STATIC AND FREE VIBRATION ANALYSES OF PLATES ON ELASTIC FOUNDATION USING A CELL-BASED SMOOTHED THREE-NODE MINDLIN PLATE ELEMENT (CS-MIN3)

Phung Van Phuc
Division of Computational Mechanics, Ton Duc Thang University
Nguyen Thoi Trung
Faculty of Mathematics & Computer Science, University of Science, VNU-HCM
Nguyen Xuan Hung
Faculty of Mathematics & Computer Science, University of Science, VNU-HCM
Tran Trung Dung
Faculty of Civil & Electrical Engineering, Ho Chi Minh City Open University

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ABSTRACT

A cell-based smoothed three-node Mindlin plate element (CS-MIN3) was recently proposed to improve the performance of the existing three-node Mindlin plate element (MIN3) for static and free vibration analyses of Mindlin plates. In this paper, the CS-MIN3 is incorporated with spring systems for treating more complicated static and free vibration analyses of Mindlin plates on the elastic foundation. The plate-foundation system is modeled as a discretization of triangular plate elements supported by discrete springs at the nodal points representing the elastic foundation. The accuracy and reliability of the proposed method are verified by comparing with those of others available numerical results.

Keywords: Smoothed finite element methods (S-FEM), Reissner-Mindlin plate, cell-based smoothed three-node Mindlin plate element (CS-MIN3), elastic foundation, elastic supports.

Introduction

Plates on elastic foundations can be found in several types of engineering structures and real life applications such as basement foundations of building, traffic highways, airport runways, etc.

In the analysis of plates resting on elastic foundation, most of the researcher used the Winkler elastic foundation model and different numerical methods were adopted, such as finite element method [1], boundary element method [2], and others method [3]. A large amount of research has been conducted on analysis of free vibration of structures on elastic foundation by many researchers. For example, Kenney [4] studied vibration of anisotropic plate assemblies with Winler foundation. Raju [5] discussed the effect of mode shape change in the stability problem and vibration behaviour of simply-supported orthotropic rectanggular plates on elastic foundation. More recently, Matsunaga [6] investigated the vibration and stability of thick plates on elastic foundation. Huang et al. [7] studied plate resting on elastic supports and elastic foundation by finite strip method.

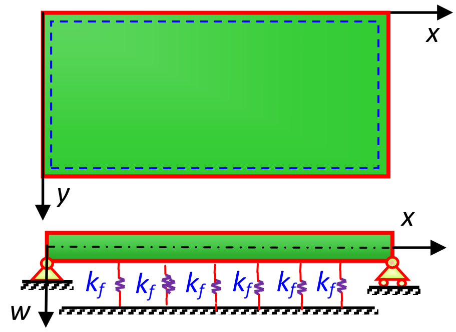
In the other frontier of developing advanced finite element technologies, Liu and Nguyen-Thoi [8] have applied a strain smoothing technique of meshfree methods by Chen [9] into the conventional FEM using linear interpolations to formulate a series of smoothed finite element methods (S-FEM) including the cell-based smoothed FEM (CS-FEM) [10] which shows some interesting properties in the problems of solid mechanics. Extending the idea of the CS-FEM to plate structures, Nguyen-Thoi et al. [11] have recently formulated a cell-based smoothed three-node Mindlin plate element (CS-MIN3) for static and free vibration analyses of isotropic Mindlin plates by incorporating the CS-FEM with the original MIN3 element [12]. In the CS-MIN3, each triangular element will be divided into three sub-triangles, and in each sub-triangle, the stabilized MIN3 is used to compute the strains. Then the strain smoothing technique on whole the triangular element is used to smooth the strains on these three sub-triangles. The numerical results showed that the CS–MIN3 is free of shear locking and achieves the high accuracy compared to the exact solutions and others existing elements in the literature.

This paper hence extends the triangular plate element CS-MIN3 for static and free vibration analyses of plates on elastic foundation. The plate-foundation system is modeled as a discretization of triangular plate elements supported by discrete springs at the nodal points representing the elastic foundation. The accuracy and reliability of the proposed method are verified by comparing with those of others available numerical results.

Weakform for Mindlin plates on elastic foundation

Consider a Mindlin plate on elastic foundation as shown in Figure 1. The elastic foundation is modeled by springs with foundation stiffness coefficient k_f.

Figure 1. Model of a Mindlin thick plate on elastic foundation



The middle (neutral) surface of plate is chosen as the reference plane that occupies a domain $\Omega \subset R^2$ as shown in Figure 2. Let w be the deflection, and $\beta^T=[\beta_x \quad \beta_y]$ be the vector of rotations, where β_x, β_y are the rotations of the middle plane around y-axis and x–axis, respectively, with the positive directions defined as shown in Figure 2.

The unknown vector of three independent field variables at any point in the problem domain of the Mindlin plates can be written as $\mathbf{u}^T = \begin{bmatrix} w & \beta_x & \beta_y \end{bmatrix}$. The bending and shear strains $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ of the deflected plate are defined, respectively, as

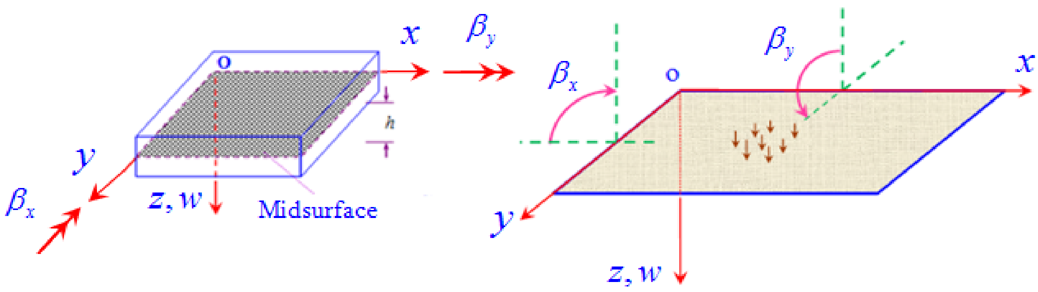
$$\boldsymbol{\kappa} = \mathbf{L}_d \boldsymbol{\beta} \quad ; \quad \boldsymbol{\gamma} = \nabla w + \boldsymbol{\beta} \qquad (1)$$

where $\nabla = \begin{bmatrix} \partial / \partial x & \partial / \partial y \end{bmatrix}^T$ and $\mathbf{L}_d$ is a differential operator matrix. The standard Galerkin weakform of the transient analysis of Mindlin plates on elastic foundation can now be written as Huang [7]

$$\int_\Omega \delta\boldsymbol{\kappa}^T \mathbf{D}^b \boldsymbol{\kappa}\, d\Omega + \int_\Omega \delta\boldsymbol{\gamma}^T \mathbf{D}^s \boldsymbol{\gamma}\, d\Omega + \int_\Omega \delta\mathbf{u}^T \mathbf{m}\, \ddot{\mathbf{u}} d\Omega + \int_\Omega \delta w^T k_f w\, d\Omega = \int_\Omega \delta\mathbf{u}^T \mathbf{b}\, d\Omega \qquad (2)$$

where $\mathbf{b} = \begin{bmatrix} b(x,y) & 0 & 0 \end{bmatrix}^T$, in which $b(x,y)$ is the distributed load applied on the plate; $\mathbf{m}$ is the matrix containing the mass density of the material ρ and thickness t; $\mathbf{D}^b$ and $\mathbf{D}^s$ are the material matrices related to the bending and shear deformations.

Figure 2. Positive directions of displacement w and two rotations β_x, β_y in Mindlin plate



Formulation of the CS-MIN3 for Mindlin plates on elastic foundation

FEM formulation for Mindlind plates on elastic foundation [7]

Now, discretize the bounded domain Ω into N_e finite elements such that $\Omega = \bigcup_{e=1}^{N_e} \Omega_e$ and $\Omega_i \cap \Omega_j \neq \varnothing$, $i \neq j$, then the finite element solution $\mathbf{u}^h = \begin{bmatrix} w & \beta_x & \beta_y \end{bmatrix}^T$ of a displacement model for the Mindlin plates is expressed as:

$$\mathbf{u}^h = \sum_{I=1}^{N_n} \begin{bmatrix} N_I(\mathbf{x}) & 0 & 0 \\ 0 & N_I(\mathbf{x}) & 0 \\ 0 & 0 & N_I(\mathbf{x}) \end{bmatrix} \mathbf{d}_I = \mathbf{N}\mathbf{d} \quad (3)$$

where N_n is the total number of nodes of problem domain discretized; $N_I(\mathbf{x})$ is shape function at node I; $\mathbf{d}_I = [w_I \ \beta_{xI} \ \beta_{yI}]^T$ is the displacement vector of the nodal degrees of freedom of $\mathbf{u}^h$ associated to node I, respectively.

The bending and shear strains can be then expressed in the matrix forms as:

$$\boldsymbol{\kappa} = \sum_I \mathbf{B}_I \mathbf{d}_I, \qquad \boldsymbol{\gamma}^s = \sum_I \mathbf{S}_I \mathbf{d}_I \quad (4)$$

where

$$\mathbf{B}_I = \begin{bmatrix} 0 & N_{I,x} & 0 \\ 0 & 0 & N_{I,y} \\ 0 & N_{I,y} & N_{I,x} \end{bmatrix} \qquad \mathbf{S}_I = \begin{bmatrix} N_{I,x} & N_I & 0 \\ N_{I,y} & 0 & N_I \end{bmatrix} \quad (5)$$

in which $N_{I,x}$ and $N_{I,y}$ are the derivatives of the shape functions in x-direction and y-direction, respectively. The discretized system of equations of Mindlin plates on elastic foundation using the FEM for transient analysis then can be expressed as

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F} \quad (6)$$

where $\mathbf{K}$ is the global stiffness matrix given by

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{D}^b \mathbf{B} \, d\Omega + \int_\Omega \mathbf{S}^T \mathbf{D}^s \mathbf{S} \, d\Omega + \int_\Omega \mathbf{N}_w{}^T k_f \mathbf{N}_w \, d\Omega \quad (7)$$

in which $\mathbf{N}_w = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 \end{bmatrix}^T$; $\mathbf{F}$ is the load vector defined by

$$\mathbf{F} = \int_{\Omega} p\mathbf{N}\,\mathrm{d}\Omega + \mathbf{f}^{b} \tag{8}$$

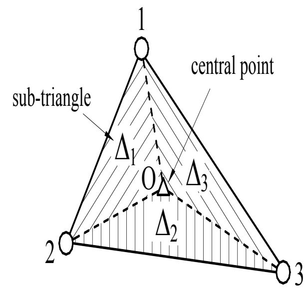
in which $\mathbf{f}^{b}$ is the remaining part of $\mathbf{F}$ subjected to prescribed boundary loads, and $\mathbf{M}$ is the global mass defined by

$$\mathbf{M} = \int_{\Omega} \mathbf{N}^{T}\mathbf{m}\mathbf{N}\ \mathrm{d}\Omega \tag{9}$$

Formulation of CS-MIN3 for Mindlin plates on elastic foundation

In the CS-MIN3 [11], the domain discretization is the same as that of the MIN3 using N_n nodes and N_e triangular elements. However in the formulation of the CS-MIN3, each triangular element Ω_e is further divided into three sub-triangles Δ_1, Δ_2 and Δ_3 by connecting the central point O of the element to three field nodes as shown in Figure 3

Figure 3. Three sub-triangles (Δ_1, Δ_2 and Δ_3) created from the triangle 1-2-3 in the CS-MIN3 by connecting the central point O with three field nodes 1, 2 and 3



In the CS-MIN3, we assume that the displacement vector $\mathbf{d}_{eO}$ at the central point O is the simple average of three displacement vectors $\mathbf{d}_{e1}$, $\mathbf{d}_{e2}$ and $\mathbf{d}_{e3}$ of three field nodes.

$$\mathbf{d}_{eO} = \frac{1}{3}\left(\mathbf{d}_{e1} + \mathbf{d}_{e2} + \mathbf{d}_{e3}\right) \tag{10}$$

Using the MIN3 method for the sub-triangle Δ_1, the bending and shear strains $\boldsymbol{\kappa}^{\Delta_1}$ and $\boldsymbol{\gamma}^{\Delta_1}$ can be approximated by

$$\boldsymbol{\kappa}^{\Delta_1} = \underbrace{\begin{bmatrix}\mathbf{b}_1^{\Delta_1} & \mathbf{b}_2^{\Delta_1} & \mathbf{b}_3^{\Delta_1}\end{bmatrix}}_{\mathbf{b}^{\Delta_1}}\begin{bmatrix}\mathbf{d}_{eO}\\ \mathbf{d}_{e1}\\ \mathbf{d}_{e1}\end{bmatrix} = \mathbf{b}^{\Delta_1}\mathbf{d}^{\Delta_1};\ \boldsymbol{\gamma}^{\Delta_1} = \underbrace{\begin{bmatrix}\mathbf{s}_1^{\Delta_1} & \mathbf{s}_2^{\Delta_1} & \mathbf{s}_3^{\Delta_1}\end{bmatrix}}_{\mathbf{s}^{\Delta_1}}\begin{bmatrix}\mathbf{d}_{eO}\\ \mathbf{d}_{e1}\\ \mathbf{d}_{e1}\end{bmatrix} = \mathbf{s}^{\Delta_1}\mathbf{d}^{\Delta_1} \tag{11}$$

where $\mathbf{b}^{\Delta_1}$ and $\mathbf{s}^{\Delta_1}$ are, respectively, computed similarly as the matrices $\mathbf{B}$ and $\mathbf{S}$ of the MIN3 [11]. Substituting $\mathbf{d}_{eO}$ in Eq. (10) into Eq. (11), we obtain

$$\boldsymbol{\kappa}^{\Delta_1} = \underbrace{\begin{bmatrix}\frac{1}{3}\mathbf{b}_1^{\Delta_1} + \mathbf{b}_2^{\Delta_1} & \frac{1}{3}\mathbf{b}_1^{\Delta_1} + \mathbf{b}_3^{\Delta_1} & \frac{1}{3}\mathbf{b}_1^{\Delta_1}\end{bmatrix}}_{\mathbf{B}^{\Delta_1}}\begin{bmatrix}\mathbf{d}_{e1} & \mathbf{d}_{e2} & \mathbf{d}_{e3}\end{bmatrix}^{T} = \mathbf{B}^{\Delta_1}\mathbf{d}_e \tag{12}$$

$$\boldsymbol{\gamma}^{\Delta_1} = \underbrace{\begin{bmatrix}\frac{1}{3}\mathbf{s}_1^{\Delta_1} + \mathbf{s}_2^{\Delta_1} & \frac{1}{3}\mathbf{s}_1^{\Delta_1} + \mathbf{s}_3^{\Delta_1} & \frac{1}{3}\mathbf{s}_1^{\Delta_1}\end{bmatrix}}_{\mathbf{S}^{\Delta_1}}\begin{bmatrix}\mathbf{d}_{e1} & \mathbf{d}_{e2} & \mathbf{d}_{e3}\end{bmatrix}^{T} = \mathbf{S}^{\Delta_1}\mathbf{d}_e \tag{13}$$

Similarly, by using cyclic permutation, we easily obtain the bending and shear strains $\boldsymbol{\kappa}^{\Delta_j}$, $\boldsymbol{\gamma}^{\Delta_j}$ and matrices $\mathbf{B}^{\Delta_j}$, $\mathbf{S}^{\Delta_j}$, $j=2,3$, for the second sub-triangle Δ_2 (triangle O-2-3) and third sub-triangle Δ_3 (triangle O-3-1), respectively.

Now, applying the cell-based strain smoothing operation in the CS-FEM [10], the constant bending and shear strains $\boldsymbol{\kappa}^{\Delta_j}$ and $\boldsymbol{\gamma}^{\Delta_j}$, $j=1,2,3$ are, respectively, used to create a *smoothed* bending and shear strains $\tilde{\boldsymbol{\kappa}}_e$ and $\tilde{\boldsymbol{\gamma}}_e$ on the element Ω_e such as:

$$\tilde{\boldsymbol{\kappa}}_e = \tilde{\mathbf{B}}_e \mathbf{d}_e \;\; ; \;\; \tilde{\boldsymbol{\gamma}}_e = \tilde{\mathbf{S}}_e \mathbf{d}_e \quad (14)$$

where $\tilde{\mathbf{B}}_e$ and $\tilde{\mathbf{S}}_e$ are the smoothed bending and shear strain gradient matrices given by

$$\tilde{\mathbf{B}}_e = \frac{1}{A_e}\sum_{j=1}^{3} A_{\Delta_j}\mathbf{B}^{\Delta_j} \;\; ; \;\; \tilde{\mathbf{S}}_e = \frac{1}{A_e}\sum_{j=1}^{3} A_{\Delta_j}\mathbf{S}^{\Delta_j} \quad (15)$$

Therefore the global stiffness matrix of the CS-MIN3 are assembled by

$$\tilde{\mathbf{K}} = \sum_{e=1}^{N_e} \tilde{\mathbf{K}}_e \quad (16)$$

where $\tilde{\mathbf{K}}_e$ is the smoothed element stiffness for plate given by

$$\begin{aligned}\tilde{\mathbf{K}}_e &= \int_{\Omega_e} \tilde{\mathbf{B}}^T \mathbf{D}^b \tilde{\mathbf{B}}\, d\Omega + \int_{\Omega_e} \tilde{\mathbf{S}}^T \hat{\mathbf{D}}^s \tilde{\mathbf{S}}\, d\Omega + \int_{\Omega_e} \mathbf{N}_w^T k_f \mathbf{N}_w\, d\Omega \\ &= \tilde{\mathbf{B}}^T \mathbf{D}^b \tilde{\mathbf{B}} A_e + \tilde{\mathbf{S}}^T \hat{\mathbf{D}}^s \tilde{\mathbf{S}} A_e + \int_{\Omega_e} \mathbf{N}_w^T k_f \mathbf{N}_w\, d\Omega\end{aligned} \quad (17)$$

Note that for convenience in numerical computation, the foundation stiffness coefficient k_f in Eq. (17) can be derived from the following equation ref in Huang [7]

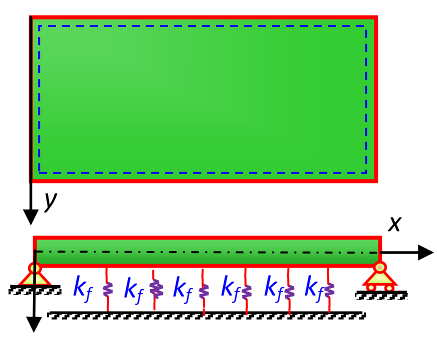
$$k_f = KD/B^4 \quad (18)$$

where K is the non-dimensional elastic foundation coefficient; B is the shorter dimension of the plate; and $D = Et^3/\left(12(1-\nu)\right)$ is the bending stiffness of the plate.

Numerical results

Free vibration analysis of Mindlin plate on elastic foundations

Figure 4. Model of a plate on elastic foundation



Consider the free vibration analysis of a rectangular plate rested on the elastic foundation with the non-dimensional elastic foundation coefficient given by $K = 100$. The dimensions of the plate are given by length L=30m, width B=10m and thickness t=0.5m as shown in Figure 4. The boundary conditions of plate are simply supported along four edges of plate and the density of plate is given by ρ = 2500 kg/m^3.

Figure 5. Five lowest frequencies of the plate on elastic foundation discretized by mesh 15×5

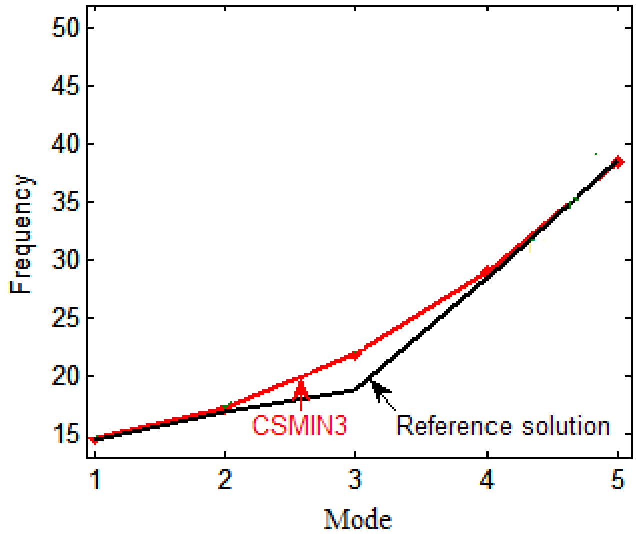
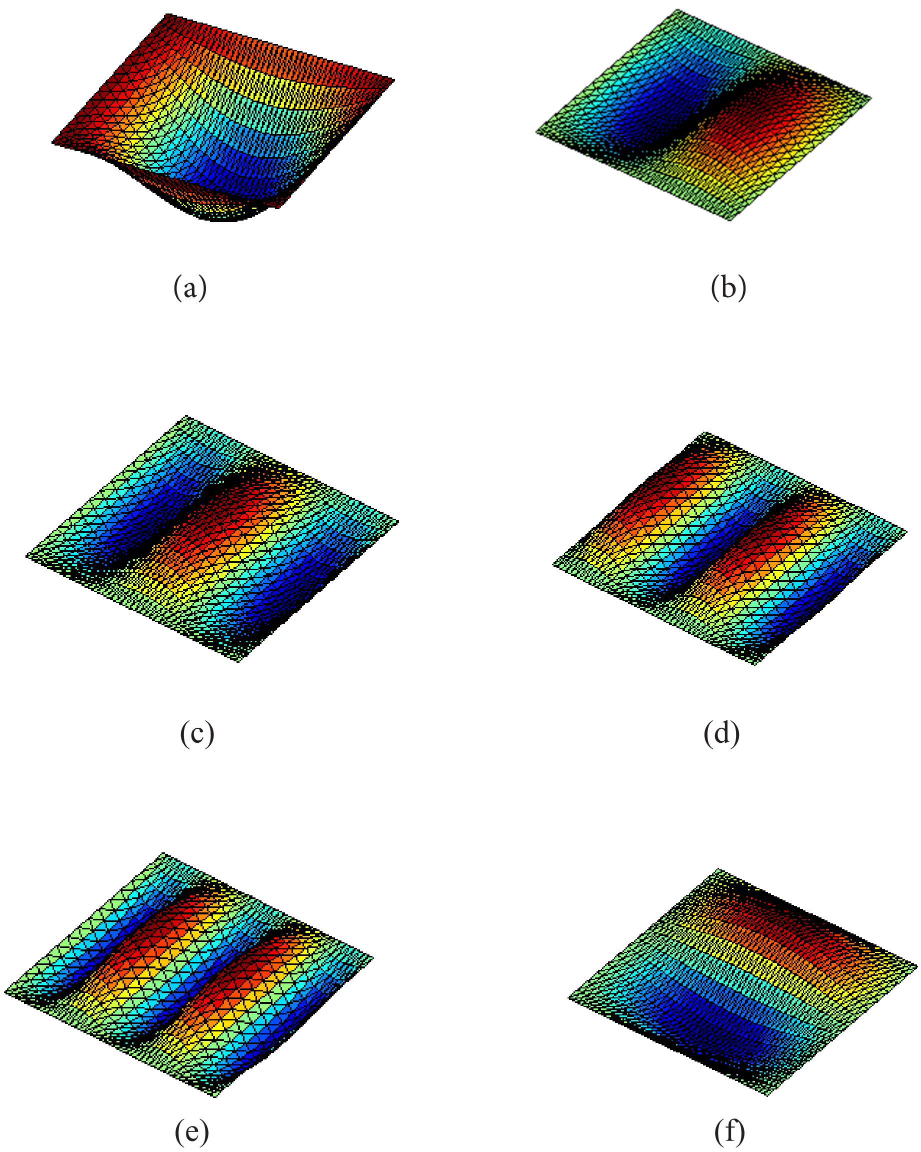


Figure 6. Six lowest eigenmodes of the plate on elastic foundation (mesh 45×15) by the CS-MIN3. (a) Mode 1; (b) Mode 2; (c) Mode 3; (d) Mode 4; (e) Mode 5; (f) Mode 6



(a) (b) (c) (d) (e) (f)

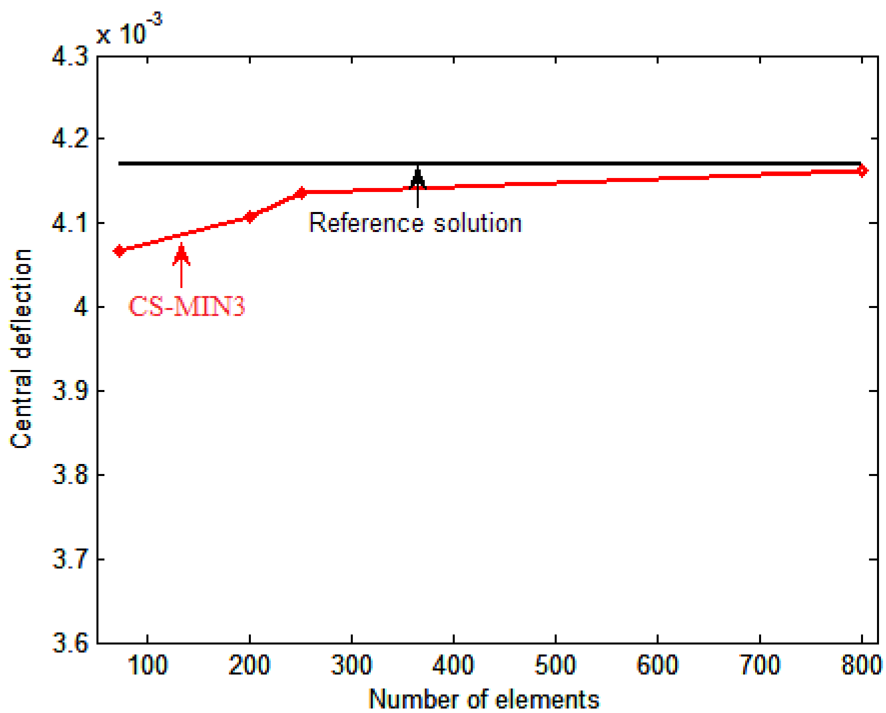
Figure 5 plots five lowest frequencies of plate by CS-MIN3 methods for the meshes 15×5. It is observed that the results of CS-MIN3 agree excellently with the reference solution [7]. In addition, Figure 6 plots the shape of six lowest eigenmodes of plate on the elastic foundation using the CS-MIN3. It is seen that the shapes of eigen-modes reveal the real physical modes.

Static analysis of Mindlin plate on elastic foundations

Now, the static analysis of the plate on the elastic foundation is considered. The model of plate in Figure 4 is still chosen to analysis, however in order to compare the results with those of reference solution [7], the dimensions of the plate are reset into the length *L*=50m, width *B*=10m, thickness *t*=0.02m and the non-dimensional elastic foundation coefficient is given by $K = 1000$. In addition, the plate is subjected to a concentrated load *P*=1000N at the center of plate. The plate is free along two longer edges and is simply supported along the two remaining edges. In addition, Young's modulus of material is given by $E = 31\times10^9\,\mathrm{N/m^2}$ and Poisson's ratio of material is also given by v = 0.2. Four uniform discretizations of plate corresponding to 72, 200, 252 and 800 elements are used.

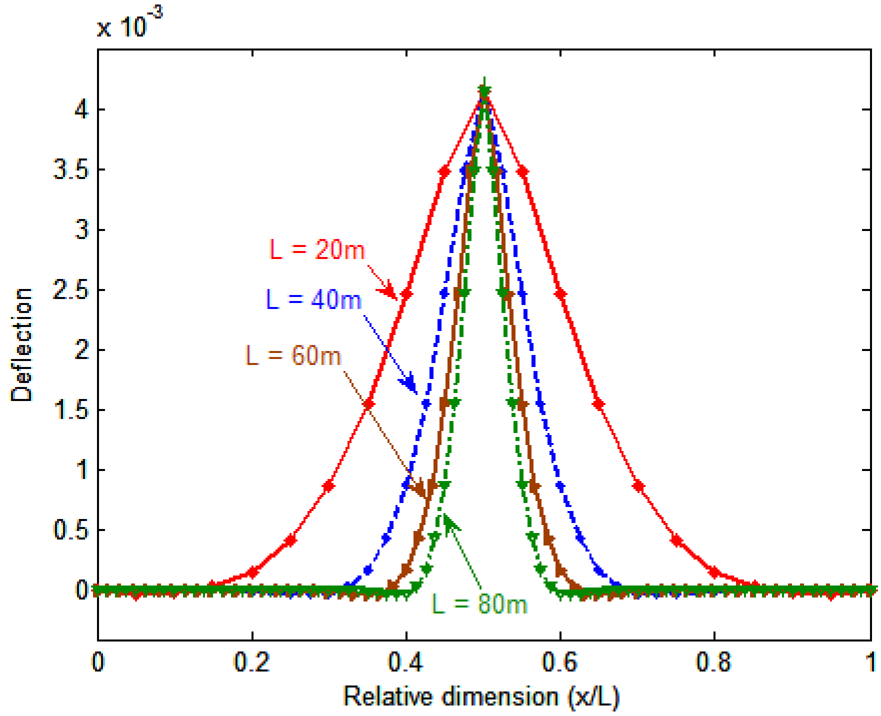
The convergence of deflection is first studied. Figure 7 compares the convergence of central deflection $\bar{w} = wD/(PB^2)$ of plate. It is seen that the solution of the CS-MIN3 is the closest to the reference solution.

Figure 7. Convergence of central deflection $\bar{w} = wD/(PB^2)$ of plate on the elastic foundation



Next the effects of plate dimention ratio *B/L* are studied by keeping the plate width *B*=10m fixed, while the plate length *L* is changed. Figure 8 shows the deflection of the middle-line along the longitudinal axis using the CS-MIN3 when the length *L* of plate is changed. It can be seen that when the length *L* inceases, the region of distribution of deflection is narrower. In addition, this result is quite similar to that in Huang and Thambiratriam [7].

Figure 8. Deflection of the middle-line along the longitudinal axis by the CS-MIN3 when the plate length is changed



Conclusion

The paper presents an incorporation of the original CS-MIN3 with spring systems for treating more complicated static and free vibration analyses of Mindlin plates on the elastic foundation. The plate-foundation system is modeled as a discretization of triangular plate elements supported by discrete springs at the nodal points representing the elastic foundation. The accuracy and reliability of the proposed method are verified by comparing with those of others available numerical results.

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