EXISTENCE OF SOLUTIONS OF SET CONTROL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In [1-8], authors studied the existence and comparison of solutions of the set differential equation (SDE)

$$D_H X(t) = F(t, X(t)),$$

where $X(t_0) = X_0 \in K_c(R^n), X(t) \in K_c(R^n), t \in [t_0, T] = I \subset R_+$

and $F : I \times K_c(R^n) \to K_c(R^n)$.

In this paper, we give the so-called set control differential equation (SCDE) in the form

$$D_H X(t) = F(t, X(t), U(t)),$$

where $X(t_0) = X_0 \in K_c(R^n), X(t) \in K_c(R^n), U(t) \in K_c(R^p),\ t \in [t_0, T] = I \subset R_+$ $F : I \times K_c(R^n) \times K_c(R^p) \to K_c(R^n)$,

and study existence of solutions and comparison solutions of SCDE.

The paper is organized as follows: we recall some basic concepts and notations which are useful in next sections in section 2 and some results on SDE in section 3. Existence results on solutions and comparison of solutions of SCDE are presented in section 4.

2. PRELIMINARIES

We recall some notations and concepts presented in detail in recent series works of V. Lakshmikantham et al. See [1-8].

Let $K_C(R^n)$ denote the collection of all nonempty, compact and convex subsets of R^n. Given A, B in $K_C(R^n)$. The Hausdorff distance between *A* and *B* defined as

$$D[A, B] = \max\left\{\sup_{a\in A}\inf_{b\in B} \|a - b\|,\ \sup_{b\in B}\inf_{a\in A} \|a - b\|\right\}, \qquad (2.1)$$

where $\|.\|$ denotes the Euclidean norm in R^n.

It is known that $\left(K_C(R^n), D\right)$ is a complete metric space and if the space $K_C(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_C(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space. See [15].

The Hausdorff metric satisfies some below properties.

$$D\left[A + C, B + C\right] = D[A, B] \text{ and } D[A, B] = D[B, A], \qquad (2.2)$$

$$D\left[\lambda A, \lambda B\right] = \lambda D[B, A], \qquad (2.3)$$

$$D[A,B] \le D\left[A,C\right] + D\left[C,B\right], \tag{2.4}$$

$$D[A+A',B+B'] \le D[A,B] + D\left[A',B'\right] \tag{2.5}$$

for all $A,B,C \in K_c(R^n)$ and $\lambda \in R_+$

Let $A,B \in K_c(R^n)$. The set $C \in K_c(R^n)$ satisfying $A = B + C$ is known as the geometric difference of the sets A and B and is denoted by the symbol $A - B$. Given an interval $I = \left[t_0, T\right] \subset R_+$. Function $F : I \to K_c(R^n)$ is said to be continuous at $t^* \in I$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t^*) > 0$ such that

$$D\left[F(t), F(t^*)\right] < \varepsilon \quad \text{for all} \quad t \in I \quad \text{with} \quad \left|t - t^*\right| < \delta .$$

We say that the mapping F has a Hukuhara derivative $D_H F(\tau)$ at a point $\tau \in I$, if

$$\lim_{h\to 0+} \frac{F(\tau+h) - F(\tau)}{h} \text{ and } \lim_{h\to 0+} \frac{F(\tau) - F(\tau-h)}{h}$$

exist in the topology of $K_C(R^n)$ and are equal to $D_H F(\tau)$.

The Hukuhara integral of F is given by

$$\int_I F(s)ds = \left\{ \int_I f(s)ds : \ f \ \text{ is a continuous selector of } \ F \right\}$$

for any compact set $I \subset R_+$.

We have the following properties of the Hukuhara integral.

If $F : I \to K_c(R^n)$ is integrable, one has

$$\int_{t_0}^{t_2} F(s)ds = \int_{t_0}^{t_1} F(s)ds + \int_{t_1}^{t_2} F(s)ds, \ t_0 \le t_1 \le t_2 \tag{2.6}$$

and

$$\int_{t_0}^{t} \lambda F(s)ds = \lambda \int_{t_0}^{t} F(s)ds, \lambda \in R .$$

If $F,G : I \to K_c(R^n)$ are integrable, then $D\left[F(.),G(.)\right] : I \to R$ is integrable and

$$D\left[\int_{t_0}^{t} F(s)ds, \int_{t_0}^{t} G(s)ds\right] \le \int_{t_0}^{t} D\left[F(s),G(s)\right]ds . \tag{2.7}$$

Let us denote $D\left[A, \theta^n\right] = \|A\| = sup\left\{\|a\| : a \in A\right\}$

for $A \in K_c(R^n)$, where θ^n is the zero element of R^n which is regarded as a one point set. More details in continuity, Hukuhara derivative, Hukuhara integral of the mapping $F : I \to K_c(R^n)$, please see [1-8].

3. SET DIFFERENTIAL EQUATION

In [1-8], authors considered the set differential equation (SDE) as following.

$$D_H X(t) = F(t, X(t)), \tag{3.1}$$

where

$X(t_0) = X_0 \in K_c(R^n), X(t) \in K_c(R^n), t \in [t_0, T] = I$, state $X(t) \in K_c(R^n)$

and $F : I \times K_c(R^n) \to K_c(R^n)$.

The mapping $X \in C^1\left[I, K_c(R^p)\right]$ is said to be a solution of (3.1) on I if it satisfies (3.1) on I. Since $X(t)$ is continuously differentiable, we have

$$X(t) = X_0 + \int_{t_0}^{t} D_H X(s)ds,\ t \in I.$$

We associate with the initial value problem (3.1) the following

$$X(t) = X_0 + \int_{t_0}^{t} F(s, X(s))ds,\ t \in I \qquad (3.2)$$

where the integral is the Hukuhara integral. Observe that $X(t)$ is a solution of (3.1) if only if it satisfies (3.2) on I.

We recall some results in [1-8].

The local existence result on solution of SDE is the following.

Theorem 3.1. *Assume that*

(i) $F \in C\left[R_0, K_c(R^n)\right],\ D\left[F(t, X(t)), \theta^n\right] \le M_0,$ *on* $R_0 = I \times B(X_0, b)$ *where*

$$B(X_0, b) = \left\{X(t) \in K_c(R^n) : D\left[X(t), X_0\right] \le b\right\} \text{ and}$$

(ii) $g \in C\left[I \times [0, 2b], R_+\right],\ 0 \le g(t, w) \le M_1$ *on* $I \times [0, 2b], g(t, 0) = 0,\ g(t, w)$ *is nondecreasing in w for each* $t \in I$ *and* $w(t) \equiv 0$ *is the unique solution of*

$$w' = g(t, w),\ w(t_0) = 0 \text{ on } I. \tag{3.3}$$

(iii) $D\left[F(t, \overline{X}(t)), F(t, X(t))\right] \le g\left(t, D\left[\overline{X}(t), X(t)\right]\right)$ *on* R_0.

Then, the (3.1) has a unique solution $X(t) = X(t, X_0)$ *on* $[t_0, t_0 + \eta]$, *where* $\eta = min\left\{a, \frac{b}{M}\right\},\ M = max\left\{M_0, M_1\right\}$.

The global existence result on solution of SDE is as follows.

Theorem 3.2. *Assume that* $F \in C\left[R_+ \times K_c(R^n), K_c(R^n)\right]$ *and*

$$D\left[F(t,X(t)),\theta^n\right] \le g\left(t,D\left[X(t),\theta^n\right]\right),(t,X(t)) \in R_+ \times K_c(R^n),$$

where $g \in C\left[R_+^2, R_+\right]$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in R_+$ *and the maximal solution* $r(t,t_0,w_0)$ *of*

$w' = g(t,w)$, $w(t_0)=w_0 \ge 0$ *exists on* $\left[t_0,+\infty\right)$. *Suppose further that* F *is smooth enough to guarantee local existence of solution of (3.1) for any* $(t_0,X_0) \in R_+ \times K_c(R^n)$. *Then the largest interval of existence of any solution* $X(t) = X(t,t_0,X_0)$ *of (3.1) such that* $D\left[X_0,\theta^n\right] \le w_0$ *is* $\left[t_0,+\infty\right)$.

The below theorem is an existence result on solutions of SDE.

Theorem 3.3. *Assume that* $F \in C\left[R_+ \times K_c(R^n), K_c(R^n)\right]$ *and*

(i) $D\left[F(t,X(t)),\theta^n\right] \le g\left(t,D\left[X(t),\theta^n\right]\right)$, $(t,X(t)) \in \left[t_0,T\right] \times K_c(R^n)$,

where $g \in C\left[\left[t_0,T\right] \times R_+, R_+\right]$, $g(t,w)$ *is nondecreasing in* (t,w);

(ii) *the maximal solution* $r(t,t_0,w_0)$ *of*

$$w' = g(t,w),\ w(t_0)=w_0 \ge 0$$

exists on I.

Then there exists a solution $X(t) = X(t,t_0,X_0)$ *to the (3.1) which satisfies*

$$D\left[X(t),X_0\right] \le r(t,w_0),\ t \in I,\ \text{where } w_0 = D\left[X_0,\theta^n\right].$$

Some results on comparison of solutions of SDE and stability of solutions of SDE were studied in [1, 4, 5].

4. MAIN RESULTS

In this paper, we provide a set control differential equation (SCDE) as following

$$D_H X(t) = F(t,X(t),U(t)),\ X(t_0) = X_0 \in K_c(R^n), \tag{4.1}$$

where $F : I \times K_c(R^n) \times K_c(R^p) \to K_c(R^n)$, state $X(t) \in K_c(R^n)$, control $U(t) \in K_c(R^p)$.

The $U : I \to K_c(R^p)$ is integrable, is called an admissible control. Let U be a set of all admissible controls. The mapping $X \in C^1\left[I, K_c(R^n)\right]$ is said to be a solution of (4.1) on I if it satisfies (4.1) on *I*. Since $X(t)$ is continuously differentiable, we have

$$X(t) = X_0 + \int_{t_0}^{t} D_H X(s)ds,\ t \in I.$$

We associate with the initial value problem (4.1) the following

$$X(t) = X_0 + \int_{t_0}^{t} F(s, X(s), U(s))ds, t \in I$$

where the integral is the Hukuhara integral. Observe that $X(t)$ is a solution of (4.1) if only if it satisfies (4.2) on *I*.

Now, based on the theorems 3.1-3.3 of SDE we have some existence results on solutions of SCDE.

Firstly, we have a unique existence of solution of SCDE as following.

Theorem 4.1. *Assume that*

(i) $F \in C\left[R_0, K_c(R^n)\right]$, $D\left[F(t, X(t), U(t)), \theta^n\right] \le M_0$, *on* $R_0 = I \times B(X_0, b) \times \mathrm{U}$,

where $B(X_0, b) = \left\{X(t) \in K_c(R^n) : D\left[X(t), X_0\right] \le b\right\}$ *and*

(ii) $g \in C\left[I \times [0, 2b], R_+\right]$, $0 \le g(t, w) \le M_1$ *on* $I \times [0, 2b], g(t, 0) = 0$, $g(t, w)$ *is nondecreasing in w for each* $t \in I$ *and* $w(t) \equiv 0$ *is a unique solution of*

$$w' = g(t, w), w(t_0) = 0 \text{ on } I. \tag{4.3}$$

(iii) $D\left[F(t, \overline{X}(t), \overline{U}(t)), F(t, X(t), U(t))\right] \le g\left(t, D\left[\overline{X}(t), X(t)\right]\right)$ *on* R_0.

Then, the (4.1) has a unique solution $X(t) = X(t, X_0, U(t))$ *on* $[t_0, t_0 + \eta]$, *where* $\eta = min\left\{a, \frac{b}{M}\right\}$, $M = max\left\{M_0, M_1\right\}$.

Proof. Function $U(t)$ is of variable t. Set $h(t, X(t)) = F(t, X(t), U(t))$ plays the role of function $F(t, X(t))$ in theorems 3.1 and consider $U(t)$ as parameter, then using theorems 3.1, we have theorem 4.1. **!**

Then, we have the global existence of solution of SCDE as below.

Theorem 4.2. *Assume that* $F \in C\left[R_+ \times K_c(R^n) \times K_c(R^p), K_c(R^n)\right]$ *and*

$$D\left[F(t, X(t), U(t)), \theta^n\right] \le g(t, D\left[X(t), \theta^n\right]), (t, X(t), U(t)) \in R_+ \times K_c(R^n) \times \mathrm{U},$$

where $g(t, w)$ *is nondecreasing in w for each* $t \in R_+$ *and the maximal solution* $r(t, t_0, w_0)$ *of*

$$w' = g(t, w), w(t_0) = w_0 \ge 0$$

exists on $[t_0, +\infty)$. *Suppose further that f is smooth enough to guarantee local existence of solution of (4.1) for any* $(t_0, X_0, U(t_0)) \in R_+ \times K_c(R^n) \times \mathrm{U}$. *Then the largest interval of existence of any solution* $X(t) = X(t, t_0, X_0, U(t))$ *of (4.1) such that* $D\left[X_0, \theta^n\right] \le w_0$ *is* $[t_0, +\infty)$.

This theorem also holds for $J = \left[t_0, T\right], T > t_0$.

Proof. Using theorem 3.2 and the proof is similar the proof of theorem 4.1. !

We adapt theorem 3.3 of SDE to below theorem of SCDE.

Theorem 4.3. *Assume that* $F \in C\left[R_+ \times K_c(R^n) \times K_c(R^p), K_c(R^n)\right]$ *and*

(i) $D\left[F(t, X(t), U(t)), \theta^n\right] \le g\left(t, D\left[X(t), \theta^n\right]\right),$

$(t, X(t), U(t)) \in [t_0, T] \times K_c(R^n) \times \mathrm{U}$,

where $g \in C\left[[t_0, T] \times R_+, R_+\right]$, $g(t, w)$ *is nondecreasing in* (t, w)*;*

(ii) *the maximal solution* $r(t, t_0, w_0)$ *of*

$$w' = g(t, w),\ w(t_0) = w_0 \ge 0$$

exists on I.

Then there exists a solution $X(t) = X(t, t_0, X_0, U(t))$ *to the (4.1) which satisfies*

$$D\left[X(t), X_0\right] \le r(t, w_0),\ t \in I,\ \text{where}\ w_0 = D\left[X_0, \theta^n\right].$$

For comparison solutions of SCDE we need the following assumption.

Assumption

$F : R \times K_c(R^n) \times K_c(R^p) \to K_c(R^n)$ satisfies the condition

$$D\left[F(t, \bar{X}(t), \bar{U}(t)), F(t, X(t), U(t))\right] \le c(t)\left\{ D\left[\bar{X}(t), X(t)\right] + D\left[\bar{U}(t), U(t)\right]\right\} \quad (4.4)$$

For $t \in I; \bar{X}(t), X(t) \in K_c(R^n); \bar{U}(t), U(t) \in K_c(R^p)$, where $c(t)$ is a positive and integralble on I.

Let $C = \int_{t_0}^{T} c(t)dt$. Because $c(t)$ is integrable on I, it is bounded almost everywhere by a positive constant K.

Theorem 4.4 *Suppose that* F *satisfies assumption (4.4) and* $\bar{X}(t), X(t)$ *are solutions of SCDE (3.1) starting at* $\bar{X}_0, X_0$ *and of* $\bar{U}(t), U(t)$*, respectively. Then one has*

$$D\left[\bar{X}(t), X(t)\right] \le \varepsilon \quad if \quad D\left[\bar{U}(t), U(t)\right] \le \delta(\varepsilon) \quad and \quad D\left[\bar{X}_0, X_0\right] \le \delta(\varepsilon).$$

Proof. The solutions of SCDE (3.1) for controls $\bar{U}(t), U(t)$ originating at $\bar{X}_0, X_0$, respectively, are equivalent to the following integral forms

$$\bar{X}(t) = \bar{X}_0 + \int_{t_0}^{t} F(s, \bar{X}(s), \bar{U}(s))ds$$

$$X(t) = X_0 + \int_{t_0}^{t} F(s,X(s),U(s))ds\ .$$

We estimate

$$\begin{aligned}
&D\left[\bar{X}(t),X(t)\right] \\
&= D\left[\bar{X}_0 + \int_{t_0}^{t} F(s,\bar{X}(s),\bar{U}(s))ds, X_0 + \int_{t_0}^{t} F(s,X(s),U(s))ds\right] \\
&\le D\left[\bar{X}_0, X_0\right] + D\left[\int_{t_0}^{t} F(s,\bar{X}(s),\bar{U}(s))ds, \int_{t_0}^{t} F(s,X(s),U(s))ds\right] \\
&\le D\left[\bar{X}_0, X_0\right] + \int_{t_0}^{t} D\left[F(s,\bar{X}(s),\bar{U}(s)), F(s,X(s),U(s))\right]ds \\
&\le D\left[\bar{X}_0, X_0\right] + \int_{t_0}^{t} c(s)\left\{D\left[\bar{X}(s),X(s)\right] + D\left[\bar{U}(s),U(s)\right]\right\}ds \\
&\le D\left[\bar{X}_0, X_0\right] + \int_{t_0}^{t} c(s)D\left[\bar{X}(s),X(s)\right]ds + \int_{t_0}^{t} c(s)D\left[\bar{U}(s),U(s)\right]ds\ .
\end{aligned}$$

If $D\left[\bar{U}(t),U(t)\right] \le \delta(\varepsilon)$ and $D\left[\bar{X}_0, X_0\right] \le \delta(\varepsilon)$, then

$$D\left[\bar{X}(t),X(t)\right] \le (K+1)\,\delta(\varepsilon) + \int_{t_0}^{t} c(s)D\left[\bar{X}(s),X(s)\right]ds\ .$$

Here we have used (2.5), (2.7).

Using Gronwall inequality, we have

$$D\left[\bar{X}(t),X(t)\right] \le (K+1)\,\delta(\varepsilon)\,exp(C)\ .$$

It follows the proof if we choose $0 < \delta(\varepsilon) \le \dfrac{\varepsilon}{(K+1)exp(C)}$.

5. CONCLUSION

In this paper we gave a new concept of set control differential equation and studied its existence of solutions. A comparison of two solutions was considered. Some more results on existence and comparison of solutions of set control differential equation will be presented in next works.

REFERENCES

[1]. Gnana Bhaskar T, Vasundhara Devi J, *Stability criteria for set differential equations, J. Mathematical and computer modelling*, Vol 4, pp1371-1378, (2005).

[2]. Gnana Bhaskar T, Lakshmikantham V, Vasundhara Devi J; *Nonlinear variation of parameters formula for set differential equations in a metric space*, J. Nonlinear Analysis, Vol 63, pp 735-744, (2005) .

[3]. Galanis G. N, Gnana Bhaskar T, Lakshmikantham V, Palamides P.K; *Set value functions in Frechet spaces: Continuity, Hukuhara differentiability and applications to set differential equations,* J. Nonlinear Analysis Vol 61, pp 559- 575, (2005).

[4]. Lakshmikantham V; *Set differential equations versus fuzzy differential equations*, J. Applied Mathematics and Computation, Vol 164, pp 277-294, (2005).

[5]. Lakshmikantham V, Gnana Bhaskar T, Vasundhara Devi J; *Theory of set differential equations in metric spaces*, Cambridge Scientific Publisher, UK, (2006).

[6]. Lakshmikantham V, Mohapatra R; *Theory of fuzzy differential equations and inclusions,* Taylor & Francis, London, (2003).

[7]. LakshmikanthamV., Leela S; *Fuzzy differential systems and the new concept of stability,* J. Nonlinear Dynamics and Systems Theory, 1(2), pp 111-119, (2001).

[8]. LakshmikanthamV., Leela S., Vatsala A. S; *Interconnection between set and fuzzy differential equations,* J. Nonlinear Analysis, Vol 54, pp 351-360, (2003).

[9]. Phu N. D; *Genaral views in theory of systems*, VNU–Publishing House, HCM City, (2003).

[10]. Phu N.D., Huong N.T; *Multivalued Differential Equations*, VNU – Publishing House, HCM City, (2005).

[11]. Phu N. D, Tung T.T; *Sheaf optimal control problems in fuzzy type,* J. Science and Technology Development, Vol 8 (12), pp 5 -11, (2005).

[12]. Phu N. D, Tung T.T; *The comparison of sheaf- solutions in fuzzy control problems,* J. Science and Technology Development, Vol 9 (2), pp 5 -10, (2006).

[13]. Phu N. D., Tung T.T., *Some Results on Sheaf solutions of Sheaf fuzzy Control Problems*, Electronic Journal of Differential Equations Vol (2006), N.108, pp 1- 8.

[14]. Phu N. D., Tung T.T., *Some Properties of Sheaf solutions of Sheaf set Control Problems,* J. Nonlinear Analysis, Vol 67(2007), pp 1309-1315.

[15]. Phu N. D., Tung T.T, E*xistence of Solutions of Fuzzy Control Differential Equations*, J. Science and Technology Development (In Press).

[16]. Tolstonogov. A; *Differential Inclusions in Banach Space,* Kluwer Academic Publishers, Dordrecht, (2000).