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UNIFORM IN TIME CONVERGENCE OF A TAMED-ADAPTIVE EULER-MARUYAMA SCHEME FOR SDES WITH MARKOVIAN SWITCHING, UNDER NON-LIPSCHITZ CONDITIONS

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Abstract. In this paper, we introduce a tamed-adaptive Euler-Maruyama approximation scheme for stochastic differential equations with Markovian switching. We show that the scheme converges in L^1-norm when applying for SDEs with locally Lipschitz continuous drift and locally Hölder continuous diffusion. The rates of convergence are obtained for both finite and infinite time interval approximations.
Keywords: tamed-adaptive Euler-Maruyama, Stochastic differential equations, one-sided Lipschitz drift, locally Hölder continuous diffusion, uniform in time approximation.

1. Introduction

In this paper, we focus on the numerical approximation for the process $(X_t)_{t\geq 0}$, which is a solution to the stochastic differential equation (SDE) with Markovian switching

$$X_t = x_0 + \int_0^t b(\theta_s, X_s)ds + \int_0^t \sigma(\theta_s, X_s)dW_s, \tag{1.1}$$

where $x_0 \in \mathbb{R}$ is a fixed initial value, $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion, $(\theta_t)_{t\geq 0}$ is a Markov chain on a finite state space $S = \{1, 2, \cdots, N\}$. Here, the processes W and θ are defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W and θ, augmented by all the null sets in $\mathcal{F}$ so that it satisfies the usual conditions. The Markovian switching $(\theta_t)_{t\geq 0}$

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has the transition matrix that can be written as

$$\mathbb{P}[\theta_{t+u} = j|\theta_t = i] = \begin{cases} \gamma_{ij}u + o(u) & \text{if } i \neq j \\ 1 + \gamma_{ii}u + o(u) & \text{if } i = j \end{cases}, \qquad \sum_{j \in S} \gamma_{ij} = 0,$$

and

$$\begin{cases} \lim\limits_{\Delta t \to 0} \dfrac{\mathbb{P}[\theta_{t+\Delta t} = j|\theta_t = i]}{\Delta t} = \gamma_{ij} & \text{if } i \neq j \\ \lim\limits_{\Delta t \to 0} \dfrac{\mathbb{P}[\theta_{t+\Delta t} \neq i|\theta_t = i]}{\Delta t} = -\gamma_{ii} & \text{if } i = j \end{cases}.$$

One important note is that, throughout this paper, we only consider the case where the Markovian switching $\{\theta_t\}_{t\geq 0}$ is independent of the Brownian motion $\{W_t\}_{t\geq 0}$.

Regarding the motivation of this topic, SDEs have become versatile tools for modeling practical systems. More specifically, SDEs with Markovian switching are particularly suitable for those systems possessing unexpected alterations in their construction and parameters caused by different reasons such as component disruption, changing subsystem relations, or sudden environmental interruption. For examples of such regime switching, see, e.g., [1, 2, 3] and references therein.

Since the randomness of the Markovian switching makes the equations even harder to solve, numerical methods are more needed in dealing with this kind of equation. Up until now, there have been many works that have been done on SDEs with Markovian switching, both with regular conditions and irregular conditions on coefficients. In 2004, X. Mao and C. Yuan ([3]) wrote the very first papers on numerical methods for SDEs with Markovian switching under the global Lipschitz condition and more general Lipschitz condition. In this paper, Mao and Yuan have shown the convergence of Euler-Maruyama solutions to the exact solution in L^2, and also the rate of convergence is 1/2. After that, there have been several studies on numerical methods for many classes of SDEs with Markovian switching (see [4, 5, 6]) and references therein. In [7, 8, 9], the authors consider higher-order approximation schemes for Markovian switching SDEs. However, most of the works that have been done so far mainly considered the numerical simulation of SDEs with Markovian switching on finite time interval $[0, T]$, which leads to the fact that the bound for the error of the estimation solutions still depends on T. This might become a problem when we want to simulate the SDEs with Markovian switching on a larger time interval. For SDEs without switching, W. Fang and B. Giles introduced in [10] an adaptive version of the classical Euler-Maruyama scheme for SDEs whose drift coefficient is not globally Lipschitz and studied the convergence of that adaptive scheme on an infinite time interval.

Having received inspiration from the work [11], in this paper, we introduce a tamed-adaptive Euler-Maruyama scheme for SDEs with Markovian switching to handle the case of locally Lipschitz continuous drift, locally Hölder continuous diffusion, and both coefficients possess polynomial growth rate, in both finite and infinite intervals.

The remaining part of this paper will be organized as follows. In Section 2., we will list down the conditions that are needed, the tamed-adaptive Euler-Maruyama scheme in use, and all the main results of our work. All the proofs of the results can be found in Section 3.

2. Model assumptions and Main results

2.1. Assumptions

We consider the following conditions on the drift and diffusion coefficients.

(A1) There exist $\gamma \in \mathbb{R}, \eta \in [0,+\infty)$ and $p_0 \in [2,+\infty)$ such that
$xb(i,x) + \frac{p_0-1}{2}|\sigma(i,x)|^2 \leq \gamma|x|^2 + \eta$, for any $x \in \mathbb{R}, i \in S$.

(A2) There exists a constant L_1 such that
$(x-y)(b(i,x)-b(i,y)) \leq L_1|x-y|^2$, for any $x,y \in \mathbb{R}, i \in S$.

(A3) There exist positive constant l and L_2 such that
$|b(i,x)-b(i,y)| \leq L_2(1+|x|^l+|y|^l)|x-y|$, for any $x,y \in \mathbb{R}, i \in S$.

(A4) There exist positive constants m, L_3 and $\alpha \in \left[0,\frac{1}{2}\right]$ such that
$|\sigma(i,x)-\sigma(i,y)| \leq L_3(1+|x|^m+|y|^m)|x-y|^{\alpha+\frac{1}{2}}$, for any $x,y \in \mathbb{R}, i \in S$.

From [12], we know that there exists a sequence $\{\tau_k\}_{k\geq 0}$ of stopping time such that $0=\tau_0<\tau_1<\ldots<\tau_k\to\infty$ and θ_t is constant on every interval $[\tau_k,\tau_{k+1})$, i.e. for every $k\geq 0$, $\theta_t=\theta_{\tau_k}$ for every $t\in[\tau_k,\tau_{k+1})$.
It may then follow from Theorem 2.1 in [13] that if Assumptions **(A1)**, **(A3)** and **(A4)** hold for $p_0 \geq 4l+4$, then Equation (1.1) has a unique strong solution.

2.2. Tamed-adaptive Euler-Maruyama scheme

For each $\Delta \in (0,1)$, we define the tamed-adaptive Euler-Maruyama scheme of Equation 1.1 as follows

$$\begin{cases} t_0 = 0, \widehat{X}_0 = x_0, t_{k+1} = t_k + h_\Delta\left(\widehat{X}_{t_k}\right), \\ \widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + b\left(\theta_{t_k}, \widehat{X}_{t_k}\right)(t_{k+1}-t_k) + \sigma_\Delta\left(\theta_{t_k}, \widehat{X}_{t_k}\right)\left(W_{t_{k+1}} - W_{t_k}\right), \end{cases} \tag{2.1}$$

where

$$h_\Delta(x) = \frac{\Delta}{\left(1+\sum_{i\in S}|b(i,x)|+\sum_{i\in S}|\sigma(i,x)|+|x|^l\right)^2}, \tag{2.2}$$

for some function σ_Δ defined later.

The following is proof of well-defindedness of the Euler-Maruyama scheme (2.1).

Suppose that, for the coefficients b, σ, and σ_Δ defined as above, there exist positive constants L and β satisfying the following conditions

(T1) $|b(i,x)| \vee |\sigma(i,x)| \leq L\left(1+|x|^\beta\right)$;

(T2) $x\left(b(i,x)-b(i,0)\right) \leq L|x|^2$;

(T3) $|\sigma_\Delta(i,x)| \leq L|\sigma(i,x)|$;

(T4) $|\sigma_\Delta(i,x)| \leq \frac{L}{\sqrt{\Delta}}$.

for any $x \in \mathbb{R}$ and $i \in S$. Then

$$\lim_{k\to+\infty} t_k = +\infty.$$

Under the assumptions of Proposition 2.2., we consider for each $t > 0$ a stopping time $\underline{t} := \max\{t_n : t_n \leq t\}$, and define

$$\hat{X}_t = \hat{X}_{\underline{t}} + b\left(\theta_{\underline{t}}, \hat{X}_{\underline{t}}\right)(t-\underline{t}) + \sigma_\Delta\left(\theta_{\underline{t}}, \hat{X}_{\underline{t}}\right)\left(W_t - W_{\underline{t}}\right). \tag{2.3}$$

Therefore, $\hat{X}_t$ is the solution of the SDE

$$d\hat{X}_t = b\left(\theta_{\underline{t}}, \widehat{X}_{\underline{t}}\right)dt + \sigma_\Delta\left(\theta_{\underline{t}}, \widehat{X}_{\underline{t}}\right)dW_t, \quad \hat{X}_0 = x_0.$$

Remark 2.1. *We can easily verify that if the Assumptions* **(A1)**-**(A4)** *are satisfied, then the following function*

$$\sigma_\Delta(i,x) = \frac{\sigma(i,x)}{1+\Delta^{\frac{1}{2}}|\sigma(i,x)|}$$

satisfies all conditions of Proposition 2.2..

In the following, we consider the L^p-norm of the exact and approximate solutions. Assume that Condition **(A1)** is satisfied and σ is bounded on every compact subset of $\mathbb{R}$. Then, for any $p \in [0, p_0]$,

$$\mathbb{E}\left[|X_t|^p\right] \leq \begin{cases} \left|x_0^2.e^{2\gamma t} + \dfrac{\eta}{\gamma}\left(e^{2\gamma t}-1\right)\right|^{\frac{p}{2}} & \text{if } \gamma \neq 0 \\ |x_0^2 + 2\eta t|^{\frac{p}{2}} & \text{if } \gamma = 0 \end{cases}.$$

We will omit the proof of Proposition 2.2. since it can be followed from the argument in the proof of Proposition 2.3 in [11].

If Conditions **(T1)-(T4)** are satisfied, and there exists $\gamma \in \mathbb{R}, \eta \in [0, \infty)$ and $p_0 \in [2, \infty)$ such that for all $x \in \mathbb{R}, i \in S$ and $t \geq 0$, it holds

$$xb(i,x) + \frac{p_0 - 1}{2}\sigma_\Delta^2(i,x) \leq \gamma |x|^2 + \eta. \tag{2.4}$$

Then, for an positive integer $k \leq \frac{p_0}{2}$, there exists a positive constant $C = C(x_0, k, \eta, \gamma, L)$ which depends neither on t nor Δ such that

$$\mathbb{E}\left[|\widehat{X}_t|^{2k}\right] \vee \mathbb{E}\left[|\widehat{X}_{\underline{t}}|^{2k}\right] \leq \begin{cases} Ce^{2k\gamma t} & \text{if } \gamma > 0 \\ C(1+t)^k & \text{if } \gamma = 0 \\ C & \text{if } \gamma < 0 \end{cases}. \tag{2.5}$$

The next Theorem about the convergence rate of the Euler-Maruyama scheme will be the most contribution of this paper. Let Assumptions **(A1)-(A4)** hold and $p_0 \leq 2l \vee (2 + 4\alpha + 4m)$. Suppose that functions b, σ and σ_Δ satisfy all conditions of Theorem 2.2., and

$$|\sigma(i,x) - \sigma_\Delta(i,x)| \leq L_4 \Delta^{1/2} |\sigma(i,x)|^2, \tag{2.6}$$

for $i \in S$ and $L_4 > 0$.
Then, for any $T > 0$, there exists a positive constant $C_T = C_T(x_0, L, L_1, L_2, L_3, L_4, \gamma, \eta, T)$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[\left|\widehat{X}_t - X_t\right|\right] \leq \begin{cases} C_T \Delta^\alpha & \text{if } 0 < \alpha < \frac{1}{2}, \\ \dfrac{C_T}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{2.7}$$

Moreover, if $L_1 < 0$ and $\gamma < 0$, then there exists a positive constant $C = C(x_0, L, L_1, L_2, L_3, L_4, \gamma, \eta)$, which does not depend in T such that

$$\sup_{t \geq 0} \mathbb{E}\left[\left|\widehat{X}_t - X_t\right|\right] \leq \begin{cases} C \Delta^\alpha & \text{if } 0 < \alpha < \frac{1}{2}, \\ \dfrac{C}{\log \frac{1}{\Delta}} & \text{if } \alpha = 0. \end{cases} \tag{2.8}$$

3. Proofs of main results

The following result, a consequence of the strong Markov property of W, is needed throughout the proofs in this paper. We have that

$$\mathbb{E}\left[(W_s - W_{\underline{s}})^k \mid \mathcal{F}_{\underline{s}}\right] = \begin{cases} 0 & \text{if } k \text{ is an odd integer,} \\ \kappa(k)(s - \underline{s})^{k/2} & \text{if } k \text{ is an even integer,} \end{cases} \tag{3.1}$$

for some positive constant $\kappa(k)$ depending on k.

3.1. Yamada and Watanabe approximation

In this paper, we apply the approximation technique of Yamada and Watanabe to handle the super-linear growing rate of the diffusion function. The function $\phi_{\delta\varepsilon}$ in this paper is defined exactly as that in [11].
$\phi_{\delta\varepsilon}$ has the following useful properties for any $x \in \mathbb{R}$,:

(YW1) $\phi'_{\delta\varepsilon}(x) = \frac{x}{|x|}\phi'_{\delta\varepsilon}(|x|)$,

(YW2) $0 \leq |\phi'_{\delta\varepsilon}(x)| \leq 1$,

(YW3) $|x| \leq \varepsilon + \phi_{\delta\varepsilon}(x)$,

(YW4) $\frac{\phi_{\delta\varepsilon}(|x|)}{|x|} \leq \frac{\delta}{\varepsilon}$,

(YW5) $\phi''_{\delta\varepsilon}(|x|) = \psi_{\delta\varepsilon}(|x|) \leq \frac{2}{|x|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|x|) \leq \frac{2\delta}{\varepsilon\log\delta}$.

3.2. Proof of Proposition 2.2.

For each $H > 0$, let

$$\begin{cases} t_0^H = 0, \quad t_{k+1}^H = t_k^H + h_\Delta^H\left(\hat{X}_{t_k^H}^H\right) \\ \hat{X}_{t_{k+1}^H}^H = \hat{X}_{t_k^H}^H + b_H\left(\theta_{t_k^H}, \hat{X}_{t_k^H}^H\right)h_\Delta^H\left(\hat{X}_{t_k^H}^H\right) + \sigma_\Delta\left(\theta_{t_k^H}, \hat{X}_{t_k^H}^H\right)\left(W_{t_{k+1}^H} - W_{t_k^H}\right) \end{cases},$$

where $h_\Delta^H(x) = \begin{cases} h_\Delta(x) & \text{if } |x|^\beta \leq H \\ \frac{\Delta}{L(1+H)} & \text{if } |x|^\beta > H \end{cases}$ and $b_H(i,x) = \begin{cases} b(i,x) & \text{if } |x|^\beta \leq H \\ \frac{x}{1+|x|^2} & \text{if } |x|^\beta > H \end{cases}$.

It is straightforward to verify that b_H and h_Δ^H satisfy the following properties. Assume that Conditions **(T1) - (T4)** hold. Then there exists a positive constant L so that for all Δ, the functions b, h_Δ, b_H and h_Δ^H satisfy the following inequalities

(P1) $|b_H(i,x)|h_\Delta(x) \leq L\Delta$ and $|b_H(i,x)|^2 h_\Delta(x) \leq L\Delta$,

(P2) $x(b_H(i,x) - b(i,0)) \leq L|x|^2$,

(P3) $h_\Delta^H(x) \geq \frac{\Delta}{L(1+H)}$,

for any $x \in \mathbb{R}$ and $i \in S$.

Proof of Proposition 2.2.. Throughout this proof, we will denote by C a constant, whose value may change from one line to another. However, the value of C does not depend on H. Applying Itô's formula for $\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right)$ gives

$$\phi_{\delta\varepsilon}\left(\hat{X}_t^H\right) = \phi_{\delta\varepsilon}(x_0) + \int_0^t \Big[\phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right) + \frac{1}{2}\phi''_{\delta\varepsilon}\left(\hat{X}_s^H\right)\left|\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right)\right|^2\Big]ds + \int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right)dW_s.$$

By **(YW5)**, we have

$$\frac{1}{2}\phi''_{\delta\varepsilon}\left(\hat{X}_s^H\right)\left|\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right)\right|^2 \le \frac{1}{2}\frac{2\delta}{\varepsilon\log\delta} = \frac{\delta}{\varepsilon\log\delta}.$$

Also,

$$\phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right) = \left(\phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right) - \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right)\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right) + \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right).$$

We shall evaluate the terms $\phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right)$ and $\left(\phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right) - \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right)\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right)$.

First, by **(YW1)**, we have

$$\begin{aligned}\phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right) &= \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right)\left(b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right) - b_H\left(\theta_{\underline{s}^H}, 0\right)\right) \\ &\quad + \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right) b_H\left(\theta_{\underline{s}^H}, 0\right) \\ &\le \frac{\phi'_{\delta\varepsilon}\left(|\hat{X}_{\underline{s}^H}^H|\right)}{|\hat{X}_{\underline{s}^H}^H|} L|\hat{X}_{\underline{s}^H}^H|^2 + \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right) b_H\left(\theta_{\underline{s}^H}, 0\right) \le L|\hat{X}_{\underline{s}^H}^H| + \sum_{i=1}^N |b(i,0)|.\end{aligned}$$

Next, we apply the Taylor's expansion for $\phi'_{\delta\varepsilon}$ to have that there exists an $(\mathcal{F}_s)$-adapted process $\xi = (\xi_s)$ such that

$$\phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right) = \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right) + \phi''_{\delta\varepsilon}(\xi_s)\left(\hat{X}_s^H - \hat{X}_{\underline{s}^H}^H\right).$$

This implies

$$\left(\phi'_{\delta\varepsilon}\left(\hat{X}_s^H\right) - \phi'_{\delta\varepsilon}\left(\hat{X}_{\underline{s}^H}^H\right)\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right) = \phi''_{\delta\varepsilon}(\xi_s)\left(\hat{X}_s^H - \hat{X}_{\underline{s}^H}^H\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}_{\underline{s}^H}^H\right).$$

But we have

$$\hat{X}_t^H = \hat{X}_{\underline{t}^H}^H + b_H\left(\theta_{\underline{t}^H}, \hat{X}_{\underline{t}^H}^H\right)\left(t - \underline{t}^H\right) + \sigma_\Delta\left(\theta_{\underline{t}^H}, \hat{X}_{\underline{t}^H}^H\right)\left(W_t - W_{\underline{t}^H}\right).$$

Thus,

$$\begin{aligned}
&\phi''_{\delta\varepsilon}(\xi_s)\left(\hat{X}^H_s - \hat{X}^H_{\underline{s}^H}\right) b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\\
&= \phi''_{\delta\varepsilon}(\xi_s)\,|b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)|^2\left(s - \underline{s}^H\right)\\
&\quad + \phi''_{\delta\varepsilon}(\xi_s)\, b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\left(W_s - W_{\underline{s}^H}\right)\\
&\le \frac{2\delta}{\varepsilon\log\delta}L + \frac{2\delta}{\varepsilon\log\delta}|b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\left(W_s - W_{\underline{s}^H}\right)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\hat{X}^H_t| &\le \varepsilon + \phi_{\delta\varepsilon}(x_0)\\
&\quad + \int_0^t \left[\frac{2\delta}{\varepsilon\log\delta}L + \frac{2\delta}{\varepsilon\log\delta}\left|b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\left(W_s - W_{\underline{s}^H}\right)\right|\right.\\
&\quad \left. + L|\hat{X}^H_{\underline{s}^H}| + \sum_{i=1}^N |b(i,0)|\right] ds + \int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}^H_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right) dW_s.
\end{aligned}$$

Moreover, thanks to (3.1), **(P1)** and Condition **(T4)**, we have

$$\begin{aligned}
&\mathbb{E}\Big[|b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\left(W_s - W_{\underline{s}^H}\right)|\Big]\\
= \;&\mathbb{E}\Big[|b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)|\mathbb{E}|W_s - W_{\underline{s}^H}| \mid \mathcal{F}_{\underline{s}^H}\Big]\\
\le \;&\mathbb{E}\Big[|b_H\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)\left(s - \underline{s}^H\right)^{\frac{1}{2}}\Big] \le C.
\end{aligned}$$

Therefore, we have

$$|\hat{X}^H_t| \le C + L\int_0^t |\hat{X}^H_{\underline{s}^H}|ds + \int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right) dW_s. \tag{3.2}$$

Take supremum of both sides for t from 0 to T, we get

$$\sup_{0\le t\le T}|\hat{X}^H_t| \le C + L\sup_{0\le t\le T}\int_0^t |\hat{X}^H_{\underline{s}^H}|ds + \sup_{0\le t\le T}\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right) dW_s.$$

It follows that

$$\begin{aligned}
\sup_{0\le s\le t}|\hat{X}^H_s| &\le C + L\int_0^t \sup_{0\le u\le s}|\hat{X}^H_u|ds\\
&\quad + \sup_{0\le t\le T}\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right) dW_s.
\end{aligned} \tag{3.3}$$

Taking expectations both sides of (3.3) gives

$$\mathbb{E}\Big[\sup_{0\le s\le t}|\hat{X}^H_s|\Big] \le C + L\int_0^t \mathbb{E}\Big[\sup_{0\le u\le s}|\hat{X}^H_u|\Big]ds \\ + \mathbb{E}\Big[\sup_{0\le t\le T}\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H},\hat{X}^H_{\underline{s}^H}\right)dW_s\Big]. \tag{3.4}$$

By Burkholder-Davis-Gundy Inequality, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H},\hat{X}^H_{\underline{s}^H}\right)dW_s\Big|\Big] \le C. \tag{3.5}$$

Hence, combining (3.4), (3.5) and using Gronwall's inequality gives

$$\mathbb{E}\Big[\sup_{0\le s\le t}|\hat{X}^H_s|\Big] \le C + L\int_0^t \mathbb{E}\Big[\sup_{0\le u\le s}|\hat{X}_{u^H}|\Big]ds \le Ce^{Lt} \le C.$$

Now, for any $T > 0$

$$\mathbb{P}[t_k < T] = \mathbb{P}\Big[t_k < T, \sup_{0\le s\le T}|\hat{X}^H_s| > H^{\frac{1}{\beta}}\Big] + \mathbb{P}\Big[t_k < T, \sup_{0\le s\le T}|\hat{X}^H_s| \le H^{\frac{1}{\beta}}\Big].$$

Through Markov's inequality, we have

$$\mathbb{P}\Big[t_k < T, \sup_{0\le s\le T}|\hat{X}^H_s| > H^{\frac{1}{\beta}}\Big] \le \frac{1}{H^{\frac{1}{\beta}}}\mathbb{E}\left[\sup_{0\le s\le T}|\hat{X}^H_s|\right] \le \frac{C}{H^{\frac{1}{\beta}}}.$$

On the other hand, on the set $\{\sup_{0\le s\le T}|\hat{X}^H_s| \le H^{\frac{1}{\beta}}\}$, $\hat{X}^H_s = \hat{X}_s$ for all $s \le T$, which means $t^H_k = t_k$ if $t_k < T$.

By **(P3)**, we have that $t^H_{k+1} - t^H_k = h^H_\Delta \ge \dfrac{\Delta}{L(1+H)}$, we may then follows $t^H_k \ge \dfrac{k\Delta}{L(1+H)}$ which tends to infinity as k goes to infinity.

Thus,

$$\limsup_{k\to\infty}\mathbb{P}\left[t_k < T, \sup_{0\le s\le T}\left|\hat{X}^H_s\right| \le H^{\frac{1}{\beta}}\right] \le \limsup_{k\to\infty}\mathbb{P}\left[t^H_k < T\right] = 0,$$

which deduces that

$$\limsup_{k\to\infty}\mathbb{P}\left[t_k < T\right] \le \frac{1}{H^{\frac{1}{\beta}}}\mathbb{E}\left[\sup_{0\le s\le T}|\hat{X}^H_s|\right] \le \frac{C}{H^{\frac{1}{\beta}}},$$

for any $H > 0$. By letting $H \to \infty$, we get $\limsup_{k\to\infty}\mathbb{P}\left[t_k < T\right] = 0$ for any $T > 0$. This implies that $\lim_{k\to\infty} t_k = +\infty$ almost surely, and our proof is complete. □

3.3. Proof of Theorem 2.2.

To begin with, we will prove the following key estimation on moments of $\widehat{X}$. If the Conditions **(T1)-(T4)** are satisfied, then for any $p > 0$ and $T > 0$, there exists a positive constant $C(p, L, T, x_0, \Delta) < \infty$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\hat{X}_t\right|^p\right] \le C\left(p, L, T, x_0, \Delta\right).$$

Proof. From (3.2), we may deduce that for any $p > 1$

$$\begin{aligned}|\hat{X}_t^H|^p \le C(x_0, \delta, \varepsilon, L, T, \sum_{i=1}^{N}|b\left(i,0\right)|, \Delta, p) + L^p\left(\int_0^t |\hat{X}^H_{\underline{s}^H}|ds\right)^p \\ + \left(\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}^H}, \hat{X}^H_{\underline{s}^H}\right)dW_s\right)^p.\end{aligned} \tag{3.6}$$

Note that thanks to Hölder's Inequality, one can have

$$\left(\int_0^t \left|\hat{X}^H_{\underline{s}^H}\right| ds\right)^p \le \left(\int_0^t 1^{\frac{p}{p-1}}\, ds\right)\left(\int_0^t |\hat{X}^H_{\underline{s}^H}|^p ds\right) \le T^{p-1}\int_0^t |\hat{X}^H_{\underline{s}^H}|^p ds.$$

Now by using (3.6) and doing similar to the way we get (3.4), we can obtain the following

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|\hat{X}^H_s\right|^p\right] \le C\left(x_0, \delta, \varepsilon, L, T, \sum_{i=1}^{N}|b(i,0)|, \Delta, p\right) \\ + L^pT^{p-1}\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}\left|\hat{X}^H_u\right|^p\right]ds \\ + \mathbb{E}\left[\sup_{0\le t\le T}\left(\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}}{}^H, \hat{X}^H_{\underline{s}^H}\right)dW_s\right)^p\right].\end{aligned} \tag{3.7}$$

Applying Burkholder-Davis-Gundy Inequality, we get

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t \phi'_{\delta\varepsilon}\left(\hat{X}_s\right)\sigma_\Delta\left(\theta_{\underline{s}}{}^H, \hat{X}^H_{\underline{s}^H}\right)dW_s\right|^p\right] \le C(\Delta, T, L, p). \tag{3.8}$$

Combining (3.8), (3.7) and applying Gronwall's Inequality gives

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|\hat{X}^H_s\right|^p\right] \le C\left(x_0, \delta, \varepsilon, L, T, \sum_{i=1}^{N}|b(i,0)|, \Delta, p\right).$$

Therefore

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\hat{X}_t^H\right|^p\right]\le C\left(x_0,\delta,\varepsilon,L,T,\sum_{i=1}^N|b(i,0)|,\Delta,p\right). \tag{3.9}$$

Now by using Markov's Inequality and (3.9), we obtain that for any $T>0, i\in\{1,...,N\}$ and $H>0$,

$$\begin{aligned}\mathbb{P}\left[\sup_{0\le t\le T}\left|\hat{X}_t\right|\neq\sup_{0\le t\le T}\left|\hat{X}_t^H\right|\right]&\le\mathbb{P}\Big[\sup_{0\le t\le T}\left|\hat{X}_t^H\right|>H^{\frac{1}{\beta}}\Big]\\&\le\frac{C\left(x_0,\delta,\varepsilon,L,T,\sum_{i=1}^N|b(i,0)|,\Delta,\beta\right)}{H^2}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{H=1}^{\infty}\mathbb{P}\Big[\sup_{0\le t\le T}\left|\hat{X}_t\right|\neq\sup_{0\le t\le T}\left|\hat{X}_t^H\right|\Big]\\&\qquad\le\sum_{H=1}^{\infty}\frac{C\left(x_0,\delta,\varepsilon,L,T,\sum_{i=1}^N|b(i,0)|,\Delta,\beta\right)}{H^2}<\infty,\end{aligned}$$

by Borel-Cantelli Lemma, we may deduce that

$$\lim_{H\to\infty}\sup_{0\le t\le T}\left|\hat{X}_t^H\right|=\sup_{0\le t\le T}\left|\hat{X}_t\right| \text{ a.s.}$$

From (3.9), we apply Fatou's Lemma to get

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t|^p\Big]\le C\left(x_0,\delta,\varepsilon,L,T,\sum_{i=1}^N|b\left(i,0\right)|,\Delta,p\right).$$

□

Proof of Theorem 2.2.. By Hölder's inequality, the proof is reduced to showing that (2.5) holds for positive integer k and $k\le p_0/2$. Firstly, we will prove for the case $k=1$. Applying Itô's formula for $e^{-2\gamma t}\widehat{X}_t^2$, we have

$$\begin{aligned}e^{-2\gamma t}\widehat{X}_t^2=x_0^2&+\int_0^t\left\{e^{-2\gamma s}\left(-2\gamma\widehat{X}_{\underline{s}}^2+2\widehat{X}_{\underline{s}}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)+\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right)\right\}ds\\&+\int_0^t\left\{2e^{-2\gamma s}\widehat{X}_{\underline{s}}\sigma_\Delta\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right\}dW_s.\end{aligned} \tag{3.10}$$

On the other hand, it follows from (2.3) that

$$\widehat{X}_s^2=\widehat{X}_{\underline{s}}^2+2\widehat{X}_{\underline{s}}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(s-\underline{s})+b^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(s-\underline{s})^2$$

$$+ \sigma_\Delta^2 \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}})^2 + 2\widehat{X}_{\underline{s}} \sigma_\Delta \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}})$$
$$+ 2b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s}) \sigma_\Delta \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}}).$$

Thanks to **(T3)**, (3.1) and (2.2),

$$\max \left\{ \left| \widehat{X}_{\underline{s}} b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s}) \right| ; b^2 \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s})^2; \right.$$
$$\left. \mathbb{E} \left[\sigma_\Delta^2 \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}})^2 \mid \mathcal{F}_{\underline{s}} \right] \right\} \leq C\Delta. \tag{3.11}$$

Therefore,

$$\begin{aligned}
\mathbb{E} \left[-2\gamma \widehat{X}_s^2 \right] &= \mathbb{E} \left[-2\gamma \widehat{X}_{\underline{s}}^2 \right] + \mathbb{E} \left[-2\gamma b^2 \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s})^2 \right] \\
&\quad + \mathbb{E} \left[-2\gamma \sigma_\Delta^2 \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}})^2 \right] \\
&\quad + 2\mathbb{E} \left[-2\gamma \widehat{X}_{\underline{s}} b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s}) \right] \\
&\quad + 2\mathbb{E} \left[-2\gamma \widehat{X}_{\underline{s}} \sigma_\Delta \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}}) \right] \\
&\quad + 2\mathbb{E} \left[-2\gamma b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s}) \sigma_\Delta \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}}) \right] \\
&\leq \mathbb{E} \left[-2\gamma \widehat{X}_{\underline{s}}^2 \right] + C|\gamma|\Delta.
\end{aligned} \tag{3.12}$$

A similar argument yields

$$\begin{aligned}
\mathbb{E} \left[2\widehat{X}_s b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) \right] &= \mathbb{E} \left[2\widehat{X}_{\underline{s}} b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) \right] + \mathbb{E} \left[2b^2 \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (s - \underline{s}) \right] \\
&\quad + \mathbb{E} \left[2b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) \sigma_\Delta \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) (W_s - W_{\underline{s}}) \right] \\
&\leq \mathbb{E} \left[2\widehat{X}_{\underline{s}} b \left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right) \right] + C|\gamma|\Delta.
\end{aligned} \tag{3.13}$$

It then follows from (2.5), (3.10), (3.12), and (3.13) that

$$\mathbb{E} \left[e^{-2\gamma t} \widehat{X}_t^2 \right] \leq |x_0|^2 + C(\eta + \Delta + |\gamma|\Delta) \int_0^t e^{-2\gamma s} ds. \tag{3.14}$$

By Hölder's inequality and (3.11), we have

$$\begin{aligned}
\mathbb{E} \left[\left| \widehat{X}_{\underline{t}} \right|^p \right] &= \mathbb{E} \left[\left| \widehat{X}_t - b \left(\theta_{\underline{t}}, \widehat{X}_{\underline{t}}\right) (t - \underline{t}) - \sigma_\Delta \left(\theta_{\underline{t}}, \widehat{X}_{\underline{t}}\right) (W_t - W_{\underline{t}}) \right|^p \right] \\
&\leq 3^{p-1} \left[\mathbb{E} \left[|\widehat{X}_t|^p \right] + \mathbb{E} \left[|b(\theta_{\underline{t}}, \widehat{X}_{\underline{t}})(t - \underline{t})|^p \right] \right. \\
&\quad \left. + \mathbb{E} \left[|\sigma_\Delta(\theta_{\underline{t}}, \widehat{X}_{\underline{t}})(W_t - W_{\underline{t}})|^p \right] \right]
\end{aligned}$$

$$\leq 3^{p-1}\left[\mathbb{E}\left[|\widehat{X}_t|^p\right]+C\Delta^p+C\Delta^{p/2}\right]. \tag{3.15}$$

Therefore, it follows from (3.14) and (3.15) that (2.5) holds for the case $k=1$.

Secondly, assume that (2.5) holds for any $k \leq k_0 \leq [p_0/2]-1$, we shall try to point out that (2.5) still holds for $k=k_0+1$.
By applying Itô's formula for $e^{-p\gamma s}\left|\widehat{X}_t\right|^p$ with $p=2(k_0+1)$ being an even integer, we have

$$\begin{aligned} e^{-p\gamma s}\left|\widehat{X}_t\right|^p &= |x_0|^p+\int_0^t\left\{pe^{-p\gamma s}\Big[-\gamma\left|\widehat{X}_{\underline{s}}\right|^p+\widehat{X}_{\underline{s}}^{p-1}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right.\\ &\quad\left.+\frac{p-1}{2}\left|\widehat{X}_{\underline{s}}\right|^{p-2}\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\Big]\right\}ds\\ &\quad+\int_0^t\left\{pe^{-p\gamma s}\left|\widehat{X}_s\right|^{p-2}\widehat{X}_s\sigma_\Delta\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right\}dW_s. \end{aligned} \tag{3.16}$$

From (2.3), we apply the Newton expansion formula to see that for any positive integer q,

$$\begin{aligned} |\widehat{X}_s|^q &= \left|\widehat{X}_{\underline{s}}+b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(s-\underline{s})+\sigma_\Delta\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(W_s-W_{\underline{s}})\right|^q\\ &= \sum_{0\leq i,j,r\leq q, i+j+r=q}\frac{q!}{i!j!r!}\left|\widehat{X}_{\underline{s}}\right|^i\left|b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(s-\underline{s})\right|^j\\ &\qquad\times\left|\sigma_\Delta\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(W_s-W_{\underline{s}})\right|^r. \end{aligned} \tag{3.17}$$

Thanks to (3.1), we have

$$\begin{aligned} \mathbb{E}\left[-\gamma|\widehat{X}_s|^p|\mathcal{F}_{\underline{s}}\right] &= -\gamma\left(\widehat{X}_{\underline{s}}\right)^p+p\gamma\widehat{X}_{\underline{s}}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(s-\underline{s})\left|\widehat{X}_{\underline{s}}\right|^{p-2}\\ &\quad+\sum_{0\leq i\leq p-2, i+j+2r=p}\frac{-\gamma p!}{i!j!(2r)!}\left(\widehat{X}_{\underline{s}}\right)^i|b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)(s-\underline{s})|^j\\ &\qquad\times\left[\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)^{2r}\alpha_{2r}(s-\underline{s})^r\right]. \end{aligned}$$

Taking expectation both sides and using (3.11), we have

$$\mathbb{E}\left[-\gamma\left|\widehat{X}_s\right|^p\right]\leq\mathbb{E}\left[-\gamma\left|\widehat{X}_{\underline{s}}\right|^p\right]+C|\gamma|\sum_{i=0}^{p-2}E\left[\left|\widehat{X}_{\underline{s}}\right|^i\right]. \tag{3.18}$$

Similarly, choose $q=p-1$ and $q=p-2$ in (3.17), by the same argument, we also have

$$\mathbb{E}\left[\left|\widehat{X}_s\right|^{p-1}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right]\leq\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^{p-2}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)+C\sum_{i=0}^{p-2}\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^i\right]\right] \tag{3.19}$$

and

$$\mathbb{E}\left[\frac{p-1}{2}\left|\widehat{X}_s\right|^{p-2}\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right]$$
$$\leq \frac{p-1}{2}\left|\widehat{X}_{\underline{s}}\right|^{p-2}\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)+C\sum_{i=0}^{p-2}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^i\right]. \tag{3.20}$$

Combining (3.18), (3.19) and (3.20), we get

$$\begin{aligned}
&\mathbb{E}\left[-\gamma\left|\widehat{X}_s\right|^p+\widehat{X}_s^{p-1}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)+\frac{p-1}{2}\left|\widehat{X}_s\right|^{p-2}\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right]\\
&\leq \mathbb{E}\left[-\gamma\left|\widehat{X}_{\underline{s}}\right|^p+\left|\widehat{X}_{\underline{s}}\right|^{p-1}b\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)+\frac{p-1}{2}\left|\widehat{X}_{\underline{s}}\right|^{p-2}\sigma_\Delta^2\left(\theta_{\underline{s}},\widehat{X}_{\underline{s}}\right)\right]\\
&\quad+C\sum_{i=0}^{p-2}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^i\right]\\
&=\eta\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^{p-2}\right]+C\sum_{i=0}^{p-2}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^i\right]\leq C\sum_{i=0}^{p-2}\mathbb{E}\left[\left|\widehat{X}_{\underline{s}}\right|^i\right].
\end{aligned} \tag{3.21}$$

From (3.15), (3.16), (3.21) and the inductive assumption, we may conclude that (2.5) holds for $k=k_0+1$. From this, we use Hölder's inequality to complete our proof. □

3.4. Proof of Theorem 2.2.

First, we shall prove prove the following uniformly in a time bound for the difference between $\widehat{X}_t$ and $\widehat{X}_{\underline{t}}$. Assume that all the conditions of Theorem 2.2. are satisfied, then there exists a positive constant $C_p=C(p,L)$ such that

$$\sup_{t\geq 0}\mathbb{E}\left[\left|\widehat{X}_t-\widehat{X}_{\underline{t}}\right|^p\right]\leq C_p\Delta^{p/2},$$

for any $p\geq 0$.

The proof of this lemma is similar to the one of Lemma 3.2 in [11] and will be skipped.

Proof of Theorem 2.2.. Let $Y_t=\widehat{X}_t-X_t$. By condition **(YW3)**, we have

$$e^{-L_1t}|Y_t|\leq e^{-L_1t}\varepsilon+e^{-L_1t}\phi_{\delta\varepsilon}(Y_t).$$

Apply Itô's formula for $e^{-L_1t}\phi_{\delta\varepsilon}(Y_t)$, we obtain

$$e^{-L_1t}|Y_t|\leq e^{-L_1t}\varepsilon+\int_0^t\left\{e^{-L_1s}\left[-L_1\phi_{\delta\varepsilon}(Y_s)\right.\right.$$

$$+ \phi'_{\delta\varepsilon}(Y_s)\left(b\left(\theta_s, X_s\right) - b\left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right)\right)$$
$$+ \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma\left(\theta_s, X_s\right) - \sigma_\Delta\left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right)\right|^2\Bigg]\Bigg\}ds$$
$$+ \int_0^t \left\{e^{-L_1 s}\phi'_{\delta\varepsilon}(Y_s)\left(\sigma\left(\theta_s, X_s\right) - \sigma_\Delta\left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right)\right)\right\}dW_s. \tag{3.22}$$

Let

$$J_1(s) = \phi'_{\delta\varepsilon}(Y_s)\left(b\left(\theta_s, X_s\right) - b\left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right)\right)$$

and

$$J_2(s) = \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\left|\sigma\left(\theta_s, X_s\right) - \sigma_\Delta\left(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}\right)\right|^2.$$

We shall evaluate $J_1(s)$ and $J_2(s)$. From **(YW1)**, **(YW2)**, **(A2)**, **(A3)**, we obtain

$$\begin{aligned}
J_1(s) &= \phi'_{\delta\varepsilon}(Y_s)\left(b(\theta_s, X_x) - b(\theta_s, \widehat{X}_s)\right) + \phi'_{\delta\varepsilon}(Y_s)\left(b(\theta_s, \widehat{X}_s) - b(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right)\\
&\leq \frac{\phi'_{\delta\varepsilon}(|Y_s|)}{|Y_s|}Y_s\left(b(\theta_s, X_x) - b(\theta_s, \widehat{X}_s)\right) + \left|b(\theta_s, \widehat{X}_s) - b(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right|\\
&\leq L_1\frac{\phi'_{\delta\varepsilon}(|Y_s|)}{|Y_s|}|Y_s|^2 + \left|b(\theta_s, \widehat{X}_s) - b(\theta_s, \widehat{X}_{\underline{s}})\right| + \left|b(\theta_s, \widehat{X}_{\underline{s}}) - b(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right|\\
&\leq L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + L_2\left(1 + |\widehat{X}_s|^l + |\widehat{X}_{\underline{s}}|^l\right)|\widehat{X}_s - \widehat{X}_{\underline{s}}|\\
&\quad + \left(|b(\theta_s, \widehat{X}_{\underline{s}})| + |b(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})|\right)\mathbb{I}_{\{\theta_s \neq \theta_{\underline{s}}\}}\\
&\leq L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + L_2\frac{\left(1 + |\widehat{X}_s|^l + |\widehat{X}_{\underline{s}}|^l\right)^2\Delta^{\frac{1}{2}} + |\widehat{X}_s - \widehat{X}_{\underline{s}}|^2\Delta^{\frac{-1}{2}}}{2}\\
&\quad + C\left(1 + |\widehat{X}_{\underline{s}}|^{l+1}\right)\mathbb{I}_{\{\theta_s \neq \theta_{\underline{s}}\}}\\
&\leq L_1\phi'_{\delta\varepsilon}(|Y_s|)|Y_s| + C\left(1 + |\widehat{X}_{\underline{s}}|^{l+1}\right)\mathbb{I}_{\{\theta_s \neq \theta_{\underline{s}}\}}\\
&\quad + \frac{3}{2}L_2\Delta^{\frac{1}{2}}\left(1 + |\widehat{X}_s|^{2l} + |\widehat{X}_{\underline{s}}|^{2l}\right) + \frac{1}{2}L_2\Delta^{\frac{-1}{2}}|\widehat{X}_s - \widehat{X}_{\underline{s}}|^2.
\end{aligned} \tag{3.23}$$

Next, we will write

$$\begin{aligned}
J_2(s) &= \frac{1}{2}\phi''_{\delta\varepsilon}(Y_s)\\
&\times\left|\sigma(\theta_s, X_s) - \sigma(\theta_s, \widehat{X}_s) + \sigma(\theta_s, \widehat{X}_s) - \sigma(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}) + \sigma(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}) - \sigma_\Delta(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right|^2.
\end{aligned}$$

From **(YW5)**, **(A4)**, and (2.6), we have that

$$J_2(s) \leq \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\Bigg(\left|\sigma(\theta_s, X_s) - \sigma(\theta_s, \widehat{X}_s)\right|^2$$

$$
\begin{aligned}
&\quad + \left|\sigma(\theta_s, \widehat{X}_s) - \sigma(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right|^2 + \left|\sigma(\theta_{\underline{s}}, \widehat{X}_{\underline{s}}) - \sigma_\Delta(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right|^2 \Bigg) \\
&\le \frac{3}{|Y_s|\log\delta}\mathbb{I}_{[\frac{\varepsilon}{\delta};\varepsilon]}(|Y_s|)\Bigg[3L_3^2\left(1+|X_s|^{2m}+|\widehat{X}_s|^{2m}\right)|X_s-\widehat{X}_s|^{1+2\alpha} \\
&\quad + L_4^2\Delta\left|\sigma(\theta_{\underline{s}}, \widehat{X}_{\underline{s}})\right|^4 + 6L_3^2\left(1+|\widehat{X}_s|^{2m}+|\widehat{X}_{\underline{s}}|^{2m}\right)|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{1+2\alpha} \\
&\quad + C\left(1+|\widehat{X}_{\underline{s}}|^{2m+2\alpha+1}\right)\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}\Bigg] \\
&\le \frac{9L_3\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|\widehat{X}_s|^{2m}\right) + \frac{C\delta\Delta\left(1+|\widehat{X}_{\underline{s}}|^{4m+4\alpha+2}\right)}{\varepsilon\log\delta} \\
&\quad + \frac{3C\delta}{\varepsilon\log\delta}\left(1+|\widehat{X}_{\underline{s}}|^{2m+2\alpha+1}\right)\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}} \\
&\quad + \frac{9L_3^2\delta}{\varepsilon\log\delta}\left(1+|\widehat{X}_s|^{2m}+|\widehat{X}_{\underline{s}}|^{2m}\right)^2\Delta^{\frac{1}{2}+\alpha} + \frac{9L_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{2+4\alpha}\Delta^{\frac{-1}{2}-\alpha} \\
&\le \frac{9L_3\varepsilon^{2\alpha}}{\log\delta}\left(1+|X_s|^{2m}+|\widehat{X}_s|^{2m}\right) + \frac{C\delta\Delta\left(1+|\widehat{X}_{\underline{s}}|^{4m+4\alpha+2}\right)}{\varepsilon\log\delta} \\
&\quad + \frac{27L_3^2\delta}{\varepsilon\log\delta}\left(1+|\widehat{X}_s|^{4m}+|\widehat{X}_{\underline{s}}|^{4m}\right)\Delta^{\frac{1}{2}+\alpha} \\
&\quad + \frac{9L_3^2\delta}{\varepsilon\log\delta}|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{2+4\alpha}\Delta^{\frac{-1}{2}-\alpha} \\
&\quad + \frac{3C\delta}{\varepsilon\log\delta}\left(1+|\widehat{X}_{\underline{s}}|^{2m+2\alpha+1}\right)\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}.
\end{aligned}
\tag{3.24}
$$

Combining (3.22), (3.23) and (3.24), take expectation both sides and apply the property $-L_1\phi_{\delta\varepsilon}(x) + L_1\phi'_{\delta\varepsilon}(|x|)|x| \le \max\{L_1\varepsilon; 0\}$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[e^{-L_1 t}|Y_t|\right] \\
&\le e^{-L_1 t}\varepsilon + \int_0^t e^{-L_1 s}\Bigg[\max\{L_1\varepsilon;0\} + \frac{3}{2}L_2\Delta^{\frac{1}{2}}\left(1+\mathbb{E}\left[|\widehat{X}_s|^{2l}\right]+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{2l}\right]\right) \\
&\quad + \frac{1}{2}L_2\Delta^{\frac{-1}{2}}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^2\right] + C\mathbb{E}\left[\left(1+|\widehat{X}_{\underline{s}}|^{l+1}\right)\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}\right] \\
&\quad + \frac{9L_3\varepsilon^{2\alpha}}{\log\delta}\left(1+\mathbb{E}\left[|X_s|^{2m}\right]+\mathbb{E}\left[|\widehat{X}_s|^{2m}\right]\right) + \frac{C\delta\Delta\left(1+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{4m+4\alpha+2}\right]\right)}{\varepsilon\log\delta} \\
&\quad + \frac{27L_3^2\delta}{\varepsilon\log\delta}\left(1+\mathbb{E}\left[|\widehat{X}_s|^{4m}\right]+\mathbb{E}\left[|\widehat{X}_{\underline{s}}|^{4m}\right]\right)\Delta^{\frac{1}{2}+\alpha} \\
&\quad + \frac{9L_3^2\delta}{\varepsilon\log\delta}\mathbb{E}\left[|\widehat{X}_s-\widehat{X}_{\underline{s}}|^{2+4\alpha}\right]\Delta^{\frac{-1}{2}-\alpha}
\end{aligned}
$$

$$+\frac{3C\delta}{\varepsilon\log\delta}\mathbb{E}\left[\left(1+|\widehat{X}_{\underline{s}}|^{2m+2\alpha+1}\right)\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}\right]\Bigg]ds. \tag{3.25}$$

Here, we give a key estimation to deal with the indicator function $\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}$.

$$\begin{aligned}\mathbb{E}\left[\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}|\mathcal{F}_{\underline{s}}\right]&=\mathbb{E}\left[\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}|\theta_{\underline{s}}\right]=\sum_{i\in S}\mathbb{I}_{\{\theta_{\underline{s}}=i\}}\mathbb{P}\left[\theta_s\neq i|\theta_{\underline{s}}=i\right]\\&=\sum_{i\in S}\mathbb{I}_{\{\theta_{\underline{s}}=i\}}\sum_{j\neq i}\mathbb{P}\left[\theta_s=j|\theta_{\underline{s}}=i\right]\\&=\sum_{i\in S}\mathbb{I}_{\{\theta_{\underline{s}}=i\}}\sum_{j\neq i}\left(\gamma_{ij}(s-\underline{s})+o(s-\underline{s})\right)\\&\leq C\left(\max_{i\in S}(-\gamma_{ii})(s-\underline{s})+o(s-\underline{s})\right)\sum_{i\in S}\mathbb{I}_{\{\theta_{\underline{s}}=i\}}\leq C\Delta.\end{aligned}$$

Thus,

$$\mathbb{E}\left[\left(1+|\widehat{X}_{\underline{s}}|^{l+1}\right)\mathbb{I}_{\{\theta_s\neq\theta_{\underline{s}}\}}\right]\leq C\Delta. \tag{3.26}$$

Thanks to Assumption $p_0\geq 2l\vee(2+4m+4\alpha)$, Theorem 2.2., Proposition 2.2., Lemma 3.4., and estimation (3.26), there exists a positive constant $C_T=C_T(x_0,L,L_1,L_2,L_3,L_4,\gamma,\eta,T)$ such that for any $t\in[0,T]$, it holds that

$$\begin{aligned}\mathbb{E}\left[e^{-L_1t}|Y_t|\right]&\leq e^{-L_1t}\varepsilon\\&+C_T\left[\varepsilon+\Delta^{\frac{1}{2}}+\Delta+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta\Delta^{\frac{1}{2}+\alpha}}{\varepsilon\log\delta}+\frac{\delta\Delta}{\varepsilon\log\delta}\right]\int_0^t e^{-L_1s}ds.\end{aligned} \tag{3.27}$$

Multiply both sides of (3.27) by e^{L_1t}, we obtain that

$$\mathbb{E}\left[|Y_t|\right]\leq C_T\left[\varepsilon+\Delta^{\frac{1}{2}}+\Delta+\frac{\varepsilon^{2\alpha}}{\log\delta}+\frac{\delta\Delta^{\frac{1}{2}+\alpha}}{\varepsilon\log\delta}+\frac{\delta\Delta}{\varepsilon\log\delta}\right]\left(\frac{e^{L_1t}-1}{L_1}\right).$$

- If $\alpha\in\left(0,\frac{1}{2}\right]$, we shall choose $\varepsilon=\Delta^{\frac{1}{2}},\delta=2$ and obtain that

$$\begin{aligned}&\mathbb{E}\left[|\widehat{X}_t-X_t|\right]\\&\leq C_T\left[\Delta^{\frac{1}{2}}+\Delta^{\frac{1}{2}}+\Delta+\frac{\Delta^{\alpha}}{\log 2}+\frac{2\Delta^{\frac{1}{2}+\alpha}}{\Delta^{\frac{1}{2}}\log 2}+\frac{2\Delta}{\Delta^{\frac{1}{2}}\log 2}\right]\left(\frac{e^{L_1t}-1}{L_1}\right).\end{aligned}$$

  Then

$$\sup_{t\in[0,T]}\mathbb{E}\left[|\widehat{X}_t-X_t|\right]\leq C_T\Delta^{\alpha}.$$

- If $\alpha = 0$, we shall choose $\varepsilon = \Delta^{\frac{1}{4}}, \delta = \Delta^{\frac{-1}{4}}$ and obtain that

$$\mathbb{E}\left[|\widehat{X}_t - X_t|\right] \le C_T\left[\Delta^{\frac{1}{4}} + \Delta^{\frac{1}{2}} + \Delta + \frac{1}{\log\left(\frac{1}{\Delta^{\frac{1}{4}}}\right)} + \frac{\Delta^{\frac{-1}{4}}\Delta^{\frac{1}{2}}}{\Delta^{\frac{1}{4}}\log\left(\frac{1}{\Delta^{\frac{1}{4}}}\right)} + \frac{\Delta^{\frac{-1}{4}}\Delta}{\Delta^{\frac{1}{4}}\log\left(\frac{1}{\Delta^{\frac{1}{4}}}\right)}\right]\left(\frac{e^{L_1 t} - 1}{L_1}\right).$$

Then

$$\sup_{t\in[0,T]} \mathbb{E}\left[|\widehat{X}_t - X_t|\right] \le \frac{C_T}{\log\left(\frac{1}{\Delta}\right)}.$$

From the proof above, we obtain (2.7). If $L_1 < 0$ and $\gamma < 0$, the constant C_T in (3.27) can be chosen such that it is independent of T. Hence, we obtain (2.8), and Theorem 2.2. is completely proved. □

4. Simulation

We consider the SDE

$$dX_s = b(\theta_s, X_s)ds + \sigma(\theta_s, X_s)dW_s,$$

where the Markovian switching has two states $\{1; 2\}$, and the generator is $\begin{pmatrix} -0.1 & 0.1 \\ 0.2 & -0.2 \end{pmatrix}$.

To simulate the convergence rate of the Euler-Maruyama scheme, we borrow the idea from the Multilevel Monte Carlo method and consider

$$me(l) = \frac{1}{K}\sum_{k=1}^{K}\left|\widehat{X}_1^{(l,k)} - \widehat{X}_1^{(l+1,k)}\right|,$$

where for each $l \ge 2$, $\{\widehat{X}^{(l,k)}\}_{1\le k\le K}$ is a sequence of independence copies of $\widehat{X}^{(l)}$ defined by (2.1) and (2.2) with $\Delta = 2^{-l}$. Also, $\{\widehat{X}^{(l+1,k)}\}_{1\le k\le K}$ are defined similarly, but with $\Delta = 2^{-l-1}$. One important note is that for each k and l, $\widehat{X}_1^{(l,k)}$ and $\widehat{X}_1^{(l+1,k)}$ are generated on the same Brownian motion. For this, we use Algorithm 1 in [10].

Moreover, since the equation of interest is SDE with Markovian switching, we also recall the algorithm to construct the Markovian switching $(\theta_t)_{t\ge 0}$. First, we will start with $\theta_0 = 1$. Then for each time point $t \ge 0$, we take another time point $s < t$ such that $t - s = u$. Assume that θ_s is at state 1. From this, we generate a uniform $[0, 1]$

pseudo-random number V, then if $0.1u < V$, θ_t will switch to state 2. Otherwise, θ_t will stay at state 1.

We consider the following cases for functions b and σ.

Case 1:

$$\begin{cases} b(1,x) = -1 + x - x^3 \\ b(2,x) = -1 + x - x^3 \end{cases}, \text{ and } \begin{cases} \sigma(1,x) = 1 + (1+x)x^{2/3} \\ \sigma(2,x) = 1 + \sqrt{\dfrac{x^4 + x^{4/3}}{14}} \end{cases}.$$

For the functions b and σ defined as above, we can choose $p_0 = 15, L_1 = 1, \gamma = 13/3, \eta = 18075, l = 2, m = 2$, and $\alpha = 1/6$.

Case 2:

$$\begin{cases} b(1,x) = -1 + x - x^3 \\ b(2,x) = -1 - x - x^{7/3} \end{cases}, \text{ and } \begin{cases} \sigma(1,x) = 1 + \sqrt{\dfrac{x^4 + x^{4/3}}{14}} \\ \sigma(2,x) = 1 + \sqrt{\dfrac{2x^2 + x^{10/3} + x^{4/3}}{14}} \end{cases}.$$

For Case 2, the constants can be chosen as $p_0 = 15, L_1 = 1, \gamma = 13/3, \eta = 5/6, l = 2, m = 2$, and $\alpha = 1/6$.

Case 3:

$$\begin{cases} b(1,x) = -1 + x - x^3 \\ b(2,x) = -1 - x - x^{7/3} \end{cases}, \text{ and } \begin{cases} \sigma(1,x) = 1 + (1+x)x^{2/3} \\ \sigma(2,x) = 1 + \sqrt{\dfrac{2x^2 + x^{10/3} + x^{4/3}}{14}} \end{cases}.$$

In this case, we choose $p_0 = 15, L_1 = 1, \gamma = 13/3, \eta = 18075, l = 2, m = 4/3$, and $\alpha = 1/6$.

Case 4:

$$\begin{cases} b(1,x) = -1 - x - x^{7/3} \\ b(2,x) = -1 - 9x - x^{11/3} \end{cases}, \text{ and } \begin{cases} \sigma(1,x) = 1 + \sqrt{\dfrac{x^{10/3} + x^{4/3}}{14}} \\ \sigma(2,x) = 1 + x^{2/3} + x \end{cases}.$$

Similarly, we can choose the constants as $L_1 = -1, \gamma = -1/6, \eta = 90, l = 4/3, m = 1$, and $\alpha = 1/6$.

Verifying that the two functions b and σ satisfy the Conditions **(A1)** - **(A4)** in all cases with the constants listed above is straightforward. Since $p_0 \geq 2l \vee (2 + 4m + 4\alpha)$, from Theorem 2.2. above, the tamed-adaptive Euler-Maruyama scheme should converge in L_1-norm at the rate of order α in any finite time interval. Moreover, in Case 4, the functions b and σ are chosen such that $L_1 < 0$ and $\gamma < 0$. Therefore, the scheme will converge in L_1-norm at the rate of order α in infinite time interval.

We draw the regression line to estimate the empirical rate of convergence in all four cases. In all the cases, the empirical rates are better than the theoretical rate, which assures our results in this paper. One important note is that in Case 4, we run the simulation algorithm in a much longer time interval, but the convergence result is still the same. This is also strong support for the result in Theorem 2.2..

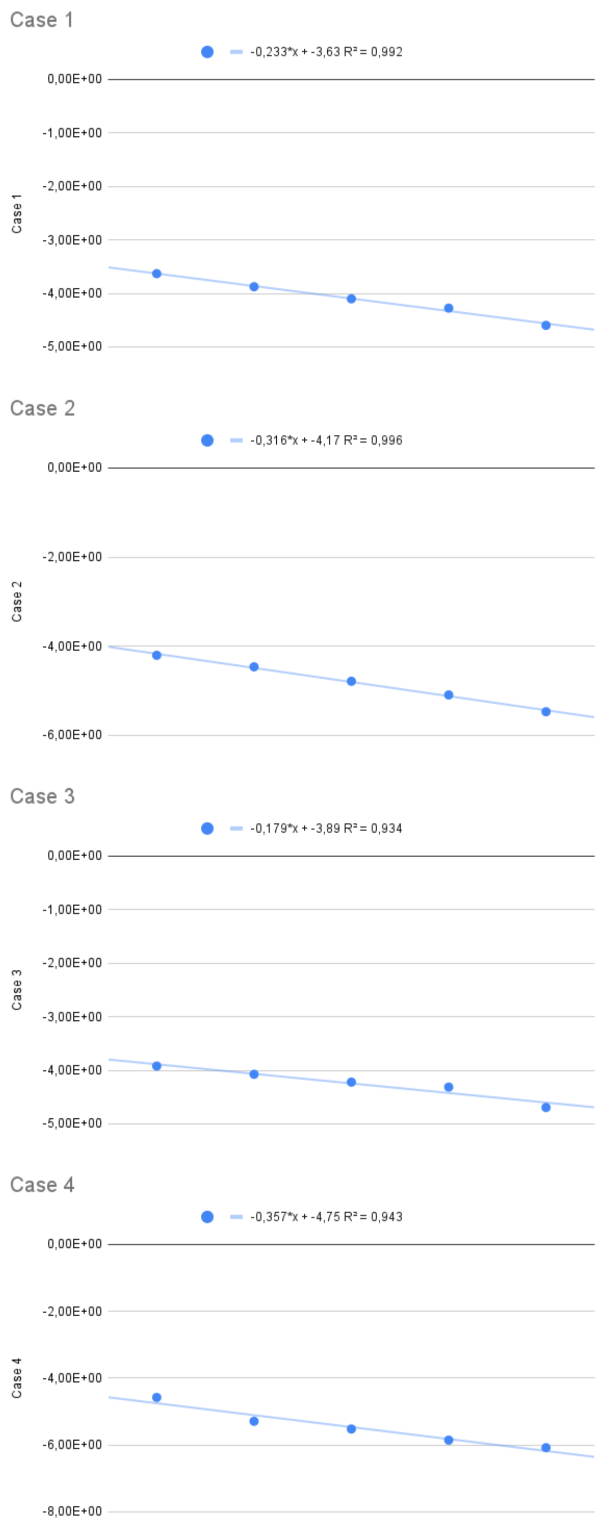

Figure 1. Simulation results of $\log_2 me(l)$ ***for*** $l = 2, ..., 6$ ***(cases 1-4)***

5. Conclusions

In this paper, inspired by the work [11], we develop the tamed-adaptive Euler-Maruyama scheme to construct an approximate solution to SDE with Markovian switching. Under Lipschitz conditions on the drift function and $(\alpha + 1/2)$-Hölder continuous diffusion, we pointed out in this paper that the proposed Euler-Maruyama scheme will converge in L^1-norm to the exact solution with a rate of α.

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