THE COMPARISON OF SHEAF- SOLUTIONS IN FUZZY CONTROL PROBLEM

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(Manuscript Received on December 14th, 2005, Manuscript Revised March 8th, 2006)

ABSTRACT: *In* [2] *the author considered the Sheaf-Optimal Control Problem (SOCP) by differential equations:* $\frac{dx(t)}{dt} = f(t, x(t), u(t))$,

where $x_0 \in Q \subset R^n, u \in U \subset R^p, t \in [0,T] \subset R^+$*, and sheaf of solutions:*

$$H_{t,u} = \{x(t) = x(t, x_0, u(t)) \mid x_0 \in H_0 \subseteq Q, t \in I = [0,T] \subset R^+, u(t) \in U\}$$

with the goal function $I(u) \to \min$.

In [5]*, we have offered the neccesary conditions of Sheaf-Optimal Control Problem in Fyzzy type (SOFCP), that means the controls* $u(t) \in U \subset E^p$ *not belong to* R^p .

This paper shows some comparison of sheaf-solutions $H_{t,u}$ *and* $H_{t,\bar{u}}$ *for many kinds of fuzzy controls* $u(t), \bar{u}(t) \in U \subset E^p$ *in Sheaf Fuzzy Control Problem(SFCP)*

Keywords: *Fuzzy Theory, Optimal Control Theorey, Differential Equations.*

1. INTRODUCTION :

For Sheaf-Optimal Control Problem (SOCP) many controls $u(t)$ and $\bar{u}(t) = u(t) + \Delta u$ are considered with $\|\Delta u\| = \|\bar{u}(t) - u(t)\| \le \delta$, where $u(t), \bar{u}(t) \in U \subset R^p$ [2]. For Sheaf-Optimal Control Problem in Fuzzy Type (SOFCP) we have fuzzy controls $u(t)$ and $\bar{u}(t) \in U \subset E^p$ with $\|\bar{u}(t) - u(t)\| \le T\sqrt{p}$ [5].

For the Sheaf Fuzzy Control Problem (SFCP) we have the same fuzzy controls $u(t)$ and $\bar{u}(t) \in U \subset E^p$, that was defined by definition 5 in [5]. The paper is organized as follows:

In the second section, offering the Sheaf Fuzzy Control Problem (SFCP) we get estimations of the norms $\| \bullet \|_C$ and $\| \bullet \|_L$ of

$$\Delta x = x(t, x_0, \bar{u}(t)) - x(t, x_0, u(t)) \text{ and}$$
$$\Delta f = f(t, x(t, x_0, \bar{u}(t)), \bar{u}(t)) - f(t, x(t, x_0, u(t)), u(t))$$

In section 3, we study some comparisons of sheaf solutions $H_{t,u}$ in many kinds of fuzzy controls $u(t), \bar{u}(t) \in U \subset E^p$, that means we have to compare the measure $\left| \mu(H_{T,\bar{u}}) - \mu(H_{T,u}) \right|$

2. THE SHEAF FUZZY CONTROL PROBLEM (SFCP)

As we know, the solutions of differential equations depend locally on initial, right hand side and parameters. Now, we consider a control system of differential equations

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)) \qquad (1)$$

where $x(0) = x_0 \in H_0 \subset Q \subset R^n, x(t) \in Q \subset R^n$, $u(t) \in U \subset E^p$, $t \in I = [0,T] \subset R^+$ and $f : I \times R^n \times E^p \to R^n$.

Definition 1. The sheaf - solution (or sheaf-trajectory) $\left[x(t,x_0,u)\right]$ which gives at the time t a set

$$H_{t,u}=\left\{x(t)=x(t,x_0,u)\,|\,x_0\in H_0\subset Q, x(t)-\text{solution of}(1)\right\}, \qquad (2)$$

where $x_0\in H_0\subset Q\subset R^n$, $u(t)\in U\subset E^p$, $t\in I$.

In the case, when a control u(t) is fuzzy, we have Sheaf Fuzzy Control Problem (SFCP).

Suppose at time $t=0, u(0)=0$ and $x(0)=x_0\in H_0$. For two admissible controls $u(t)$ and $\overline{u}(t)\in U\subset E^p$, we have two sets of sheaf-solutions

$$H_{t,u}=\left\{x(t)=x(t,x_0,u)\,|\,x_0\in H_0\subset Q, x(t)-\text{a solution of}(1)\text{ by control }u(t)\right\}$$

$$H_{t,\overline{u}}=\left\{\overline{x}(t)=x(t,x_0,\overline{u}(t))\,|\,x_0\in H_0\subset Q, \overline{x}(t)-\text{a solution of}(1)\text{ by control }\overline{u}(t)\right\},$$

where $t\in I$. (See fig.1)

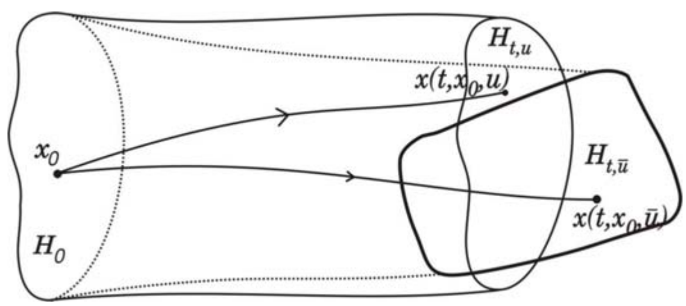

Fig. 1. The sheaf-solutions of Sheaf Fuzzy Control Problem (SFCP).

If $\mu(H_{t,u})$ is a measure of the set $H_{t,u}$ then $\mu(H_{t,u})$ is called a cross-area of sheaf trajectory at (t,u), in particular it is a square of set $H_{t,u}$.That is $\mu(H_{t,u})=\int\limits_{H_{t,u}} dx_t$ and $\mu(H_{t,\overline{u}})=\int\limits_{H_{t,\overline{u}}} d\overline{x}_t$ is a square of $H_{t,\overline{u}}$.

Assumption 1. Suppose that the vector function $f(t,x(t),u(t))$ satisfies

i) $$\left\|\frac{\partial f}{\partial x}\Delta x(t)+\frac{\partial f}{\partial u}\Delta u(t)\right\| \le M(\|\Delta x(t)\|+\|\Delta u(t)\|) \qquad (3)$$

ii) $$\sum_{k=2}^{+\infty}\frac{1}{k!}\left\|d^k f\right\|\le m \qquad (4)$$

iii) $$\left|sp\frac{\partial f(t,x(t,x_0,u(t)),u(t)}{\partial x}\right|=L(\|u(t)\|) \qquad (5)$$

for all $x(t)\in Q\subset R^n$, $u(t),\overline{u}(t)\in U\subset E^p$, $t\in I$, where M, m, L are real positive constants and spA is trace of matrix A.

Lemma 1. *For the fuzzy controls* $u(t)$ *and* $\overline{u}(t)\in U\subset E^p$, *the norm of* $\Delta u=\overline{u}(t)-u(t)$ *is estimated as follows:*

a) $$\|\Delta u\|_C\le\sqrt{p} \qquad (6)$$

b) $$\|\Delta u\|_L=\int_0^T\|\Delta u(t)\|\,dt\le T\sqrt{p}, \qquad (7)$$

Proof of Lemma 1: Let $u(t), \bar{u}(t) \in U \subset E^p$ are fuzzy controls. In [5], we defined a fuzzy function $u : I \to U \subset E^p = E \times E \times \ldots \times E$, that means $u(t) = (u_1(t), u_2(t), \ldots, u_p(t))$. Because every $u_k(t)$ satisfies $|u_k(t)| \le 1$ (k=1,2,..p) then a norm of

a) $$\|\Delta u\|_C = \max\left\{\|\bar{u}(t) - u(t)\| : t \in I\right\}$$
$$\le \max\left\{\sqrt{\sum_{i=1}^{p} |\bar{u}_i(t) - u_i(t)|^2} \ : t \in I\right\} \le \sqrt{p}$$

where $u(t), \bar{u}(t) \in U \subset E^p$

b) $$\|\Delta u\|_L = \int_0^T \|\Delta u(t)\| \, dt \le \sqrt{p} \int_0^T dt \le T\sqrt{p}$$ (■)

Theorem 1. *Suppose that* $u(t), \bar{u}(t) \in U \subset E^p$ *are fuzzy controls. If the function* $f(t, x(t), u(t))$ *satisfies* (3) *and* (4) *then the norm of* $\Delta x = x(t, x_0, \bar{u}(t)) - x(t, x_0, u(t))$ *is estimated as follows:*

a) $$\|\Delta x\|_C \le (T m + M\sqrt{p}\ ) \exp(MT) \quad (8)$$

b) $$\|\Delta x\|_L \le T^2 (m + M\sqrt{p}\ ) \exp(MT) \quad (9)$$

Proof of Theorem 1: Let $u(t), \bar{u}(t) \in U \subset E^p$ are fuzzy controls with $\Delta u = \bar{u}(t) - u(t)$ satisfies (6) or (7) .

a) The solutions of (1) are equivalent the following integrals:

$$x(t) = x_0 + \int_0^t f(s, x(s), u(s))\, ds \quad \text{and} \quad \bar{x}(t) = x_0 + \int_0^t f(s, \bar{x}(s), \bar{u}(s))\, ds .$$

Estimating $\|\Delta x(t)\|$ as follows $\|\Delta x(t)\| \le \int_0^t \|f(s, \bar{x}(s), \bar{u}(s) - f(s, x(s), u(s))\| ds$

$$\le \int_0^t \left\| \frac{\partial f}{\partial x}(s, x(s), \bar{u}(s)) dx + \frac{\partial f}{\partial u}(s, x(s), u(s)) du + \sum_{k=2} d^k f(s, x(s), u(s)) \right\| ds$$
$$\le M\int_0^t \|dx + du + ..\| ds \le M\int_0^t \|\Delta x(s)\| ds + M\int_0^t \|\Delta u(s)\| ds + mT$$
$$\le M\int_0^t \|\Delta x(s)\| ds + MT\sqrt{p} + mT$$

By Gronwall-Bellmann's Lemma, it implies that

$$\|\Delta x\|_C = \max_{t \in [0,T]} \|\Delta x(t)\| \le T(m + M\sqrt{p}\ ) \exp(MT)$$

b) $$\|\Delta x(t)\| \le M\int_0^t \|\Delta x(s)\| ds + M\int_0^t \|\Delta u(s)\| ds + mT$$
$$\|\Delta x(t)\| \le M\int_0^t \|\Delta x(s)\| ds + MT\sqrt{p} + mT$$
$$\le T(m + M\sqrt{p}\ ) \exp(MT)$$

For $\|\Delta x\|_L = \int_0^T \|\Delta x(t)\| dt \le T^2 (M\sqrt{p} + m) \exp(MT)$ we have (9) (■)

Theorem 2. *Suppose that* $u(t), \bar{u}(t) \in U \subset E^p$ *are fuzzy controls, if the function* $f(t, x(t), u(t))$ *satisfies* (3) *and* (4) *then the norm of*

$$\Delta f = f(t, x(t, x_0, \bar{u}(t)), \bar{u}(t)) - f(t, x(t, x_0, u(t)), u(t))$$

is *estimated as follows:*

a) $$\|\Delta f\|_C \le MT[(M\sqrt{p} + m)\exp(MT) + \sqrt{p}] + m \qquad (10)$$

b) $$\|\Delta f\|_L \le T\left\{M\left[T(m + M\sqrt{p})\exp(MT) + \sqrt{p}\right] + m\right\} \qquad (11)$$

Proof of Theorem 2:

a) For $$\le \max\left\{\left\|df + \frac{1}{2!}d^2 f + \frac{1}{3!}d^3 f + \ldots\right\| : t \in I\right\}$$

$$\le \max\left\{\|df\| + \sum_{k=2}^{+\infty}\frac{1}{k!}\|d^k f\| : t \in I\right\}$$

$$\le \max\left\{\left\|\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u}du\right\| + \sum_{k=2}^{+\infty}\frac{1}{k!}\|d^k f\| : t \in I\right\}$$

$$\le M(\|\Delta x\|_C + \|\Delta u\|_C) + m$$

$$\le M[T(M\sqrt{p} + m)\exp(MT) + T\sqrt{p}] + m$$

$$\le MT[(M\sqrt{p} + m)\exp(MT) + \sqrt{p}] + m$$

b) For $$\|\Delta f\|_L = \int_0^T \|f(s, \bar{x}(s, x_0, \bar{u}(s)), \bar{u}(s)) - f(s, x(s, x_0, u(s)), u(s))\| ds$$

$$\le M\left(\int_0^T \|\Delta x(t)\| dt + \int_0^T \|\Delta u(t)\| dt\right) + m\int_0^T dt$$

$$\le M(\|\Delta x\|_L + \|\Delta u\|_L) + mT$$

$$\le M\left[T^2(m + M\sqrt{p})\exp(MT) + T\sqrt{p}\right] + mT$$

$$\le T\left\{M\left[T(m + M\sqrt{p})\exp(MT) + \sqrt{p}\right] + m\right\}$$ **(■)**

3. THE COMPARISON OF SHEAF SOLUTIONS IN THE SFCP

Lemma 2. *For* $A, B \ge 0$ *there exists a real number K such that* $e^A - e^B \le K e^{A-B}$**.**

Proof of Lemma 2: We have $e^A - e^B = e^B(e^{A-B} - 1) \le K e^{A-B}$, $K > e^B$ **(■)**

Now, suppose that $\mu(H_0)$ is given. There are many following results of comparison of sheaf- solutions :

Theorem 3. *Suppose that* $u(t), \bar{u}(t) \in U \subset E^p$ *are fuzzy controls. If the function* $f(t, x(t), u(t))$ *satisfies* (3) ,(4) *and* (5) *then we have the following estimation:*

$$|\mu(H_{T,\bar{u}}) - \mu(H_{T,u})| \le \mu(H_0)\exp(LT\sqrt{p}) \qquad (12)$$

Proof of Theorem 3: We have $\mu(H_{t,u}) = \int_{H_{t,u}} dx_t = \int_{H_0}\left|\det\frac{\partial x(t, x_0, u)}{\partial x_0}\right| dx_0$,

where

$$\left|\det\frac{\partial x(t,x_0,u)}{\partial x_0}\right| = \exp\left(\int_0^T sp\frac{\partial f(\gamma,x(\gamma,x_0,u),u(\gamma)}{\partial x}d\gamma\right),$$

that means

$$\|\Delta f\|_C = \max\left\{\|f(t,x(t,x_0,\overline{u}(t)),\overline{u}(t)) - f(t,x(t,x_0,u(t)),u(t))\| : t\in I\right\}$$

$$\mu(H_{t,u}) = \int_{H_0}\left|\det\frac{\partial x(t,x_0,u)}{\partial x_0}\right|dx_0 = \int_{H_0}\left|\exp\left(\int_0^T sp\frac{\partial f(\gamma,x(\gamma,x_0,u),u(\gamma))}{\partial x}d\gamma\right)\right|dx_0$$

$$\mu(H_{T,u}) = \mu(H_0)\exp(L\int_0^T\|u(t)\|dt).$$

It is analogous of proof a) above, we have $\mu(H_{T,u}) = \mu(H_0)\exp(L\int_0^T\|\overline{u}(t)\|dt)$.

Estimating $|\mu(H_{T,\overline{u}}) - \mu(H_{T,u})|$ we have

$$|\mu(H_{T,\overline{u}}) - \mu(H_{T,u})| \le \mu(H_0)\left[\exp(L\int_0^T\|\overline{u}(t)\|dt) - \exp(L\int_0^T\|u(t)\|dt)\right]$$

$$\le \mu(H_0)K\exp[L\int_0^T(\|\overline{u}(t)\| - \|u(t)\|)dt]$$

$$\le \mu(H_0)K\exp[L\int_0^T\|\Delta u(t)\|dt] \le \mu(H_0)K\exp[LT\sqrt{p}]$$

where $K \ge \exp(LT\sqrt{p})$. (■)

Corollary 1 *Suppose that* $u(t),\overline{u}(t)\in U\subset E^p$ *are fuzzy controls. If the function* $f(t,x(t),u(t))$ *satisfies* (3) *and* (4), *then for* (1) *when n=1 we have the following estimation:*

$$|\mu(H_{T,\overline{u}}) - \mu(H_{T,u})| \le (b_0 - a_0)\exp(2LT\sqrt{p}), \quad (13)$$

where $K = \exp(LT\sqrt{p})$.

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Proof of Corollary: When $n=1$ we have $\mu(H_0) = b_0 - a_0$, finally we get (13) (see fig.2) .

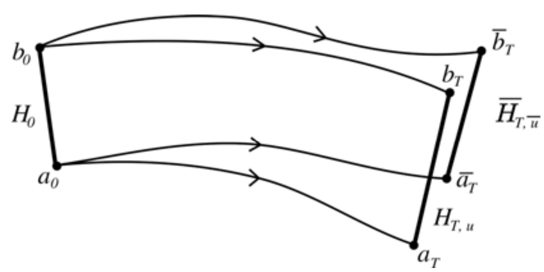

Fig. 2. The sheaf-solutions of Sheaf Fuzzy Control Problem (SFCP), when n = 1. (■)

4. CONCLUSION

In the Sheaf Fuzzy Control Problem (SFCP) for many different fuzzy controls $u(t),\overline{u}(t)\in U\subset E^p$ we have the comparison (7)-(13).There are differences between the Sheaf Fuzzy Control Problem (SFCP) and the Sheaf Optimal Control Problem in Fuzzy Type (SOFCP) what was offered in [5].

5. ACKNOWLEDGMENT

This work is a part of research supported by The Research Fund of Ministry of Educations of Vietnam, under contract NCCB 2003/18. The Authors gratefully acknowledge the financial support of this Research Fund. The Authors would like to thank the referee for his (her) careful reading and valuale remarks which improve the presentation of the paper.

SO SÁNH BÓ NGHIỆM TRONG BÀI TOÁN ĐIỀU KHIỂN MỜ

Nguyễn Đình Phư , Trần Thanh Tùng
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TÓM TẮT: *Trong* [2] *tác giả đã xét bài toán điều khiển tối ưu bó (SOCP) cho bởi hệ phương trình vi phân:*

$$\frac{dx(t)}{dt} = f(t, x(t), u(t))$$

ở đây $x_0 \in Q \subset R^n, u \in U \subset R^p, t \in [0,T] \subset R^+$*, và bó nghiệm:*

$$H_{t,u} = \{x(t) = x(t, x_0, u(t)) \mid x_0 \in H_0 \subseteq Q, t \in I = [0,T], u(t) \in U\}$$

với hàm mục tiêu $I(u) \to \min$.

Trong [5] *lại trình bày các điều kiện cần của **bài toán điều khiển tối ưu bó dạng mờ*** **(***SOFCP), với các điều khiển mờ* $u(t) \in U \subset E^p$ *thay vì thuộc* R^p**.**

Bài báo này đưa ra các so sánh các bó nghiệm $H_{t,u}$ *và* $H_{t,\bar{u}}$ *ứng với các điều khiển mờ khác nhau* $u(t), \bar{u}(t) \in U \subset E^p$ *của **bài toán điều khiển bó dạng mờ** (SFCP).*

Từ khóa: *Lý thuyết mờ, Lý thuyết điều khiển tối ưu, Phương trình Vi phân*

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