LEVEL SET EVOLUTION WITH SPEED DEPENDING ON MEAN CURVATURE: EXISTENCE OF A WEAK SOLUTION

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1. INTRODUCTION

Let Γ_0 be a smooth hypersurface which is, say, the smooth connected boundary of a bounded open subset *U* of R^n, $n \geq 2$. As time progresses we allow the surface to evolve by moving each point at a velocity equals to $(n-1)$ times the mean curvature vector plus some function *F* at that point. Assuming this evolution is smooth, we define thereby for each $t > 0$ a new hypersurface Γ_t. The primary problem is then to study geometric properties of $\{\Gamma_t\}_{t>0}$ in terms of Γ_0. We will proceed as follows: We select some continuous function $u_0 : R^n \to R$ so that its level set is Γ_0, that is

$$\Gamma_0 = \{x \in R^n \mid u_0(x) = 0\}.$$

Consider the following problem

$$u_t = \left(d_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2} \right) u_{x_i x_j} - F(x)|\nabla u| \qquad \text{in} \quad R^n \times (0, \infty), \tag{1.1}$$

with initial condition

$$u = u_0 \qquad \text{on } R^n \times \{t = 0\}. \tag{1.2}$$

Now the PDE (1.1) says that each level set of *u* evolves according to its mean curvature with forcing term *F*, at least in regions where *u* is smooth and its spatial gradient ∇u does not vanish. Similarly, we then define

$$\Gamma_t := \{x \in R^n \mid u(x,t) = 0\} \tag{1.3}$$

for each time $t > 0$.

We will show that there is a weak solution of equation (1.1) satisfying condition (1.2) in the weak sense.

2.DEFINITION AND ELEMENTARY PROPERTIES OF WEAK SOLUTIONS

In this section we concern with the definition and some properties of weak solutions of mean curvature evolution PDE (1.1). For this suppose temporarily that $u = u(x,t)$ is a smooth function whose spatial gradient $\nabla u := (u_{x_1}, \ldots, u_{x_n})$ does not vanish in some open region Ω of $R^n \times (0, \infty)$. Assume further that each level set

$$\Gamma_t = \{x \in R^n \mid u(x,t) = 0\} \quad (t \geq 0) \tag{2.1}$$

of u smoothly evolves according to its mean curvature and function F, as described in Section I.

Let $u = u(x,t)$ be a smooth unit normal vector field to $\{\Gamma_t\}_{t\geq 0}$ in Ω, and F=F(x) be a continuously differentiable function on R^n. Then

$$-\frac{1}{n-1}div(u)u$$

is the mean curvature vector field. Thus, if we fix $t \geq 0,\ x \in \Gamma_t \cap \Omega,$ the point x evolves according to the differential equation

$$\begin{cases} \dot{x} = -[div(u)u](x(s),s) + F(x(s))u(x(s),s) \\ x(t) = x. \end{cases} \tag{2.2}$$

These equations say that each level set Γ_t of u evolves along normal vector direction with velocity equal to its mean curvature plus function F. As $x(s) \in \Gamma_s\ (s \geq t)$, we have $u(x(s),s) = 0$, and so

$$0 = \frac{d}{ds}u(x(s),s) = -[(\nabla u \cdot u)div(u)](x(s),s) + F(x(s))\nabla u(x(s),s) \cdot u(x(s),s) + u_t(x(s),s).$$

Setting $s = t$, we discover

$$u_t(x,t) = (\nabla u(x,t) \cdot u(x,t))div(u)(x,t) - F(x)(\nabla u(x,t) \cdot u(x,t)).$$

Choosing $u := \frac{\nabla u}{|\nabla u|}$ it follows that

$$u_t = |\nabla u| div\left(\frac{\nabla u}{|\nabla u|}\right) - F|\nabla u| = \left(d_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)u_{x_i x_j} - F|\nabla u| \quad \text{at } (x,t). \tag{2.3}$$

2.1. Weak solutions

We consider now the level set evolution equation

$$u_t = \left(d_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)u_{x_i x_j} - F|\nabla u| \quad \text{in } R^n \times (0,\infty). \tag{2.4}$$

with initial condition

$$u = u_0 \quad \text{on } R^n \times \{t = 0\}. \tag{2.5}$$

Definition 2.1. A function $u \in C(R^n \times (0,\infty))$ is a weak subsolution of (2.4) provided that if $u - j$ has local maximum at point $(x_0,t_0) \in R^n \times (0,\infty)$ for each $j \in C^\infty(R^{n+1})$, then

$$\begin{cases} j_t \leq \left(d_{ij} - \frac{j_{x_i} j_{x_j}}{|\nabla j|^2}\right) j_{x_i x_j} - F|\nabla j| \quad \text{at } (x_0,t_0) \\ \text{if } \nabla j\,(\mathrm{x}_0,t_0) \neq 0, \end{cases}$$

and

$$\begin{cases} j_t \le \left(d_{ij} - h_i h_j\right) j_{x_i x_j} \quad \text{at } (x_0, t_0) \\ \text{for some } h \in \mathrm{R}^n \text{ with } |h| \le 1, \text{ if } \nabla j\,(\mathrm{x}_0, t_0) = 0. \end{cases}$$

Definition 2.2: A function $u \in C(R^n \times (0,\infty))$ is a weak supersolution of (2.4) provided that if

$u - j$ has local minimum at point $(x_0, t_0) \in R^n \times (0,\infty)$ for each $j \in C^\infty(R^{n+1})$, then

$$\begin{cases} j_t \ge \left(d_{ij} - \dfrac{j_{x_i} j_{x_j}}{|\nabla j|^2}\right) j_{x_i x_j} - F|\nabla j| \quad \text{at } (x_0, t_0) \\ \text{if } \nabla j\,(\mathrm{x}_0, t_0) \ne 0, \end{cases}$$

and

$$\begin{cases} j_t \ge \left(d_{ij} - h_i h_j\right) j_{x_i x_j} \quad \text{at } (x_0, t_0) \\ \text{for some } h \in \mathrm{R}^n \text{ with } |h| \le 1, \text{ if } \nabla j\,(\mathrm{x}_0, t_0) = 0. \end{cases}$$

Definition 2.3: A function $u \in C(R^n \times (0,\infty))$ is a weak solution of (2.4) provided u is both a weak subsolution and a supersolution of (2.4).

For more details of this kind of solutions, we refer to [3,4,5]. As preliminary motivation for these definitions, suppose u is a smooth function on $R^n \times (0,\infty)$ satisfying

$$u_t \le \left(d_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j} - F|\nabla u|$$

wherever $\nabla u \ne 0$. Our function u is thus a classical subsolution of (2.4) on $\{\nabla u \ne 0\}$.

Suppose now $\nabla u(x_0, t_0) = 0.$ Assume additionally that there are points $(x_k, t_k) \to (x_0, t_0)$ for which $\nabla u(x_k, t_k) \ne 0, \quad (k = 1,2,...).$ Then

$$u_t(x_k, t_k) \le (d_{ij} - h_i^k h_j^k)\, u_{x_i x_j}(x_k, t_k) - F(x_k)|\nabla u(x_k, t_k)|,$$

for

$$h^k := \frac{\nabla u(x_k, t_k)}{|\nabla u(x_k, t_k)|}.$$

Since $|h^k| \le 1 \quad (k = 1,2,...),$ we may if necessary pass to a subsequence so that $h^k \to h$ in R^n with $|h| = 1.$

Passing to the limits above, we have

$$u_t(x_0, t_0) \le (d_{ij} - h_i h_j)\, u_{x_i x_j}(x_0, t_0).$$

If, on the other hand, there do not exist such points $\{(x_k, t_k)\}_{k=1}^{\infty}$, then $\nabla u = 0$ near (x_0, t_0), and so $\nabla^2 u = 0$ and u is a function of t only, near (x_0, t_0). Moving to the edge of the set $\{\nabla u = 0\}$, we see that u is a nonincreasing function of t. Thus

$$u_t(x_0, t_0) \le (d_{ij} - h_i h_j)\, u_{x_i x_j}(x_0, t_0)$$

for any $h \in R^n$.

Further motivation for our definition of weak solution, and, particular, an explanation as to why we assume $|h| \le 1$ in the definition will be found in Section III.

2.2. Properties of weak solutions

Theorem 2.1. *(i) Assume u_k is a weak subsolution of (2.4) for k=1,2,... and $u_k \to u$ locally uniformly on $R^n \times (0, \infty)$. Then u is a weak subsolution of (2.4).*

(ii) An analogous assertion holds for weak supersolutions and solutions.

Theorem 2.2. *Assume u is a weak solution of (2.4) and $\Psi : R \to R$ is continuous. Then $v := \Psi(u)$ is also a weak solution of (2.4).*

The proofs of these theorems can be done similarly in [3,4].

3.EXISTENCE OF WEAK SOLUTIONS

3.1. Preliminaries

In this section we consider the existence of weak solution of the mean curvature flow equation (2.4) with initial condition (2.5). A weak solution will be obtained by passing to limits of classical solutions of an approximate problem. We will assume that for the moment at least, u_0 is smooth.

Our intention is to approximate (2.4), (2.5) by the partial differential equation

$$u_t^e = \left(d_{ij} - \frac{u_{x_i}^e\, u_{x_j}^e}{|\nabla u^e|^2 + e^2} \right) u_{x_i x_j}^e - F(x)\left(|\nabla u^e|^2 + e^2 \right)^{1/2} \quad \text{in} \quad R^n \times (0, \infty), \tag{3.1}$$

with initial condition

$$u^e = u_0 \quad \text{on} \quad R^n \times \{t = 0\}. \tag{3.2}$$

for $0 < e < 1$.

3.2. Solution of the approximate equations

We now investigate the approximations (3.1), (3.2) analytically. To do so, let first $0 < s < 1/2$, consider the PDE

$$u_t^{e,s} = a_{ij}^{e,s}\left(\nabla u^{e,s}\right)u_{x_i x_j}^{e,s} - F\left(\left|\nabla u^{e,s}\right|^2 + e^2\right)^{1/2} \quad \text{in} \quad R^n \times (0,\infty), \tag{3.3}$$

with initial condition

$$u^{e,s} = u_0 \quad \text{on} \quad R^n \times \{t=0\}. \tag{3.4}$$

where

$$a_{ij}^{e,d}(p) := (1+s)d_{ij} - \frac{p_i p_j}{|p|^2 + e^2}, \quad p \in R^n, 1 \le i, j \le n.$$

The smooth bounded coefficients $\{a_{ij}^{e,s}\}$ satisfy also the uniformly parabolicity condition, namely, we have

$$s|x|^2 \le a_{ij}^{e,s} x_i x_j, \quad \text{for all } x \in R^n,$$

for each $p \in R^n$, therefore, by the classical PDE theory, there exists unique smooth solution $u^{e,s}$ in $R^n \times (0,\infty)$ satisfying $u^{e,s} = u_0$ on $R^n \times \{t=0\}$.

We now consider the approximate equation in the bounded sub-domain of $R^n \times (0,\infty)$, i.e., we consider the problem

$$\begin{cases} u_t^{e,s} = \left((1+s)d_{ij} - \dfrac{u_{x_i}^{e,s}\, u_{x_j}^{e,s}}{\left|\nabla u^{e,s}\right|^2 + e^2}\right)u_{x_i x_j}^{e,s} - F(x)\left(\left|\nabla u^{e,s}\right|^2 + e^2\right)^{1/2} & in \quad B \times (0,T], \\ u^{e,s} = u_0 \quad on \quad B \times \{t=0\}, \end{cases} \tag{3.5}$$

where B is a closed ball of radius $r > 0$ centered at original, and $T > 0$.

Now we want to prove estimates for $u^{e,s}$, $u_t^{e,s}$, $\nabla u^{e,s}$ in the domain $B \times [0,T]$.

Lemma 3.1. Let $u^{e,s}$ be a solution of (3.5). Then we have the estimate

$$\left|u^{e,s}(x,t)\right| \le Ce^{lt} + Mt, \quad \text{for all } (x,t) \in B \times [0,T], \tag{3.6}$$

where $C := 2\sup_B |u_0|, \; l := \dfrac{2M}{3r}, M := 2\sup_B |F|.$

Proof. Let $j : R^n \to R$ be a function defined by

$$j(x) := m\left(2r^2 - \frac{1}{2}|x|^2\right),$$

where $m := \dfrac{1}{r^2}\sup_B |u_0|$, we see

$$j_{x_i} = -mx_i, \Delta j = -nm, j_{x_i} j_{x_j} j_{x_i x_j} = -m^3 |x|^2.$$

We define by

$$v(x,t) := j(x)e^{lt} + Mt,$$

we have

$$v_t = l\,j\,(x)e^{lt} + M,\; v_{x_i} = j_{x_i} e^{lt},\; \Delta v = \Delta j\; e^{lt} = -mne^{lt},$$

$$v_{x_i} v_{x_j} v_{x_i x_j} = e^{3lt} j_{x_i} j_{x_j} j_{x_i x_j} = -m^3 |x|^2 e^{3lt}.$$

Therefore,

$$L^{e,s}(v) := v_t - \left((1+s)d_{ij} - \frac{v_{x_i}\; v_{x_j}}{|\nabla v|^2 + e^2} \right) v_{x_i x_j} \;\; + F\left(|\nabla v|^2 + e^2\right)^{1/2}$$

$$= v_t - (1+s)\Delta v + \frac{v_{x_i} v_{x_j} v_{x_i x_j}}{|\nabla v|^2 + e^2} + F\left(|\nabla v|^2 + e^2\right)^{1/2}$$

$$= l m\left(2r^2 - \frac{1}{2}|x|^2 \right) e^{lt} + (1+s)mne^{lt} - \frac{m^3 |x|^2 e^{3lt}}{|\nabla v|^2 + e^2} + F\left(|\nabla v|^2 + e^2\right)^{1/2}$$

$$\geq \left(\frac{3}{2} l r^2 + n - 1 - M|x| \right) me^{lt} \geq \left(\frac{3}{2} l r^2 + n - 1 - Mr \right) me^{lt} > 0.$$

On the other hand,

$$v(x,0) = j\,(x) = \frac{1}{r^2}\sup_B |u_0| \left(2r^2 - \frac{1}{2}|x|^2 \right) \geq \frac{1}{r^2}\sup_B |u_0| \left(2r^2 - \frac{1}{2} r^2 \right) = \frac{3}{2}\sup_B |u_0| \geq \sup_B |u_0|$$

Therefore, $L^{e,s}(v) > 0 = L^{e,s}(u^{e,s})$, and $u^{e,s}(x,0) = u_0(x) \leq v(x,0)$ in B. By the classical maximum principle for the parabolic equation, we discover

$$u^{e,s}(x,t) \leq v(x,t) \leq Ce^{lt} + Mt.$$

The proof of the estimate for $-u^{e,s}(x,t)$ is similar as above, therefore, we get

$$|u^{e,s}(x,t)| \leq v(x,t) \leq Ce^{lt} + Mt.$$

Lemma 3.2. Let $u^{e,s}$ be a solution of (3.5). Then we have the estimate

$$\max_{B\times[0,T]} |u_t^{e,s}(x,t)| \leq C$$

where C is a constant depending only on $\sup_B |u_0|, \sup_B |\nabla u_0|, \sup_B |\nabla^2 u_0|, \sup_B |F|.$

Proof. Differentiate the equation in (3.5) with respect to t, we have

$$u_{tt}^{e,s} = \left((1+s)d_{ij} - \frac{u_{x_i}^{e,s}\; u_{x_j}^{e,s}}{|\nabla u^{e,s}|^2 + e^2} \right) u_{tx_i x_j}^{e,s}$$

$$- \frac{\left(u_{tx_i}^{e,s}\; u_{x_j}^{e,s} + u_{x_i}^{e,s}\; u_{tx_j}^{e,s} \right)\left(|\nabla u^{e,s}|^2 + e^2 \right) - 2u_{x_i}^{e,s}\; u_{x_j}^{e,s} u_{tx_i}^{e,s}\; u_{tx_j}^{e,s}}{\left(|\nabla u^{e,s}|^2 + e^2 \right)^2} u_{x_j x_j}^{e,s} - \frac{F\, u_{x_i}^{e,s}\; u_{tx_i}^{e,s}}{\left(|\nabla u^{e,s}|^2 + e^2 \right)^{1/2}}.$$

This equation is linear with respect to u_t, then we may apply the classical maximum principle, we have

$$\sup_{B\times[0,T]} |u_t^{e,s}(x,t)| \le \sup_B |u_t^{e,s}(\cdot,0)|,$$

and

$$u_t^{e,s}(x,0) = \left((1+s)d_{ij} - \frac{u_{0x_i}\ u_{0x_j}}{|\nabla u_0|^2 + e^2} \right) u_{0x_ix_j} \ - F(x)\left(|\nabla u_0|^2 + e^2\right)^{1/2}.$$

Since $0 < e < 1$ and $0 < s < 1/2$,

$$\sup_{B\times[0,T]} |u_t^{e,s}(x,t)| \le C.$$

By the transformation $u^{e,s}\ \mathbf{a}\ \frac{1}{e}u,$ we see

$$u_t = \left((1+s)d_{ij} - \frac{u_{x_i}\ u_{x_j}}{|\nabla u|^2 + 1} \right) u_{x_ix_j} \ - F\left(|\nabla u|^2 + 1\right)^{1/2} . \qquad (3.7)$$

Lemma 3.3. Let u be a solution of (3.7). Then we have the estimate

$e^{C_2u(x,t)}|\nabla u(x,t)| \le C_1e^{C_2M}$, for all $(x,t)\in B\times[0,T],$

where $M := \sup_{B\times[0,T]} |u(x,t)|;\ C_1, C_2$ are constants dependent only on $\sup_B |F(x)|$ and $\sup_B |\nabla F(x)|$.

We derived estimates for $u^{e,s}, u_t^{e,s}, \nabla u^{e,s}$ in the bounded domain $B\times[0,T]$. We note that

$$\left(1 - \frac{L^2}{L^2 + e^2}\right)|x|^2 \le a_{ij}^{e,s}(p)x_ix_j, \quad x\in R^n$$

provided $|p| \le L$. The estimates for $u^{e,s}, u_t^{e,s}, \nabla u^{e,s}$ are uniform in $0 < s < 1/2$. Consequently, uniqueness of the limit implies for each multi-index a :

$$D^a u^{e,s} \to D^a u^e$$

locally uniformly as $s \to 0$, for a smooth function u^e solving approximate equation.

3.3. Passage to limits

Theorem 3.4. Assume $u_0 : R^n \to R$ is a continuous function. Then there exists a weak solution u of (2.4),(2.5).

Proof. Suppose first u_0 is smooth. Employing estimates in Lemmas 3.1, 3.2, 3.3, we can extract a subsequence $\{u^{e_k}\}_{k=1}^{\infty} \subset \{u^e\}_{0<e<1}$ so that, as $e_k \to 0, u^{e_k} \to u$ uniformly in $B\times[0,T]$ for some Lipschitz function u in $B\times[0,T]$. Since r and T are arbitrary, we can extend r and T to infinity so that $u^e \to u$ locally uniformly in $R^n\times[0,\infty)$ for a locally Lipschitz continuous function u in $R^n\times[0,\infty)$.

We assert now that u is a weak solution of (2.4), (2.5). For this, let $j \in C^{\infty}(R^{n+1})$ and suppose $u - j$ has a strict local maximum at a point $(x_0, t_0) \in R^n \times [0, \infty)$. As $u^{e_k} \to u$ uniformly near (x_0, t_0), $u^{e_k} - j$ has a local maximum at a point (x_k, t_k), with

$$(x_k, t_k) \to (x_0, t_0) \text{ as } k \to \infty. \tag{3.8}$$

Since u^{e_k} and j are smooth, we have

$$\nabla u^{e_k} = \nabla j,\ u_t^{e_k} = j_t,\ D^2 u^{e_k} \le D^2 j \quad \text{at } (x_k, t_k). \tag{3.9}$$

Since u^{e_k} is a solution of

$$u_t^{e_k} = \left(d_{ij} - \frac{u_{x_i}^{e_k}\, u_{x_j}^{e_k}}{\left|\nabla u^{e_k}\right|^2 + e_k{}^2} \right) u_{x_i x_j}^{e_k} - F\left(\left|\nabla u^{e_k}\right|^2 + e_k{}^2 \right)^{1/2},$$

we have

$$j_t \le \left(d_{ij} - \frac{j_{x_i} j_{x_j}}{\left|\nabla j\right|^2 + e_k{}^2} \right) j_{x_i x_j} - F\left(\left|\nabla j\right|^2 + e_k{}^2 \right)^{1/2} \quad \text{at } (x_k, t_k). \tag{3.10}$$

Suppose first $\nabla j\,(x_0, t_0) \ne 0$. Then $\nabla j\,(x_k, t_k) \ne 0$ for k large enough. We consequently may pass to limits in (3.10), recalling (3.9) to deduce

$$j_t \le \left(d_{ij} - \frac{j_{x_i} j_{x_j}}{\left|\nabla j\right|^2} \right) j_{x_i x_j} - F\left|\nabla j\right| \quad \text{at } (x_0, t_0). \tag{3.11}$$

Next, assume instead $\nabla j\,(x_0, t_0) = 0$. Set

$$h^k := \frac{\nabla j\,(x_k, t_k)}{\left(\left|\nabla j\,(x_k, t_k)\right|^2 + e_k^2 \right)^{1/2}},$$

so that (3.10) becomes

$$j_t \le \left(d_{ij} - h_i^k h_j^k \right) j_{x_i x_j} \quad \text{at } (x_k, t_k). \tag{3.12}$$

Since $|h^k| \le 1$, we may assume, upon passing to a subsequence and re-indexing if necessary, that $h^k \to h$ in R^n for some $|h| \le 1$. Sending k to infinity in (3.10), we discover

$$j_t \le \left(d_{ij} - h_i h_j \right) j_{x_i x_j} \quad \text{at } (x_0, t_0). \tag{3.13}$$

If $u - j$ has a local maximum, but not necessary a strict local maximum at (x_0, t_0), we repeat the argument above with $j\,(x,t)$ replaced by

$$\hat{j}(x,t) := j\,(x,t) + |x - x_0|^4 + (t - t_0)^4,$$

again to obtain (3.11) and (3.13).

Consequently, u is a weak subsolution of (2.4),(2.5). That u is a weak suppersolution follows analogously.

Suppose at last u_0 is only continuous. We select smooth functions $\{u_0^k\}_{k=1}^{\infty}$ so that $u_0^k \to u_0$ locally uniformly on R^n. Denote by u^k the solution of (2.4),(2.5) constructed above with initial function u_0^k. According to the stability of the weak solutions[3,4] the limit $\lim_{k\to\infty} u^k = u$ exists locally uniformly in $R^n \times [0,\infty)$, according to Theorem 2.1 u is a weak solution of (2.4), (2.5).

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