STUDYING AN EFFICIENT SECOND ORDER ACCURATE SCHEME FOR SOLVING TWO-DIMENSIONAL SHALLOW FLOW MODEL

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Abstract: *The aim of this paper is to present an efficient high order accuracy numerical scheme for conservation law on structure grids. The Monotone Upstream Centered Scheme for Conservation Laws (MUSCL) procedure renders the model to preserve the well-balanced property and achieve high accuracy and efficiency for solving nonlinear two dimensional shallow water equations (2D-SWE). The effectiveness and robustness of the above scheme is shown by comparison the solution obtained by aforementioned scheme with those obtained by first order one or 1D result through three tests: 2D Riemann problem; circular dam break and run-up wave over conical island. Then, it is applied to simulate dam break flow over adverse slope which has experiment data. The Nash values are approximated 90%.*

Keywords: Finite Volume Method, 2D-SWE, second order accuracy.

1. INTRODUCTION

Two dimensional (2D) shallow water model based on hydrostatic pressure assumption has been used to simulate a wide range of surface environmental flow including dam break flow; urban flooding; tidal, tsunami hazards, etc. These applications may involve numerical calculation of very complex flow hydrodynamics such as shock-type flow discontinuities, wetting and drying over uneven bed. A robust numerical scheme is required in order to produce accurate and stable numerical solutions for these applications. Finite Volume Method (FVM), Godunov type, nowadays, is considered the most applied numerical strategy to solve 2D SWE. For most of the application, first order finite volume schemes may give rise to unacceptable numerical diffusion and hence poor numerical solution, especially for flows containing discontinuities, e.g. tsunami and dam break waves. It is therefore necessary to develop high order scheme to predict more accurately the shallow flows. The technique MUSCL for conservation law has been widely accepted and applied in solving the SWEs within the framework of finite volume Godunov-type schemes. It is able to reduce numerical diffusion without causing unphysical result. Hence, in this paper, FVM are used to solve 2D SWE on structured mesh; Harten-Lax-van Leer-Contact (HLLC) approximate Riemann solver is invoked to evaluate inter-cell fluxes and MUSCL procedure is employed to obtain high resolution.

Two well-known tests, namely, 2D Riemann problem and circular dam break are reproduced with both first order and second order accuracy schemes to indicate the effectiveness of the presented numerical scheme. And then, the sudden dam collapse flow over adverse slope example is taken to show the efficiency of the proposed scheme in handling wetting and drying problem.

2. NUMERICAL MODEL

The conservation form of 2D SWE based on pre-balance method can be written as:

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}+\frac{\partial \mathbf{G}(\mathbf{U})}{\partial y}=\mathbf{S}(\mathbf{U}) \qquad (1)$$

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Where:

$$\mathbf{U}=\begin{bmatrix}\eta\\ hu\\ hv\end{bmatrix};\ \mathbf{F}(\mathbf{U})=\begin{bmatrix}hu\\ hu^2+0.5g(\eta^2-2\eta z_b)\\ huv\end{bmatrix};$$

$$\mathbf{G}(\mathbf{U})=\begin{bmatrix}hv\\ huv\\ hv^2+0.5g(\eta^2-2\eta z_b)\end{bmatrix};$$

$$\mathbf{S}(\mathbf{U})=\begin{bmatrix}0\\ -g\eta\partial z_b/\partial x-ghS_{fx}\\ -g\eta\partial z_b/\partial y-ghS_{fy}\end{bmatrix};$$

$$S_{fx}=\frac{n^2u\sqrt{u^2+v^2}}{h^{4/3}};\ S_{fy}=\frac{n^2v\sqrt{u^2+v^2}}{h^{4/3}}$$

U is the vector of conserved variables; **F** and **G** are flux vectors and **S** is source term accounting for bed slope term and friction term; η, h and z_b are water elevation, water depth and bottom elevation, respectively; u, v are velocity components along x- and y- directions; S_{fx}, S_{fy} are friction slopes along the same directions; n is Manning roughness coefficient; g is gravity acceleration.

Based on Godunov type scheme, the flow variables are updated to a new time step by using the following equation:

$$\mathbf{U}_{i,j}^{n+1}=\mathbf{U}_{i,j}^{n}-\frac{\Delta t}{\Delta x}\left[\mathbf{F}_{i+1/2,j}-\mathbf{F}_{i-1/2,j}\right]-\frac{\Delta t}{\Delta y}\left[\mathbf{G}_{i,j+1/2}-\mathbf{G}_{i,j-1/2}\right]+\Delta t\mathbf{S}_{i,j} \quad (2)$$

where superscripts denote time levels; subscripts i and j are space indices along x- and y- directions; Δt, Δx, Δy are time step and space sizes of the computational cell.

The above formulation of the SWEs balances the flux and source term gradients by considering pressure force balancing (Liang, 2010), so it directly satisfy the C-property when the domain is fully wetted.

Interface fluxes $\mathbf{F}_{i\pm1/2,j}$ and $\mathbf{G}_{i,j\pm1/2}$ are approximated by HLLC scheme. For example:

$$\mathbf{F}_{i+1/2}=\begin{cases}\mathbf{F}_L & \text{if } s_1\ge 0,\\ \mathbf{F}_{*L} & \text{if } s_1<0\le s_2,\\ \mathbf{F}_{*R} & \text{if } s_2<0\le s_3,\\ \mathbf{F}_R & \text{if } s_3\le 0,\end{cases} \quad (3)$$

where, $\mathbf{U}_L$ and $\mathbf{U}_R$ are the left and the right states of Riemann problem, respectively; $\mathbf{F}_L=\mathbf{F}(\mathbf{U}_L)$ and $\mathbf{F}_R=\mathbf{F}(\mathbf{U}_R)$; s_1, s_2 and s_3 are estimates of the speeds of the left, contact and right waves, respectively. The middle region fluxes $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$ are the numerical fluxes in the left and the right sides of the middle region of the Riemann solution which is divided by a contact wave.

Flux vector $\mathbf{F}_*$ in the middle region that is evaluated by the following equation:

$$\mathbf{F}_*=\frac{s_3\mathbf{F}(\mathbf{U}_L)-s_1\mathbf{F}(\mathbf{U}_R)+s_1s_3(\mathbf{U}_R-\mathbf{U}_L)}{s_3-s_1} \quad (4)$$

where s_1, s_2 and s_3 are estimates of the speeds of the left, contact and right waves, respectively.

$$s_1=\begin{cases}\min\left(u_L-\sqrt{gh_L};\ u_*-\sqrt{gh_*}\right) & \text{if } h_L>0,\\ u_R-2\sqrt{gh_R} & \text{if } h_L=0,\end{cases}$$

$$s_3=\begin{cases}\max\left(u_R+\sqrt{gh_R};\ u_*+\sqrt{gh_*}\right) & \text{if } h_R>0,\\ u_L+2\sqrt{gh_L} & \text{if } h_R=0,\end{cases}$$

$$;\ s_2=\frac{s_1h_R(u_R-s_3)-s_3h_L(u_L-s_1)}{h_R(u_R-s_3)-h_L(u_L-s_1)} \quad (5)$$

u_L, u_R, h_L, h_R are the components of the left and the right initial Riemann states for a local Riemann problem, and h_* and u_* are the Roe average quantities, Le (2014).

In order to achieve second order accuracy in time and space, the MUSCL-Hancock procedure is employed. Among several slop limiters ensure the Total Variation Diminishing (TVD) property to avoid nonphysical oscillation, such as: VanLeer; VanAlbada; Minmod; Superbee, Minmod limiter is selected in this paper thanks to the effectiveness in eliminating overshoot at cell interface. The selected numerical model is written by Fortran90 and validated with several test cases (Le, 2014).

Every explicit FVM must satisfy a necessary condition which guarantees the stability and the convergence to the exact solution as the grid is refined. The stability condition is governed by the Courant–Fredrichs–Lewy (CFL) criterion, controlling the time step Δt at each time level. For Cartesian grids, CFL stability condition is given by:

$$\Delta t = Cr\left[\max\left(\frac{|\tilde{u}| + \sqrt{g\tilde{h}}}{\Delta x} + \frac{|\tilde{v}| + \sqrt{g\tilde{h}}}{\Delta y}\right)\right]^{-1} \quad (6)$$

3. RESULTS AND DISCUSSION

3.1. Circular dam break.

A cylindrical tank of 20m in diameter is located in the center of the 50m×50m domain with four open boundaries. The tank and the remaining domain are initially filled with 2m and 0,5m of still water, respectively. The tank wall is assumed to be removed instantaneously to produce a 2D circular dam break wave. This process is simulated herein to test the automatic shock-capturing capability of the current model. Fig.1 shows the 3D view of the computed water level at t=1,0s and t=2,5s on the 62,500 cells of computational domain.

Again simulations are carried out using the current model with both second and first order accuracy comparison with 1D scheme obtained by Canestrelli et al, 2009 and Hou et al, 2015 solution. Fig. 2 plots the corresponding water levels along the radial direction of y=0,0m at t=1,0s and t=2,5s. It is apparent that the second order scheme produces more accurately numerical solution than the first order one. The new 2D results agree satisfactorily with the 1D reference solution, demonstrating the capability of the model in resolving 2D shocks.

A quantitative comparison between the first and the second order schemes is carried out by calculating Nash value with reference of 1D solution. The Nash-Sutcliffe model efficiency coefficient (E) is used to quantitatively describe the accuracy of model outputs for water level at two times t=1,0s and t=2,5s by equation (7):

$$E = 1 - \frac{\sum_{i=1}^{n}\left(X_{1D,i} - X_{2Di}\right)^2}{\sum_{i=1}^{n}\left(X_{1D,i} - \overline{X_{1D}}\right)^2} \quad (7)$$

where X_{1D} is water level value along radial section computed by Canestrelli et al, 2009 and X_{2D} is value calculated by the presented 2D model. Subscript i indicates the location of cells in a haft of radial section.

Numerical diffusion can still be observed for the present schemes near the shocks as the solution accuracy is locally switched to become first order to preserve monotonicity. The shocks can be captured more precisely by refining the grid as shown in Fig. 2 where the grid size is only 0,1m.

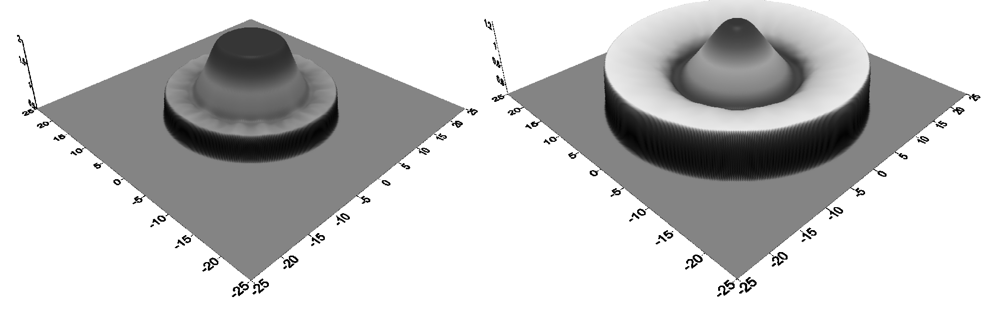


Fig. 1. 3D view of water level computed by second order scheme at t=1,0s and t=2,5s

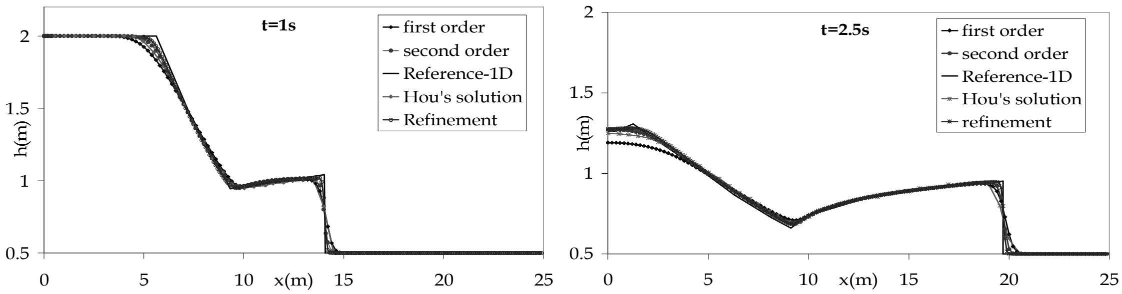


Fig. 2: Sectional view of water level at t=1s and t=2,5s

3.2. 2D Riemann problem.

This test is solved on frictionless, structured mesh of [0,200]m×[0,200]m. The initial condition including water depth and velocity components is indicated in Table 1. The grid size is 1,0m, generating to 40000 cells of computational domain. Two numerical methods: first order accuracy and second order accuracy applied to this problem is carried out the effectiveness of high order accurate in space and time.

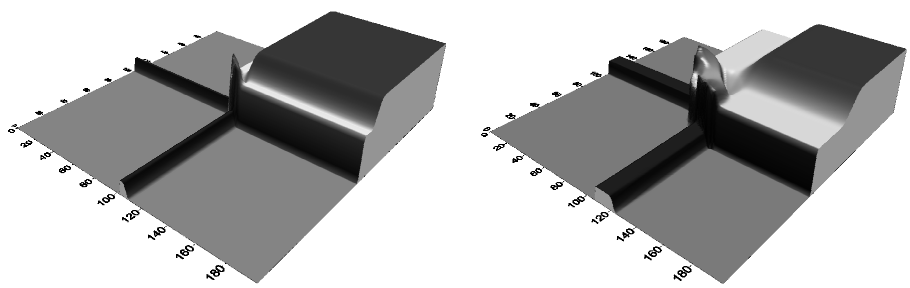


Fig. 3: Propagation of shock wave fronts at 1s and 3s obtained by 2nd order scheme

Table 1. Initial condition of 2D Riemann problem

Region	Coordinates (m)	h(m)	u(m/s)	v(m/s)
1	x≤100, y≤100	1,0	10,0	10,0
2	x>100, y≤100	1,0	0,0	10,0
3	x≤100, y>100	1,0	10,0	0,0
4	x>100, y>100	10,0	0,0	0,0

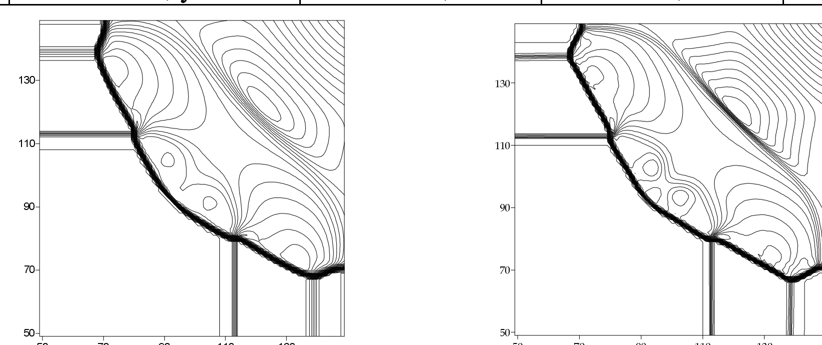


Fig. 4: Propagation of shock wave fronts at 5s obtained by: 1st order and 2nd order schemes

Equidistance of contour line is 0,25m.

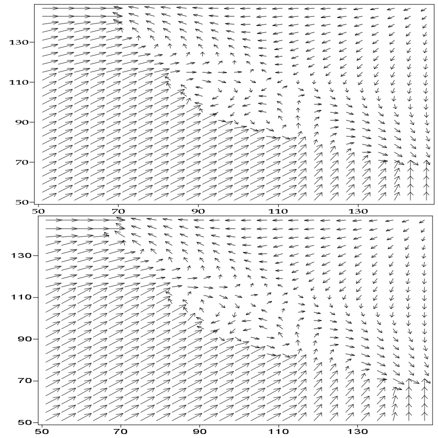


Fig. 5: Velocity maps of Fig.4

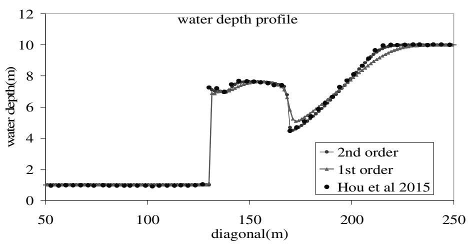


Fig. 6: Water depth profile across diagonal section at t=5s

Fig 3 illustrated the propagation of waves computed by the MUSCL scheme. The shock wave fronts are well captured by both numerical schemes, as seen in Figure 4. The vector fields of the flow velocities are compared respectively in the Fig 5. Meanwhile, Fig. 6 plots the predicted water depth profile across a diagonal section through two points (0; 0); (200; 200).

These figures show the computed results of the MUSCL scheme are less diffusive and perform slightly better in capturing steeper rarefaction waves than those obtained from the first order accuracy method. Rarefaction waves are likely to be dampened by low-order schemes. This result is also consistent with those reported in Hou et al, (2015) (see Fig. 6).

3.3. Run-up of a solitary wave on a conical island

This test illustrated the effectiveness of the presented model when comparing the numerical solution obtained first and high order schemes in simulating the solitary wave over a conical island. The domain and initial conditions are indicated clearly in Hou et al (2013).

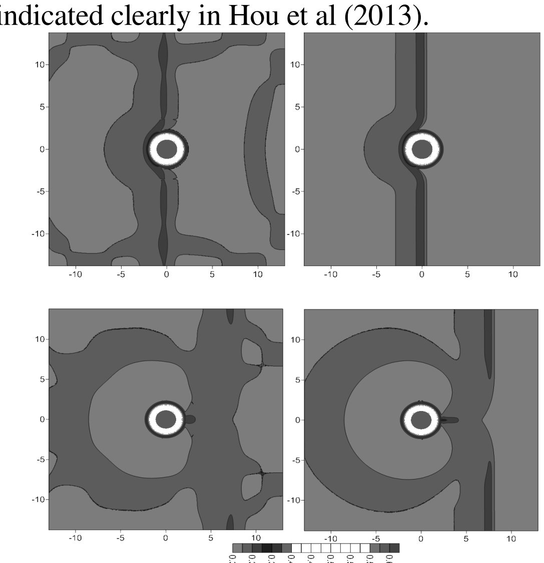


Fig. 7: Contour maps of solitary wave at t = 9s; 13s. Equidistance of contour line is 0,02m.

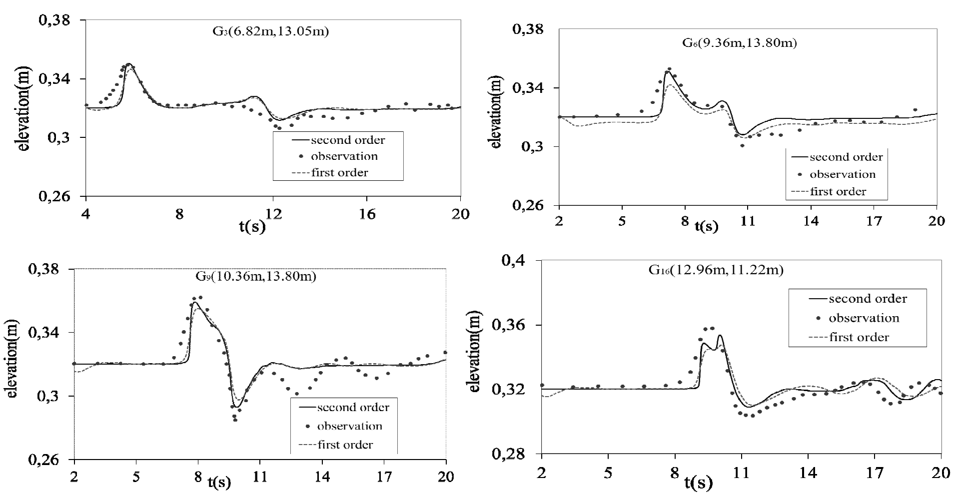


Fig.8 : Water hydrographs at different gauges

Fig. 8 shows water hydrograph at different gauges. Obviously, the biggest displacement of the peak run-up wave is seen, for instant at G_6, G_9 and G_{16}. Besides, the oscillation of first order results are much stronger than the second one according to the Fig. 7.

3.4. Flow over adverse slope.

This test was carried out by Aureli et al. (2000). The channel is prismatic, rectangular with 1,0m wide, 0,5m high and 7,0m long (see Fig. 9).

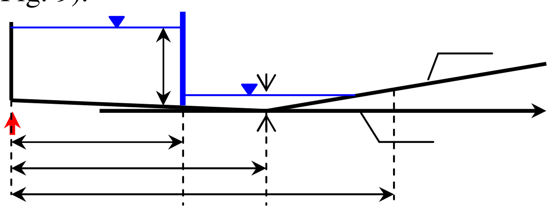
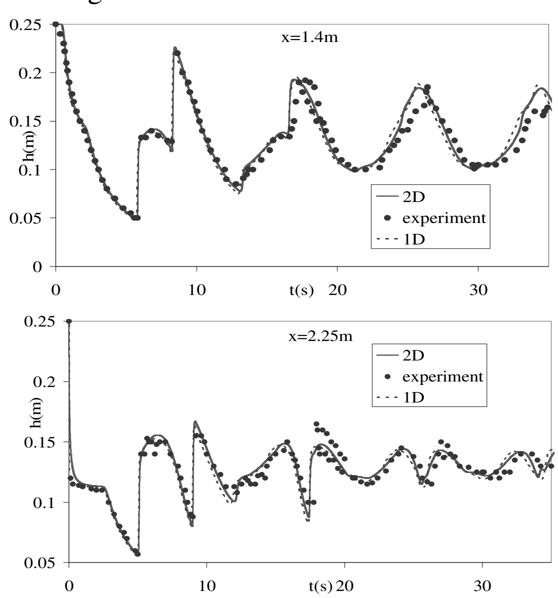


Fig. 9. Dam-break flow over adverse slope.

Manning coefficient was set to 0,01. The instantaneous dam failure was simulated by means of the sudden removal of a gate. Test case taken from this paper is: S_{01} = 0,0%; S_{02} = -10,0%. Initial water depth h_1 =0,25m; h_0 = 0.

Both 1D and 2D numerical solutions obtained by high order accurate are compared with empirical one. Water hydrograph at x=4,5m is regularly interrupted several times because of advancing and receding motion of flooding front.



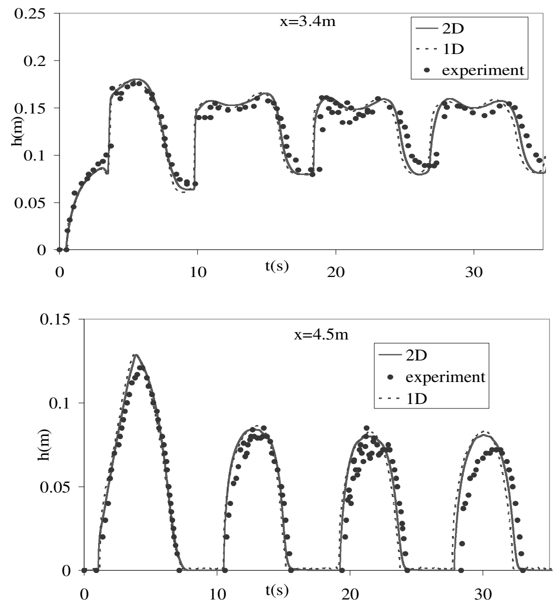


Fig. 10: Water hydrographs at: x = 1,4m; 2,25m; 3,4m and x = 4,5m.

Excellent agreement between numerical and experimental hydrographs for both schemes can be observed in Fig. 10 with Nash value at four gauges are 93,4%; 89,3%; 90,1% and 87,6%, respectively.

5. CONCLUSIONS

In this paper presents an application of high order numerical scheme FVM is used to solve 2D SWE on structured mesh. HLLC approximate Riemann solver is applied to solve flux terms. Second order accuracy is obtained by MUSCL procedure. The use of a finite volume Godunov-type scheme provides the model with automatic shock-capturing capability based on three test cases: Riemann problem, cylinder dam break, and solitary wave over conical island. The higher accuracy for general shallow flow solutions, and offers a better well-balanced property indicated by Nash values when compared with solution of first order accuracy. Besides, with experiment test of flood flow over adverse bed slope, very close agreement between numerical prediction and empirical data are observed in all 4 studied points and showing high values of Nash-Suffice (around 90%).

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Tóm tắt:

NGHIÊN CỨU TÍNH HIỆU QUẢ CỦA MỘT MÔ HÌNH TOÁN CÓ ĐỘ CHÍNH XÁC BẬC HAI GIẢI HỆ NƯỚC NÔNG HAI CHIỀU

Trong bài báo này, phương pháp thể tích hữu hạn được sử dụng để giải hệ phương trình nước nông hai chiều dạng bảo toàn trên hệ lưới có cấu trúc. Qui trình MUSCL được dùng để đảm bảo tính bảo toàn và có được kết quả chính xác bậc hai khi giải hệ phương trình nước nông phi tuyến hai chiều (2D-SWE). Tính hiệu quả của phương pháp này được đánh giá thông qua việc so sánh kết quả tính theo độ chính xác bậc hai với độ chính xác bậc nhất hay kết quả của bài toán 1 chiều thông qua 3 ví dụ: vỡ đập hình trụ tròn, bài toán Reimann và sóng lan truyền qua hình nón cụt. Sau đó tính hiệu quả của phương pháp cũng được kiểm tra thông qua ví dụ dòng chảy do vỡ đập trên kênh có độ dốc ngược. Chỉ số Nash khi so sánh kết quả của phương pháp số với số liệu thực đo đạt tới hơn 90%.

Từ khóa: Thể tích hữu hạn, 2D-SWE, độ chính xác bậc hai.

Ngày nhận bài: 11/12/2017
Ngày chấp nhận đăng: 08/3/2018