{m_{2j}}\right) > 2, \quad (1.6)$$

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} < 1. \quad (1.7)$$

Then there exists a constant C depending on the set of ramified points, but not the surface such that

$$|K(p)|^{1/2}d(p) \leq C,$$

where $K(p)$ is the Gaussian curvature of the surface at p and $d(p)$ is the geodesic distance from p to the boundary of M .

Suppose that g_2 is constant. It is easy to see that condition (1.6) is satisfied. Moreover, condition (1.8) below implies conditions (1.5) and (1.7). Applying the main Theorem, we immediately get the following corollary.

Corollary. *Suppose that M is a minimal surface in \mathbb{R}^4 and $g = (g_1, g_2)$ is the classical Gauss map of M , where g_1 or g_2 is constant, say $g_2 \equiv \text{constant}$. Let a^1, \dots, a^q be q distinct points in $\mathbb{P}^1(\mathbb{C})$. Suppose that g_1 is ramified over a^j with multiplicity at least m_j for each j such that*

$$\sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) > 3. \quad (1.8)$$

Then there exists a constant C depending on the set of ramified points, but not the surface such that

$$|K(p)|^{1/2}d(p) \leq C.$$

2. Auxiliary lemmas

2.1. Nevanlinna theory

First, we recall the standard notations of the Nevanlinna theory.

Let $f : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. For every $r \geq 0$, let $n(r, f)$ be the number of poles counting multiplicity of the meromorphic function f in the disc $|z| \leq r$. Then the Nevanlinna counting function is defined by

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r.$$

Let $\log^+(x) = \max\{\log x, 0\}$. Then the proximity function is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

The Nevanlinna characteristic is defined by

$$T(r, f) = m(r, f) + N(r, f).$$

Let $a \in \mathbb{P}^1(\mathbb{C}) \equiv \mathbb{C} \cup \{\infty\}$, we define

$$N(r, a, f) = N(r, \frac{1}{f-a}) \text{ and } m(r, a, f) = m(r, \frac{1}{f-a}),$$

for $a = \infty$, we set $N(r, \infty, f) = N(r, f)$ and $m(r, \infty, f) = m(r, f)$.

The First Fundamental Theorem of Nevanlinna theory states that for every a in the Riemann sphere,

$$T(r, f) = N(r, a, f) + m(r, a, f) + O(1),$$

where the bounded term $O(1)$ may depend on f and a .

The defect of a meromorphic function at the point a is defined by the formula

$$\delta(a, f) = \liminf_{r \rightarrow \infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a, f)}{T(r, f)}.$$

Finally, we recall the classical result of Nevanlinna's theory.

Theorem 2.1. [20, Nevanlinna] *Let $f : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C})$ be a non-constant meromorphic function. Then*

$$\sum_{a \in \mathbb{P}^1(\mathbb{C})} \delta(a, f) \leq 2.$$

2.2. Geometric orbifold

We recall some notations and some results in orbifold theory which was introduced by F. Campana [17].

Definition 2.1. Let (X, Δ) be an orbifold with $\Delta = \sum_j \left(1 - \frac{1}{m_j}\right) Z_j$, where the Z_j are distinct irreducible divisors and $m_j \in \mathbb{N} \cup \{\infty\}$. Let h be a holomorphic map from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to X .

(1) h is a (non-classical) orbifold morphism from \mathbb{D} to (X, Δ) if $h(\mathbb{D}) \not\subset \text{supp}(\Delta)$ and $\text{mult}_x(h^*Z_j) \geq m_j$ for all j and $x \in \mathbb{D}$ with $h(x) \in \text{supp}(Z_j)$. If $m_j = \infty$, we require $h(\mathbb{D}) \cap \text{supp}(Z_j) = \emptyset$.

(2) h is a classical orbifold morphism from \mathbb{D} to (X, Δ) if $h(\mathbb{D}) \not\subset \text{supp}(\Delta)$ and $\text{mult}_x(h^*Z_j)$ is a multiple of m_j for all j and $x \in \mathbb{D}$ with $h(x) \in \text{supp}(Z_j)$. If $m_j = \infty$, we require $h(\mathbb{D}) \cap \text{supp}(Z_j) = \emptyset$.

Definition 2.2. Let (X, Δ) and (X', Δ') be two orbifolds. Let Δ_1 be the union of all irreducible components of Δ with multiplicity 1. An orbifold morphism (resp. classical orbifold morphism) from (X, Δ) to (X', Δ') is a holomorphic map $f : X \setminus \Delta_1 \rightarrow X'$ such that

- (1) $f(X) \not\subset \text{supp}(\Delta')$,
- (2) for every orbifold morphism (resp. classical orbifold morphism) in the sense of Definition 2.1, $g : \mathbb{D} \rightarrow (\Delta, X)$ with $g(\mathbb{D}) \not\subset f^{-1}(\text{supp}(\Delta'))$ the composed map $f \circ g : \mathbb{D} \rightarrow X'$ defines an orbifold morphism (resp. classical orbifold morphism) from \mathbb{D} to (X', Δ') .

Definition 2.3. The orbifold Kobayashi pseudo-distance $d_{(X, \Delta)}$ (resp. the classically orbifold Kobayashi pseudo-distance $d_{(X, \Delta)}^*$) on (X, Δ) is the largest pseudo-distance on $X \setminus \Delta_1$ such that $g^*d_{(X, \Delta)} \leq d_P$ (resp. $g^*d_{(X, \Delta)}^* \leq d_P$) for every orbifold morphism (resp. classically orbifold morphism) $g : \mathbb{D} \rightarrow (X, \Delta)$, where d_P denotes the Poincaré distance on \mathbb{D} .

Definition 2.4. An orbifold (X, Δ) is hyperbolic (resp. classically hyperbolic) if $d_{(X, \Delta)}$ (resp. $d_{(X, \Delta)}^*$) is distance on $X \setminus \Delta_1$.

We need the following result due to F. Campana-J. Winkelmann [18].

Proposition 2.1 ([18]). Let $f_n : (X, \Delta) \rightarrow (X', \Delta')$ be a sequence of orbifold morphisms. Assume that $\{f_n\}$ regarded as a sequence of holomorphic maps from X to X' converge locally uniformly to a holomorphic map $f : X \rightarrow X'$. Then either $f(X) \subset \text{supp}(\Delta')$ or f is an orbifold morphism from (X, Δ) to (X', Δ') .

Definition 2.5. We say that (X, Δ) is hyperbolically embedded (resp. classically hyperbolically imbedded) in X if for any two sequences of points $(p_n), (q_n) \subset X \setminus \Delta_1$ converging two points $p, q \in X$, then

$$d_{(X, \Delta)}(p_n, q_n) \rightarrow 0 \quad (\text{resp. } d_{(X, \Delta)}^*(p_n, q_n) \rightarrow 0) \quad (n \rightarrow \infty) \Rightarrow p = q.$$

Proposition 2.2 ([21]). *Let ω be a hermitian metric on X compact. Then (X, Δ) is hyperbolically (reps. classically hyperbolically) imbedded in X iff there is a positive constant c such that $f^*\omega \leq c \cdot h_P$ for all orbifold (reps. classically orbifold) morphism $f : \mathbb{D} \rightarrow (X, \Delta)$, where h_P denotes the Poincaré metric.*

Proposition 2.3. [22, Brody's reparametrization lemma] *Let X be a compact complex manifold and $f_n : \mathbb{D} \rightarrow X$ a sequence of holomorphic maps with $\limsup \|f'_n(0)\| = +\infty$. Then there exists an increasing sequence of positive real numbers r_n and a sequence of holomorphic maps $\alpha_n : D(r_n; 0) \rightarrow \mathbb{D}$ such that $\lim r_n = +\infty$ and such that a subsequence of $f_n \circ \alpha_n$ converges locally uniformly to a holomorphic map $f : \mathbb{C} \rightarrow X$ with*

$$\sup_{z \in \mathbb{C}} \|f'(z)\| = \|f'(0)\|,$$

where $D(r_n; 0) = \{z \in \mathbb{C} : |z| < r_n\}$.

We need to prove the following proposition for the curve orbifold.

Proposition 2.4. *Let a_j be q distinct points in $\mathbb{P}^1(\mathbb{C})$ and put*

$$\Delta = \sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) a_j$$

with $q > 2$. If

$$\deg(\Delta) = \sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) > 2$$

then $(\mathbb{P}^1(\mathbb{C}), \Delta)$ is hyperbolic and hyperbolically embedded in $\mathbb{P}^1(\mathbb{C})$.

Proof. Suppose that $(\mathbb{P}^1(\mathbb{C}), \Delta)$ is not hyperbolically imbedded in $\mathbb{P}^1(\mathbb{C})$. By Proposition 2.2, we can show that there exists a sequence of orbifold morphisms $f_n : \mathbb{D} \rightarrow (\mathbb{P}^1(\mathbb{C}), \Delta)$ such that $\lim \|f'_n\| = +\infty$. Thanks to Brody reparametrization (Proposition 2.3), we obtain a sequence of orbifold morphisms $g_n : \mathbb{D}(0, r_n) \rightarrow (\mathbb{P}^1(\mathbb{C}), \Delta)$ with $r_n \rightarrow +\infty$, converging to a holomorphic map $f : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C})$. By Proposition 2.1, we can see that f is either a non-constant orbifold morphism $f : \mathbb{C} \rightarrow (\mathbb{P}^1(\mathbb{C}), \Delta)$ or a non-constant holomorphic map $f : \mathbb{C} \rightarrow \text{supp}(\Delta)$. The first case is impossible by Nevanlinna's result (Theorem 2.1). Since $\text{supp}(\Delta)$ is discrete, the second case is impossible either. Thus Proposition 2.4 is proved. \square

Proposition 2.5 ([5]). *Let ω be a hermitian metric on compact manifold X . Assume that the orbifold (X, Δ) is hyperbolic and hyperbolically imbedded in X . Then the set of all orbifold morphisms $f : \mathbb{D} \rightarrow (X, \Delta)$ is relatively compact in $\text{Hol}(\mathbb{D}, X)$, the set of all holomorphic of \mathbb{D} into X .*

3. Proof of the main theorem

First of all, we use arguments similar to those used by Ros [12, lemma 6] to prove the following lemma.

Lemma 3.1. *Let $x^{(v)} : M \rightarrow \mathbb{R}^4$ be a sequence of conformal minimal immersion, $\{g^{(v)} = (g_1^{(v)}, g_2^{(v)})\}$ the sequence of their Gauss maps and K_v the Gaussian curvature of $x^{(v)}$. Suppose that $\{g^{(v)}\}$ converges to a meromorphic function $g = (g_1, g_2)$, the sequence $\{K_v\}$ is uniformly bounded and that $\{x^{(v)}(p_0)\}$ converges for some point $p_0 \in M$. Then we have the following possibilities:*

- (i) g is a constant map, or
- (ii) a subsequence $\{K_{v'}\}$ of $\{K_v\}$ converges to zero, or
- (iii) a subsequence $\{x^{(v')}\}$ of $\{x^{(v)}\}$ converges to a conformal minimal immersion $x : M \rightarrow \mathbb{R}^4$ whose classical Gauss map is g .

Proof. Suppose that g is non-constant and that $-1 \leq K_v$ in M for each $v \in \mathbb{N}$. Let $p \in M$ be a point and (U_p, z) a complex local coordinate centered at p . Let $g_1^{(v)}, g_2^{(v)}$ and $\phi^{(v)}$ be the maps given by the Weierstrass representation of $x^{(v)}$. We put

$$M_1 = \{p : g_l(p) \neq \infty, l = 1, 2 \text{ and } p \text{ is not a branch point of } g_1 \text{ or } g_2\}.$$

Taking a point $p \in M_1$, we have $g_1(p) \neq \infty$ and $g_2(p) \neq \infty$. So $g(p) \in \mathbb{C} \times \mathbb{C}$ and $g_1'(p) \neq 0$ or $g_2'(p) \neq 0$. By choosing U_p and $\epsilon > 0$ sufficiently small, we have that g_1 or g_2 are holomorphic and without branch points on U_p . Thus

$$\frac{2|g_1'|^2}{(1 + |g_1|^2)^3(1 + |g_2|^2)} \geq 2\epsilon^2 \text{ or } \frac{2|g_2'|^2}{(1 + |g_2|^2)^3(1 + |g_1|^2)} \geq 2\epsilon^2 \text{ in } U_p.$$

As $g^{(v)} \rightarrow g$ for v large enough, we have

$$\frac{2|(g_1^{(v)})'|^2}{(1 + |g_1^{(v)}|^2)^3(1 + |g_2^{(v)}|^2)} \geq \epsilon^2 \text{ or } \frac{2|(g_2^{(v)})'|^2}{(1 + |g_2^{(v)}|^2)^3(1 + |g_1^{(v)}|^2)} \geq \epsilon^2 \text{ in } U_p.$$

Thus in U_p , we get

$$\begin{aligned} |K_v| &= \frac{2}{|\phi^{(v)}|^2(1 + |g_1^{(v)}|^2)(1 + |g_2^{(v)}|^2)} \left(\frac{|(g_1^{(v)})'|^2}{(1 + |g_1^{(v)}|^2)^2} + \frac{|(g_2^{(v)})'|^2}{(1 + |g_2^{(v)}|^2)^2} \right) \\ &\geq \frac{\epsilon^2}{|\phi^{(v)}|^2}. \end{aligned}$$

By $|K_v| \leq 1$, we have $|\phi^{(v)}| \geq \epsilon$ in U_p for large v , and then $\{\phi^{(v)}\}$ relatively compact in $\mathcal{M}(U_p)$ - the set of meromorphic functions on U_p . Therefore, the sequence of globally defined holomorphic 1-forms $\{\phi^{(v)} dz\}$ is relatively compact on M_1 , because it is relatively compact in a neighborhood of each of their points. Note that $M \setminus M_1$ is a

discrete set. Taking a subsequence if necessary, we can assume that $\{\phi^{(v)}dz\}$ converges on M_1 either to a nonzero holomorphic 1-form ϕdz or to infinity. We consider each case separately.

Case 1: $\{\phi^{(v)}dz\}$ converges to infinity on M_1 . Let p is a branch point of g_l with $g_l(p) \neq \infty$ ($l = 1, 2$). Hence, in a small disk $D(2\epsilon)$ of U_p , g_l are holomorphic and so $g_l^{(v)}$ are also holomorphic from Hurwitz theorem with v large. Thus, $\phi^{(v)}$ has not zeros on $D(2\epsilon)$ with v large and converges to infinity on $\partial D(\epsilon)$. From the maximum modulus principle, we conclude that $\{\phi^{(v)}\}$ converges to infinity on $D(\epsilon)$.

Now, suppose that $g_1(p) = \infty$ or $g_2(p) = \infty$, say g_1 . Then in a small disk $D(2\epsilon)$ of U_p , g_1 has neither zeros nor poles other than p . So $g_1^{(v)}\phi^{(v)}$ is a holomorphic map without zeros for v large. As $g_1^{(v)}\phi^{(v)}$ converges uniformly to infinity on $\partial D(\epsilon)$, the maximum modulus principle implies that $g_1^{(v)}\phi^{(v)}$ converges to infinity on $D(\epsilon)$. Therefore, it follows that $|\phi^{(v)}|^2(1 + |g_1^{(v)}|^2)(1 + |g_2^{(v)}|^2)$ converges to infinity on U_p for each $p \in M$. By the hypothesis $g^{(v)} \rightarrow g$, we conclude finally, from (1.4), that $\{K_v\}$ converges to the zero function on M .

Case 2: $\{\phi^{(v)}dz\}$ converges to a holomorphic 1-form ϕdz on M_1 . Let $p \in M \setminus M_1$ and $D(\epsilon)$ be a small disc contained in U_p , as $\phi^{(v)} \rightarrow \phi$ on $\partial D(\epsilon)$, we see that $\{\phi^{(v)}\}$ is bounded on $\partial D(\epsilon)$, then by the maximum modulus principle, it is also bounded on $D(\epsilon)$. Thus $\{\phi^{(v)}\}$ is relatively compact on $D(\epsilon)$. We conclude easily that ϕdz extends in a holomorphic way to M and that the global 1-forms $\phi^{(v)}dz$ converges to ϕdz on M . Moreover, from Hurwitz's theorem, we have that the zeros of ϕdz occur precisely at the poles of g_1 or g_2 and the order of the zero coincides with the order of the poles. Then $g_1\phi dz, g_2\phi dz$ and $g_1g_2\phi dz$ are holomorphic in M , $g_1^{(v)}\phi^{(v)}dz \rightarrow g_1\phi dz, g_2^{(v)}\phi^{(v)}dz$ and $g_1^{(v)}g_2^{(v)}\phi^{(v)}dz \rightarrow g_1g_2\phi dz$. As $\{x^{(v)}(p_0)\}$ converges for some point $p_0 \in M$, we conclude, from (1.3), that the sequence of maps $x^{(v)}$ converges uniformly on compact subsets of M to a conformal minimal immersion

$$x : M \rightarrow \mathbb{R}^4,$$

whose Weierstrass representation is given by $(\phi dz, g_1, g_2)$. In particular, $g = (g_1, g_2)$ is the classical Gauss map of x . Lemma 3.1 is proved. \square

Lemma 3.2. [10, Lemma 2.2] *Let ds_n^2 be a sequence of conformal metrics on the unit disk \mathbb{D} whose curvatures satisfy $-1 \leq K_n \leq 0$. Suppose that \mathbb{D} is a geodesic disk of radius R_n with respect to the metric ds_n^2 , where $R_n \rightarrow \infty$ and that the metrics ds_n^2 converge, uniformly on compact sets, to a metric ds^2 . Then all distances to the origin concerning ds^2 are greater than or equal to the corresponding hyperbolic distances in \mathbb{D} . In particular, ds^2 is complete.*

Theorem 3.1 ([3]). *Suppose that M is a complete non-flat minimal surface in \mathbb{R}^4 and $g = (g_1, g_2)$ is the classical Gauss map of M . Let*

$$a^{11}, \dots, a^{1q_1}; a^{21}, \dots, a^{2q_2}$$

be $q_1 + q_2$ ($q_1, q_2 > 2$) distinct points in $\mathbb{P}^1(\mathbb{C})$.

(i) In the case $g_l \not\equiv \text{constant}$ ($l = 1, 2$), if g_l is ramified over a^{lj} with multiplicity at least m_{lj} for each j ($l = 1, 2$) then

$$\gamma_1 = \sum_{j=1}^{q_1} \left(1 - \frac{1}{m_{1j}}\right) \leq 2, \text{ or } \gamma_2 = \sum_{j=1}^{q_2} \left(1 - \frac{1}{m_{2j}}\right) \leq 2, \text{ or}$$

$$\frac{1}{\gamma_1 - 2} + \frac{1}{\gamma_2 - 2} \geq 1.$$

(ii) In the case where g_1 or g_2 is constant, say $g_2 \equiv \text{constant}$, if g_1 is ramified over a^{lj} with multiplicity at least m_{lj} for each j , we have the following:

$$\gamma_1 = \sum_{j=1}^{q_1} \left(1 - \frac{1}{m_{1j}}\right) \leq 3.$$

We now prove the Main Theorem.

Suppose that the Main Theorem is not true. We will construct a non-flat complete minimal surface whose classical Gauss map is ramified by a set of distinct points, thus getting a contradiction with Theorem 3.1. So suppose the conclusion of the Main Theorem is not true; then there exists a sequence of (noncomplete) minimal surfaces $x^{(v)} : M_v \rightarrow \mathbb{R}^4$ and points $p_v \in M_v$ such that $|K_v(p_v)|^{1/2} d_v(p_v) \rightarrow \infty$, and such that the classical Gauss map $g^{(v)} = (g_1^{(v)}, g_2^{(v)})$ of $x^{(v)}$ has component maps $g_l^{(v)}$ ($l = 1, 2$) which ramify over a fixed set of q_l distinct points a^{lj} in $\mathbb{P}^1(\mathbb{C})$ with multiplicity at least m_{lj} for each j , satisfying conditions (1.5), (1.6) and (1.7) as in the Main Theorem. By R. Osserman-M. Ru's Claim 6 in [10, pp. 590-591], we can choose the surfaces M_v which satisfy condition

$$K_v(p_v) = -1, -4 \leq K_v \leq 0 \text{ on } M_v \text{ for all } v \text{ and } d_v(p_v) \rightarrow \infty. \quad (3.1)$$

By translations of \mathbb{R}^4 , we can assume that $x^{(v)}(p_v) = 0$ and M_v are simply connected by taking its universal covering if necessary. By the uniformization theorem, we can see that M_v is conformally equivalent to either the unit disk \mathbb{D} or the complex plane \mathbb{C} and we can suppose that p_v maps onto 0 for each v .

If M_v is conformally equivalent to \mathbb{C} , the maps $g_l^{(v)} : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C})$ ($l = 1, 2$) are ramified over a_{lj} with multiplicity at least m_{lj} . By the assumption, we have

$$\gamma_l = \sum_{j=1}^{q_l} \left(1 - \frac{1}{m_{lj}}\right) > 2, \quad l = 1, 2.$$

It follows from the classical result of Nevanlinna (Theorem 2.1) that the maps $g_l^{(v)}$ ($l = 1, 2$) are constant, so $K_v \equiv 0$. This contradicts the condition that $K_v(0) = -1$. Hence, M_v can not be conformally equivalent to \mathbb{C} . It implies that we have constructed a sequence of minimal surfaces, $x^{(v)} : \mathbb{D} \rightarrow \mathbb{R}^4$, satisfying (3.1).

By Proposition 2.4, the orbifolds $(\mathbb{P}^1(\mathbb{C}), \Delta_l)$ with $\Delta_l = \sum_{j=1}^{q_l} \left(1 - \frac{1}{m_{lj}}\right) a^{lj}$ ($l = 1, 2$) are hyperbolic and hyperbolically imbedded in $\mathbb{P}^1(\mathbb{C})$. Thus, according to Proposition 2.5, we obtain a subsequence of classical Gauss maps $g^{(v)}$ of $x^{(v)}$ exists—without loss generality, we can assume that $g^{(v)} = (g_1^{(v)}, g_2^{(v)}) : \mathbb{D} \rightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ converges uniformly on every compact subset of \mathbb{D} to a map $g = (g_1, g_2) : \mathbb{D} \rightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

We denote by $G^{(v)} : \mathbb{D} \rightarrow \mathbb{P}^3(\mathbb{C})$ the generalized Gauss map of minimal surface M_v . We get the classical Gauss map $g^{(v)}$ from $G^{(v)}$ by a common suitable identification for all v . Hence $G^{(v)}$ also converges uniformly on every compact subset of \mathbb{D} to a map $G : \mathbb{D} \rightarrow \mathbb{P}^3(\mathbb{C})$. Then, by R. Osserman-M. Ru's arguments in [10, pp. 591-592], we deduce that G is not constant and so g is not constant either. It implies that $x^{(v)}$ does not satisfy condition (ii) in Lemma 3.1. By (3.1), it is clear that they do not satisfy the condition (i) in this lemma either. Therefore, $x^{(v)}$ must satisfy the condition (iii), i.e., there exists a subsequence $\{x^{(v')}\}$ of $\{x^{(v)}\}$ which converges to a minimal immersion $x : \mathbb{D} \rightarrow \mathbb{R}^4$ and whose classical Gauss map is g . By (3.1) and Lemma 3.2, x is complete.

Since $g_l^{(v)} : \mathbb{D} \rightarrow (\mathbb{P}^1(\mathbb{C}), \Delta)$ is the orbifold morphism, it follows from Proposition 2.1 that $g_l : \mathbb{D} \rightarrow (\mathbb{P}^1(\mathbb{C}), \Delta_l)$ is the orbifold morphism or $g_l(\mathbb{D}) \subset \text{supp}(\Delta_l)$.

If g_1 and g_2 are non-constant orbifold morphisms, Theorem 3.1 (i) leads to a contradiction to one of the hypothesis (1.5), (1.6) and (1.7) of the Main Theorem. Thus, $g_1(\mathbb{D}) \subset \text{supp}(\Delta_1)$ or $g_2(\mathbb{D}) \subset \text{supp}(\Delta_2)$. Since $\text{supp}(\Delta_l)$ ($l = 1, 2$) are discrete, it follows that g_1 or g_2 is constant, says $g_2 \equiv \text{constant}$. Since g is non-constant, g_1 is non-constant either. Then, the conditions (1.5) and (1.7) yield

$$\sum_{j=1}^{q_1} \left(1 - \frac{1}{m_{1j}}\right) > 3.$$

So, Theorem 3.1 (ii) implies that g_1 must be constant. This is a contradiction. The proof of the Main Theorem is complete. □

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