PARALLEL PROJECTION METHODS FOR SOLVING PROBLEM OF PSEUDO-MONOTONE EQUILIBRIUM AND A FINITE SYSTEM OF NON-EXPANSIVE MAPPINGS

Đỗ Duy Thành, Đỗ Thị Hoài
Khoa Toán và KHTN, Trường Đại học Hải Phòng
Email: thanhdd@dhhp.edu.vn

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ABSTRACT: Fixed point problems and equilibrium problems have many applications and are efficient tools in science, engineering, analytic structures and many other fields. The equilibrium problem in particular is a very general mathematical problem that includes many special cases such as optimization problems, integral inequality problems, fixed point problems, etc. In this article, the authors will propose a weak convergent theorem for an algorithm for finding common solutions of a pseudomonotone equilibrium problem and a finite system of non-extended mappings in a real Hilbert space. Almost existing methods for solving this problem require a strict assumption of the strong monotonicity or Lipschitz-type continuity of the cost bifunction f. The idea of this algorithm is to combine the projection method and the parallel splitting-up technique. At each iteration step, the authors need to use one projection only and do not require to use any Lipschitz-type continuity condition of the bifunction.

Keywords: Fixed point; pseudo-monotonicity; equilibrium; parallel projection method; non-extended mapping.

PHƯƠNG PHÁP CHIẾU SONG SONG GIẢI BÀI TOÁN CÂN BẰNG GIẢ ĐƠN ĐIỆU VÀ MỘT HỌ HỮU HẠN CÁC ÁNH XẠ KHÔNG GIÃN

TÓM TẮT: Bài toán điểm bất động và bài toán cân bằng có rất nhiều ứng dụng trong khoa học, kỹ thuật, trong các cấu trúc giải tích và nhiều lĩnh vực khác. Riêng bài toán cân bằng lại là bài toán tổng quát bao gồm nhiều trường hợp riêng như bài toán tối ưu, bài toán bất đẳng thức tích phân, bài toán điểm bất động,…Trong bài báo này, chúng tôi đã đạt được định lý về sự hội tụ yếu cho thuật toán tìm nghiệm chung của bài toán cân bằng giả đơn điệu và một họ hữu hạn các ánh xạ không giãn trong một không gian Hilbert thực. Hầu hết các phương pháp hiện nay để giải bài toán này đều đòi hỏi giả thiết về tính đơn điệu mạnh hoặc liên tục kiểu Lipschitz của song hàm giá f. Ý tưởng của thuật toán là kết hợp giữa phép chiếu và kỹ thuật song song. Tại mỗi bước lặp, chúng tôi chỉ cần sử dụng một phép chiếu và không cần điều kiện liên tục kiểu Lipschitz của song hàm.

Keyword: Điểm bất động, giả đơn điệu, cân bằng, phương pháp chiếu song song, ánh xạ không giãn.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $f : C \times C \to \mathcal{R}$ be a bifunction such that $f(x,x) = 0$ for all $x \in C$. The equilibrium problem, shortly $EP(C,f)$ [3] is formulated by Find $x^* \in C$ such that $f(x^*,x) \geq 0 \ \forall x \in C.$

Let us denote the solution set of $EP(C,f)$ by $Sol(C,f)$. For each $i \in I := \{1,2,...,n\}$, let $S_i : C \to C$ be nonexpansive mappings. In this section, we consider the following problem of finding a common point of the equilibrium solution set $Sol(C,f)$ and the fixed point set $\cap_{i\in I} Fix(S_i)$:

Problem 1. Find $x^* \in \bigcap_{i\in I} Fix(S_i) \cap Sol(C,f)$.

We recall that a bifunction $f : C \times C \to \mathcal{R}$ is called

- *Strongly monotone* on C with constant $\beta > 0$,

if $f(x,y) + f(y,x) \leq -\beta \| x - y \|^2 \ \ \forall x, y \in C;$

- *Monotone* on C, if $f(x,y) + f(y,x) \leq 0 \ \ \forall x, y \in C;$

- *Pseudomonotone* on C, if $f(x,y) \leq 0 \Rightarrow f(y,x) \leq 0 \ \ \forall x, y \in C;$

- *Lipschitz – typecontinuous* on C,

if $f(x,y) + f(y,z) \leq f(x,z) - c_1 \| x - y \|^2 - c_2 \| y - z \|^2 \ \ \forall x, y, z \in C.$

2. PRELIMINARIES

Methods for solving Problem 1 have been studied extensively by many researchers; see [1], [4], [9], [10]. Almost existing methods for solving Problem 1 require a strict assumption on the strong monotonicity or Lipschitz-type continuity of the cost bifunction f and they used subgradient technique to proof the main results. Here, we only assume that f is monotone and satisfies the paramonotonicity property, and not neccesary Lipschitz-type continuous. We use ϵ-subdifferential of the convex function for this algorithm.

For solving Problem 1, we assume that for each $i \in I$, $S_i : C \to C$ is nonexpansive such that $\Gamma := \bigcap_{i\in I} Fix(S_i) \cap Sol(C,f) \neq \varnothing$ and the bifunction f and parameter sequences $\{\lambda_k\}, \{\delta_k\}$ and $\{\beta_{k,i}\}$ satisfy the following restrictions: (D_1) Jointly continuous on $C \times C$ in the sense that if $x, y \in C$, and $\{x^k\}$ and $\{y^k\}$ are two sequences in C converging weakly to x and y, respectively, then $f(x^k, y^k) \to f(x,y)$.

(D_2) Pseudomonotone on C with respect to every solution of Problem $EP(C,f)$ and satisfies the following condition, called strict paramonotonicity property:

$\{x \in Sol(C,f), y \in C, f(y,x) = 0\} \Rightarrow y \in Sol(C,f).$

(D_3) Let $L > \lambda > 0$ and $0 < a < b < 1$. The sequences $\{\delta_k\} \subset (0,1), \{\beta_{k,j}\}$ and $\{\lambda_k\}$ satisfy $\sum_{k=0}^{\infty} \delta_k = +\infty, \sum_{k=0}^{\infty} \delta_k^2 < +\infty, a \leq \beta_{k,j} \leq b \ \ \forall j \in I$, and $\{\lambda_k\} \subset [\lambda, L];$

Now the parallel projection algorithm for solving Problem 1 is formally stated as the following.

Algorithm 2.1

Initialization: *Choosing* $x^0 \in C$, *and the parameter sequences* $\{\lambda_k\}$ *and* $\{\delta_k\}$ *satisfy* (D_3), *and* $\{\epsilon_k\} \subset (0,\infty)$ *such that* $\sum_{k=0}^{\infty} \delta_k \epsilon_k < \infty$.

Iterative step $k \geq 1$,

$$\begin{cases} \text{Take } w^k \in \partial_2^{\epsilon_k} f(x^k, x^k), \gamma_k = \max\{\lambda_k, \| w^k \|\}, \alpha_k = \dfrac{\delta_k}{\gamma_k} \\ \text{Compute } y^k = Pr_C(x^k - \alpha_k w^k) \\ \text{For each } j \in I, \text{compute } u_j^k = (1-\beta_{k,j})x^k + \beta_{k,j} S_j y^k \\ \text{Set } x^{k+1} = u_{j_0}^k, \text{ where } j_0 := \operatorname{argmax}\{\| u_j^k - y^k \| : j \in I\}. \end{cases}$$

Note that for each $\epsilon > 0, x \in C$, $\partial_2^{\epsilon} f(x,y)$ stands for ϵ-subdifferential of the convex function $f(x,\cdot)$ at $y \in C$ i.e.,

$$\partial_2^{\epsilon} f(x,y) := \{w_y \in \mathcal{H} : f(x,z) - f(x,y) \geq \langle w_y, z-y \rangle - \epsilon \quad \forall z \in C\}.$$

To investigate the convergence of parallel projection method, we recall the following technical lemmas and quasi-Fej\'er convergence which will be used in the sequel.

Lemma 2.2 *([11]) Let* $\{a_k\}$ *and* $\{b_k\}$ *be a sequence of nonnegative real numbers such that* $a_{k+1} \leq a_k + b_k \quad \forall k \geq 0,$ *where* $\sum_{k=0}^{\infty} b_k < \infty$. *Then, the sequence* $\{a_k\}$ *is convergent.*

Lemma 2.3 *([8]) Let* $\mathcal{H}$ *be a real Hilbert space,* $\{\alpha_k\}$ *be a sequence of real numbers such that* $0 < a \leq \alpha_k \leq b < 1$ *for all* $k \geq 0$, *and let* $\{v^k\}, \{w^k\}$ *be sequences of* $\mathcal{H}$ *such that* $\limsup_{k\to\infty} \| v^k \| \leq c, \limsup_{k\to\infty} \| w^k \| \leq c$ *and* $\lim_{k\to\infty} \| \alpha_k v^k + (1-\alpha_k) w^k \| = c.$

Then, $\lim_{k\to\infty} \| v^k - w^k \| = 0$.

Lemma 2.4 *([2]) Let* C *be a nonempty closed convex subset in* $\mathcal{H}$. *The metric projection from* $\mathcal{H}$ *onto* C *is denoted by* Pr_C *and* $Pr_C(x) = \operatorname{argmin}\{\| x - y \| : y \in C\} \quad \forall x \in \mathcal{H}.$

The following properties hold:

(i) $\langle x - Pr_C(x), y - Pr_C(x) \rangle \leq 0 \quad \forall y \in C, x \in \mathcal{H};$

(ii) $\langle Pr_C(x) - Pr_C(y), x - y \rangle \geq \| Pr_C(x) - Pr_C(y) \|^2 \quad \forall x, y \in \mathcal{H};$

(iii) $\| x - Pr_C(x) \|^2 \leq \| x - y \|^2 - \| y - Pr_C(x) \|^2 \; \forall x \in \mathcal{H}, y \in C$

Lemma 2.5 *(Opial) Let* $\{x^k\}$ *be a sequence in* $\mathcal{H}$ *such that* $x^k \rightharpoonup \bar{x}$. *Then, for all* $y \neq \bar{x}$, *we have* $\liminf_{k\to\infty} \| x^k - \bar{x} \| < \liminf_{k\to\infty} \| x^k - y \|$.

We next deal with the so called quasi-Fej\'er convergence and its properties.

Definition 2.6 *Let* S *be a nonempty subset in* $\mathcal{H}$. *A sequence* $\{x^k\}$ *in* $\mathcal{H}$ *is said to bequasi-Fej\'er convergent to* S *if and only if for all* $x^* \in S$ *there exist* $k_0 \geq 0$ *and a sequence* $\{\alpha_k\} \subset (0,\infty)$ *such that* $\sum_{k=0}^{\infty} \alpha_k < \infty$ and $\| x^{k+1} - x^* \|^2 \leq \| x^k - x^* \|^2 + \alpha_k \quad \forall k \geq k_0$.

This definition originates in [6] and has been elaborated further in [5], [7].

Lemma 2.7 *([7], Theorem 4.1)*

Let S *be a nonempty subset in* $\mathcal{H}$. *If* $\{x^k\}$ *is quasi-Fej\'er convergent to* S, *then*

(i) The sequence $\{x^k\}$ *is bounded;*

(ii) If all weak cluster points of $\{x^k\}$ *belong to* S, *then the sequence* $\{x^k\}$ *is weakly convergent to a point of* S.

III. CONVERGENCE THEOREMS

Lemma 3.1 *Let* $f : C \times C \to \mathcal{R}$ *be pseudomonotone and the conditions* (D_3) *be satisfied. Then,*

(i) The sequence $\{x^k\}$ *generated by Algorithm 2.1 is quasi-Fej\'er convergent to* Γ *and the following inequality holds:*

$$\| x^{k+1} - x^* \|^2 \leq \| x^k - x^* \|^2 + \frac{2}{\lambda}\beta_{k,j_0}\delta_k\epsilon_k + 2\beta_{k,j_0}\delta_k^2 \quad \forall x^* \in \Gamma.$$

(ii) For each $x^* \in \Gamma$, $\limsup_{k\to\infty} f(x^k, x^*) = 0$.

Proof. (i) For each $x^* \in \Gamma$. From $x^{k+1} = u_{j_0}^k$ and $x^* \in \Gamma$, it follows $x^* \in Fix(S_{j_0})$ and

$$\begin{aligned}
\| x^{k+1} - x^* \|^2 &= \| u_{j_0}^k - x^* \|^2 = \| (1-\beta_{k,j_0})x^k + \beta_{k,j_0} S_{j_0} y^k - x^* \|^2 \\
&\leq (1-\beta_{k,j_0}) \| x^k - x^* \|^2 + \beta_{k,j_0} \| S_{j_0} y^k - x^* \|^2 \\
&\leq (1-\beta_{k,j_0}) \| x^k - x^* \|^2 + \beta_{k,j_0} \| y^k - x^* \|^2 \\
&= (1-\beta_{k,j_0}) \| x^k - x^* \|^2 \\
&\quad + \beta_{k,j_0} \left[\| x^k - x^* \|^2 - \| x^k - y^k \|^2 + 2\langle x^k - y^k, x^* - y^k \rangle \right] \\
&\leq \| x^k - x^* \|^2 + 2\beta_{k,j_0} \langle x^k - y^k, x^* - y^k \rangle. \quad (3.1)
\end{aligned}$$

Using Lemma 2.4 (i) for $x := x^k - \alpha_k w^k$ and $y := x^* \in \Gamma \subseteq C$, we get

$\langle x^k - y^k, x^* - y^k \rangle \leq \alpha_k \langle w^k, x^* - y^k \rangle$.

Combining this and (3.1), we have

$$\begin{aligned}
\| x^{k+1}-x^* \|^2 &\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\langle x^k-y^k, x^*-y^k\rangle \\
&\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\langle w^k, x^*-y^k\rangle \\
&= \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\langle w^k, x^*-x^k\rangle + 2\beta_{k,j_0}\alpha_k\langle w^k, x^k-y^k\rangle \\
&\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\langle w^k, x^*-x^k\rangle + 2\beta_{k,j_0}\alpha_k \| w^k \| . \| x^k-y^k \| \\
&\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\langle w^k, x^*-x^k\rangle \\
&+2\beta_{k,j_0}\frac{\delta_k}{\max\left\{\lambda_k, \| w^k \|\right\}} \| w^k \| . \| x^k-y^k \| \\
&\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\langle w^k, x^*-x^k\rangle \quad (3.2) \\
&\quad + 2\beta_{k,j_0}\delta_k \| x^k-y^k \| .
\end{aligned}$$

By Lemma 2.4 (ii), where $x := x^k = Pr_C(x^k)$ and $y := x^k - \alpha_k w^k$, we have

$$\langle Pr_C(x^k) - Pr_C(x^k-\alpha_k w^k), x^k - [x^k-\alpha_k w^k]\rangle \geq \| Pr_C(x^k) - Pr_C(x^k-\alpha_k w^k) \|^2$$

and hence

$$\begin{aligned}
\| x^k-y^k \|^2 &\leq \alpha_k\langle x^k-y^k, w^k\rangle \leq \alpha_k \| x^k-y^k \| . \| w^k \| \\
&= \frac{\delta_k}{\max\left\{\lambda_k, \| w^k \|\right\}} \| w^k \| . \| x^k-y^k \| \\
&\leq \delta_k \| x^k-y^k \| .
\end{aligned}$$

It implies $\| x^k-y^k \| \leq \delta_k$. (3.3)

By the pseudomonotonicity of $f, x^* \in Sol(C, f)$ and $x^k \in C$, we have

$$f(x^*, x^k) \geq 0 \Rightarrow f(x^k, x^*) \leq 0.$$

Using $w^k \in \partial_2^{\epsilon_k} f(x^k, x^k)$ and $f(x,x)=0$ for all $x \in C$, we have

$$\langle w^k, x^*-x^k\rangle \leq f(x^k, x^*) - f(x^k, x^k) + \epsilon_k = f(x^k, x^*) + \epsilon_k \leq \epsilon_k.$$

Since this, (3.2) and (3.3), we obtain

$$\begin{aligned}
\| x^{k+1}-x^* \|^2 &\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\langle w^k, x^*-x^k\rangle + 2\beta_{k,j_0}\delta_k \| x^k-y^k \| \\
&\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k[f(x^k, x^*)+\epsilon_k] + 2\beta_{k,j_0}\delta_k^2 \quad (3.4) \\
&\leq \| x^k-x^* \|^2 +2\beta_{k,j_0}\alpha_k\epsilon_k + 2\beta_{k,j_0}\delta_k^2 \\
&= \| x^k-x^* \|^2 +2\beta_{k,j_0}\frac{\delta_k}{\max\left\{\lambda_k, \| w^k \|\right\}}\epsilon_k + 2\beta_{k,j_0}\delta_k^2 \\
&\leq \| x^k-x^* \|^2 +\frac{2}{\lambda}\beta_{k,j_0}\delta_k\epsilon_k + 2\beta_{k,j_0}\delta_k^2 .
\end{aligned}$$

(ii) From (3.4), it follows that

$$2\beta_{k,j_0}\alpha_k[-f(x^k,x^*)] \le \| x^k - x^* \|^2 - \| x^{k+1} - x^* \|^2 + \frac{2}{\lambda}\beta_{k,j_0}\delta_k\epsilon_k + 2\beta_{k,j_0}\delta_k^2. \quad (3.5)$$

By (i) that the sequence $\{x^k\}$ generated by Algorithm 2.1 is quasi-Fej\'er convergent to Γ. Using this and Lemma 2.7 (i), the sequence $\{x^k\}$ is bounded. Since f is jointly continuous on C, there exists a number M such that $\| w^k \| \le M \ \forall k.$ Thus, summing the inequalities in (3.5) and using the assumption $\sum_{k=0}^{\infty}\delta_k^2 < \infty$ and $\sum_{k=0}^{\infty}\delta_k\epsilon_k < \infty$, we obtain

$$0 \le \frac{2a}{\max\{L,M\}}\sum_{k=0}^{\infty}\delta_k[-f(x^k,x^*)] \le 2a\sum_{k=0}^{\infty}\frac{\delta_k}{\max\left\{\lambda_k, \| w^k \|\right\}}[-f(x^k,x^*)]$$

$$= 2\sum_{k=0}^{\infty}a\alpha_k[-f(x^k,x^*)]$$

$$\le 2\sum_{k=0}^{\infty}\beta_{k,j_0}\alpha_k[-f(x^k,x^*)] \le \| x^0 - x^* \|^2 + 2\sum_{k=0}^{\infty}\beta_{k,j_0}\delta_k^2 + \frac{2}{\lambda}\sum_{k=0}^{\infty}\beta_{k,j_0}\delta_k\epsilon_k < \infty.$$

This implies $0 \le \sum_{k=0}^{\infty}\delta_k[-f(x^k,x^*)] < \infty.$

In Condition (D_3), we have $\sum_{k=0}^{\infty}\delta_k = \infty$ and $\delta_k > 0$ for all $k \ge 0$. Thus, we conclude $\liminf_{k\to\infty}[-f(x^k,x^*)] = 0,$ which completes the proof.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that Conditions $(D_1)-(D_3)$ are satisfied. Then, the sequences $\{x^k\}$ and $\{y^k\}$ generated by Algorithm 1 converge weakly to the same point $\bar{x} \in \Gamma$.*

Proof. By Lemma 2.7 (i), the sequence $\{x^k\}$ is quasi--Fej\'er convergent to Γ. Hence, by Lemma 3.1 (i), the sequence $\{x^k\}$ is bounded and if all the weak cluster points of $\{x^k\}$ belong to Γ, then the sequence $\{x^k\}$ converges weakly to a point of Γ. Let $\bar{x}$ be any weak cluster point of $\{x^k\}$. Then $\bar{x} \in C$ and without loss of generality, we can assume that

$$x^{k_j} \rightharpoonup \bar{x} \text{ and } \limsup_{k\to\infty} f(x^k,x^*) = \lim_{j\to\infty} f(x^{k_j},x^*).$$

It remains to prove that $\bar{x} \in \Gamma$ to get that $\{x^k\}$ converges weakly to a point in Γ, and also, thanks (3.3), that $\{y^k\}$ converges weakly to the same point. The proof of $\bar{x} \in \Gamma$ is divided into several steps.

Step1. Claim $\bar{x} \in Sol(C,f)$.

Since Condition (D_1), (3.8) and Lemma (3.1) (ii), we have

$$f(\bar{x},x^*) \geq \limsup_{j\to\infty} f(x^{k_j},x^*) \;\; = \lim_{j\to\infty} f(x^{k_j},x^*)\rangle = \limsup_{k\to\infty} f(x^k,x^*) = 0$$

Combining this and the pseudomonotonicity of f then $f(x^*,\bar{x}) \geq 0 \Rightarrow f(\bar{x},x^*) \leq 0,$

we deduce $f(\bar{x},x^*) = 0.$ Then, since f is pseudomonotone on C with respect to every solution of Problem 1 and satisfies strict paramonotonicity property in Condition (D_1), we obtain $\bar{x} \in Sol(C,f)$.

*Step*2. Claim $\bar{x} \in Fix(S_j)$ for all $j \in I$.

Applying Lemma 2.2 for $a_k := \| x^k - x^* \|^2$ and $b_k := \frac{2}{\lambda}\beta_{k,j_0}\delta_k\epsilon_k + 2\beta_{k,j_0}\delta_k^2$ and using Lemma 3.1 (i), there exists the limit $c = \lim_{k\to\infty} \| x^k - x^* \|^2 < \infty.$ (3.9)

From (3.3), it follows

$$\| S_{j_0} y^k - x^* \| = \| S_{j_0} y^k - S_{j_0} x^* \| \leq \| y^k - x^* \| \leq \| y^k - x^k \| + \| x^k - x^* \|$$

$$\leq \delta_k + \| x^k - x^* \| \to \sqrt{c} \text{ as } k \to \infty. \quad (3.10)$$

Otherwise $\lim_{k\to\infty} \| (1-\beta_{k,j_0})(x^k - x^*) + \beta_{k,j_0}(S_{j_0} y^k - x^*) \| = \lim_{k\to\infty} \| x^{k+1} - x^* \| = \sqrt{c}.$ (3.11)

Since (3.9), (3.10) and (3.11), we can conclude by Lemma 2.3 that $\lim_{k\to\infty} \| S_{j_0} y^k - x^k \| = 0.$

Thus, from (3.3) we conclude that

$$\begin{aligned} \| S_{j_0} x^k - x^k \| &\leq \| S_{j_0} x^k - S_{j_0} y^k \| + \| S_{j_0} y^k - x^k \| \\ &\leq \| x^k - y^k \| + \| S_{j_0} y^k - x^k \| \\ &\leq \delta_k + \| S_{j_0} y^k - x^k \| \to 0 \text{ as } k \to \infty. \end{aligned}$$

By the same argument, we also get

$$\begin{aligned} \| x^{k+1} - y^k \| &= \| (1-\beta_{k,j_0})x^k + \beta_{k,j_0} S_{j_0} y^k - y^k \| \leq \| x^k - y^k \| + \beta_{k,j_0} \| S_{j_0} y^k - y^k \| \\ &\leq \| x^k - y^k \| + \beta_{k,j_0}\left[\| S_{j_0} y^k - S_{j_0} x^k \| + \| S_{j_0} x^k - y^k \| \right] \\ &\leq \| x^k - y^k \| + \beta_{k,j_0}\left[\| S_{j_0} y^k - S_{j_0} x^k \| + \| S_{j_0} x^k - x^k \| + \| y^k - x^k \| \right] \\ &\leq (1+2\beta_{k,j_0}) \| x^k - y^k \| + \beta_{k,j_0} \| S_{j_0} x^k - x^k \| \to 0 \text{ as } k \to \infty. \end{aligned}$$

Then, using $\| u_j^k - y^k \| \leq \| u_{j_0}^k - y^k \| = \| x^{k+1} - y^k \|$ for every $j \in I$, we have

$$\| S_j y^k - y^k \| = \left\| \frac{u_j^k - (1-\beta_{k,j})x^k}{\beta_{k,j}} - y^k \right\| = \frac{1}{\beta_{k,j}} \left\| (u_j^k - y^k) + (1-\beta_{k,j})(y^k - x^k) \right\|$$

$$\le \frac{1}{\beta_{k,j}}\left[\| u_j^k - y^k \| + (1-\beta_{k,j})\| y^k - x^k \|\right] \le \frac{1}{a}\left[\| x^{k+1} - y^k \| + (1-a)\delta_k\right] \to 0 \text{ as } k \to \infty. \quad (3.12)$$

Now we will show that $S_j(\bar{x}) = \bar{x}$ for all $j \in I$. Indeed, suppose that there exists $i \in I$ such that $S_i(\bar{x}) \neq \bar{x}$. Since (3.12) and the Opial condition in Lemma 2.5, we have

$$\liminf_{j\to\infty} \| y^{k_j} - \bar{x} \| < \liminf_{j\to\infty} \| y^{k_j} - S_i\bar{x} \| \le \liminf_{j\to\infty}\left[\| y^{k_j} - S_i y^{k_j} \| + \| S_i y^{k_j} - S_i\bar{x} \|\right]$$

$$\le \liminf_{j\to\infty}\left[\| y^{k_j} - S_i y^{k_j} \| + \| y^{k_j} - \bar{x} \|\right] \le \liminf_{j\to\infty} \| y^{k_j} - \bar{x} \|.$$

This is a contradiction. Thus $y^{k_j} \rightharpoonup \bar{x} \in \cap_{j\in I} Fix(S_j)$ as $j \to \infty$.

CONCLUSIONS

In this paper, we introduce a parallel projection method for finding a common element of the set of solutions of the pseudomonotone equilibrium problem and the set of fixed points of a finite system of nonexpansive mappings.We propose a weak convergent in a real Hilbert space. We only assume that f is monotone and satisfies the paramonotonicity property, and not neccesary Lipschitz-type continuous.

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