

THE EXISTENCE OF THE GLOBAL ATTRACTOR FOR 2D g -Bénard PROBLEM

Trần Quang Thịnh¹, Nguyễn Đình Thi², Vũ Tuấn Anh³
^{1,2}*ĐHSP Kỹ thuật Nam Định*, ³*Đại học Hàng hải Việt Nam*
Email: thinhspktnd@gmail.com

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ABSTRACT: We consider the initial boundary value problem for 2D g -Bénard equations in a bounded domain with homogeneous Dirichlet boundary conditions. After studying the well-posedness of the problem, one is interested in asymptotic behavior of solutions when time is large or tends to infinity, as it allows us to understand and eventually predict the development trends of such systems in future, then we can make the appropriate adjustments to achieve the desired results. Specifically, for each given orbit of the system and an arbitrary time interval T , we can find an orbit lying on the global attractor whose behavior when the time is large enough of these two orbits is small enough in difference throughout the length T . In the paper, we study the existence of the global attractor of the 2D g -Bénard equations. This is a compact invariant set, which contains much information about the asymptotic behavior of solutions. Our results extend some outcomes by T.Q. Thinh and L.T. Thuy in *Hnue journal of science (2020)*

Keywords: g -Bénard; global attractor; the behavior of solution; Dirichlet boundary; g -Navier-Stokes.

SỰ TỒN TẠI TẬP HÚT TOÀN CỤC CHO BÀI TOÁN 2D g -Bénard

TÓM TẮT: Chúng tôi xét bài toán biên ban đầu cho phương trình g -Bénard 2D trong miền bị chặn với điều kiện biên Dirichlet thuần nhất. Sau khi nghiên cứu tính đặt đúng của bài toán, ta thường quan tâm đến dáng điệu tiệm cận nghiệm khi thời gian tiến ra vô cùng, vì nó cho phép ta hiểu và dự đoán được xu thế phát triển của hệ trong tương lai, sau đó chúng ta có thể thực hiện những điều chỉnh phù hợp để đạt được kết quả như mong muốn. Cụ thể, với mỗi quỹ đạo cho trước của hệ và một khoảng thời gian T tùy ý, ta đều tìm được một quỹ đạo nằm trên tập hút toàn cục mà dáng điệu khi thời gian đủ lớn của hai quỹ đạo này sai khác đủ nhỏ trên một khoảng có độ dài T . Trong bài báo này, chúng tôi nghiên cứu sự tồn tại của tập hút toàn cục của phương trình g -Bénard hai chiều. Tập hút toàn cục là tập bất biến, chứa nhiều thông tin về dáng điệu tiệm cận của nghiệm. Kết quả của chúng tôi là mở rộng kết quả của T.Q. Thịnh và L.T. Thuy trong *Hnue journal of science (2020)*.

Từ khóa: g -Bénard; tập hút toàn cục; dáng điệu nghiệm; biên Dirichlet, g -Navier-Stokes.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^2 with smooth boundary Γ . We consider the following two-dimensional (2D) autonomous g -Bénard problem

$$\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \xi \theta + f_1, & \Omega \times [0, \infty), \\
\nabla \cdot (gu) = 0, & \Omega \times [0, \infty), \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta - \kappa \Delta \theta - \frac{2\kappa}{g} (\nabla g \cdot \nabla)\theta - \frac{\kappa \Delta g}{g} \theta = f_2, & \Omega \times [0, \infty), \\
u = 0 & \text{on } (0, \infty) \times \Gamma, \\
\theta = 0 & \text{on } (0, \infty) \times \Gamma, \\
u(x, 0) = u_0(x), & x \in \Omega, \\
\theta(x, 0) = \theta_0(x), & x \in \Omega,
\end{cases} \quad (1.1)$$

where $u \equiv u(x, t) = (u_1, u_2)$ is the unknown velocity vector, θ is the temperature, $p \equiv p(x, t)$ is the unknown pressure, f_1 is the external force function, f_2 is the heat source function, $\nu > 0$ is the kinematic viscosity coefficient, ξ is a constant vector, $\kappa > 0$ is thermal diffusivity, u_0 is the initial velocity and θ_0 is the initial temperature.

As deduced and mentioned in [8], the 2D g -Bénard problem arises naturally when we study the standard 3D Bénard problem on thin domains $\Omega_g = \Omega \times (0, g)$. Here, the g -Bénard problem is a binary system consisting of the g -NVS equations and the advection distributed heat equation to model convection in the fluid. Furthermore, when $g \equiv \text{const}$ we get the usual Bénard problem and when $\theta \equiv 0$ we get the g -NVS equations. Below we will list some related results.

The existence, uniqueness of weak solutions and finite dimensional global attractors of the 2D Bénard problem have been studied in [3] in the autonomous case and in [1] in the non-autonomous case. In the three-dimensional case, the solutions are not unique and the multivalued dynamical system generated by the problem has been studied in [5].

The 2D g -NVS equations and its relationship with the 3D NVS equations in the thin domain Ω_g were introduced in [7] to derive the 2D -NVS equations from the 3D NVS equations. Since then there has been a great deal of work devoted to the study of mathematical questions related to these equations. In particular, the existence and long-term behavior of solutions to the 2D -NVS equation have been extensively studied over the years, in both autonomous and non-autonomous cases, see e.g. [2] and references therein. The existence of cyclic time solutions of the g -NVS and g -Kelvin-Voight equations was also studied more recently in [4].

For the 2D g -Bénard problem (1.1), in [8] Hitherto, M. Özlük and M. Kaya considered Boussinesq equations in bounded domain $\Omega_g = \{f(y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \Omega_2; 0 < y_3 < g\}$, where Ω_2 is a bounded region in the plane and $g = g(y_1, y_2)$ is a smooth function defined on Ω_2 . They proved the existence and uniqueness of weak solutions and derived upper bounds for the number of determining modes. More recently, in [9] M. Özlük and M. Kaya investigated the existence, uniqueness of strong solutions, and the continuous dependence of the solutions on the viscosity parameter for (1.1) problem in the non-autonomous case and the function g to be periodic with period 1 in the x_1 and x_2 directions.

The aim of this paper is to study the existence of the global attractor for the 2D g -Bénard equations with homogeneous Dirichlet boundary conditions.

To do this, we assume that the functions f_1, f_2, g satisfy the following hypothesis

$$(F) \quad f_1 \in H_g, f_2 \in L^2(\Omega, g).$$

$$(G) \quad g \in W^{1,\infty}(\Omega) \text{ such that}$$

$$0 < m_0 \leq g(x) \leq M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty^2 < \frac{cm_0^3}{4M_0},$$

where c is the Poincaré constant on Ω .

The paper is organized as follows. In Section 2, we prove the existence of a global attractor for the continuous semigroup generated by weak solutions. In particular, the obtained results extend and improve some previous results for the 2D Bénard problem in [3] and the 2D g -NVS equations in [6].

2. PRELIMINARIES

Set $\mathbb{L}^2(\Omega, g) = (L^2(\Omega, g))^2$ and $\mathbb{H}_0^1(\Omega, g) = (H_0^1(\Omega, g))^2$ be endowed, respectively.

Defines function spaces

$$\mathcal{V}_1 = \{u \in (C_0^\infty(\Omega, g))^2 : \nabla \cdot (gu) = 0\},$$

$$H_g = \text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{L}^2(\Omega, g),$$

$$V_g = \text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{H}_0^1(\Omega, g),$$

$$V'_g = \text{the dual space of } V_g,$$

$$\mathcal{V}_2 = \{\theta \in C_0^\infty(\Omega, g)\},$$

$$W_g = \text{the closure of } \mathcal{V}_2 \text{ in } H_0^1(\Omega, g),$$

$$W'_g = \text{the dual space of } W_g,$$

$$V = V_g \times W_g, H = H_g \times L^2(\Omega, g).$$

The inner product and norm in V_g, H_g are given by

$$(u, v)_g = \int_{\Omega} u \cdot v g dx, u, v \in H_g,$$

$$\text{and } ((u, v))_g = \int_{\Omega} \sum_{i,j=1}^2 \nabla u_j \cdot \nabla v_i g dx, u, v \in V_g,$$

and norms $\|u\|_g^2 = (u, u)_g, \|u\|_g^2 = ((u, u))_g$. The norms $|\cdot|_g$ and $\|\cdot\|_g$ are equivalent to the usual ones in $\mathbb{L}^2(\Omega, g)$ and $\mathbb{H}_0^1(\Omega, g)$ with u and also in $L^2(\Omega, g)$ and $H_0^1(\Omega, g)$ with θ , respectively.

We also use $\|\cdot\|_g$ for the norm in V'_g , and $\langle \cdot, \cdot \rangle$ for duality pairing between V_g and V'_g . The inclusions

$V_g \subset H_g \equiv H_{g'} \subset V_{g'}, W_g \subset L^2(\Omega, g) \subset W_{g'}$ are valid where each space is dense in the following one and the injections are continuous. By the Riesz representation theorem, it is possible to write

$$\langle f, u \rangle_g = (f, u)_g, \forall f \in H_g, \forall u \in V_g.$$

Also, we define the orthogonal projection P_g as $P_g : L^2(\Omega, g) \rightarrow H_g$ and \tilde{P}_g as $\tilde{P}_g : L^2(\Omega, g) \rightarrow W_g$. By taking into account the following equality,

$$-\frac{1}{g}(\nabla \cdot g \nabla u) = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u,$$

we define the g -Laplace operator and g -Stokes operator as $-\Delta_g u = -\frac{1}{g}(\nabla \cdot g \nabla u)$ and $A_g u = P_g[-\Delta_g u]$, respectively. For the g -Stokes operator A_g , the followings hold (see [3]):

(1) The g -Stokes operator A_g is a positive, self-adjoint operator with compact inverse, where the domain of A_g is $D(A_g) = V_g \cap H^2(\Omega, g)$.

(2) There exist countable eigenvalues of A_g satisfying

$$0 < \lambda_g \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

where $\lambda_g \geq \frac{cm_0}{M_0}$ and λ_1 is the smallest eigenvalue of A_g . In addition, there exists the corresponding collection of eigenfunctions $\{w_i\}_{i \in \mathbb{N}}$ forms an orthonormal basis for H_g .

Since the operators A_g and P_g are self-adjoint, using integration by parts we have

$$\langle A_g u, u \rangle_g = \langle P_g[-\frac{1}{g}(\nabla \cdot g \nabla)u], u \rangle_g = \int_{\Omega} (\nabla u \cdot \nabla u) g dx = \langle \nabla u, \nabla u \rangle_g.$$

Therefore, for $u \in V_g$ we can write $|A_g^{1/2} u|_g = |\nabla u|_g = \|u\|_g$.

Next, since the functional

$$\tau \in W_g \mapsto (\nabla \theta, \nabla \tau)_g \in \mathbb{R}$$

is a continuous linear mapping on W_g , we can define a continuous linear mapping \tilde{A}_g on W_g , such that $\forall \tau \in W_g, \langle \tilde{A}_g \theta, \tau \rangle_g = (\nabla \theta, \nabla \tau)_g$, for all $\theta \in W_g$.

Let us define the bilinear operator $A : V \rightarrow V'$ by

$$\langle Az, \tilde{z} \rangle_{V, V'} = v \langle A_g u, \tilde{u} \rangle + \gamma \kappa \langle \tilde{A}_g \theta, \tilde{\theta} \rangle,$$

where $z = (u, \theta), \tilde{z} = (\tilde{u}, \tilde{\theta}) \in H$ and γ is defined so that it satisfies

$$\gamma \geq \frac{2 \|\xi\|_{\infty}^2}{v \kappa \lambda_1^2} \left(1 - \frac{2 \|\nabla g\|_{\infty}}{m_0 \lambda_1^2}\right)^{-1}. \quad (2.1)$$

This constant γ is chosen so that an operator to be defined later (related to the linear part of the system of equations) is coercive under the norm defined above. We have

$$\langle Az, z \rangle_{V, V'} = \nu \|u\|^2 + \gamma \kappa \|\theta\|^2. \quad (2.2)$$

We denote the bilinear operator $B_g(u, v) = P_g[(u \cdot \nabla)v]$ and the trilinear form

$$b_g(u, v, w) = \sum_{i,j=1}^n \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

where u, v, w lie in appropriate subspaces of $L^2(\Omega, g)$. Then, one obtains $b_g(u, v, w) = -b_g(u, w, v)$, which implies that $b_g(u, v, v) = 0$.

Similarly, for $u \in V_g$ and $\theta, \tau \in W_g$ we define $\tilde{B}_g(u, \theta) = \tilde{P}_g[(u \cdot \nabla)\theta]$ and

$$\tilde{b}_g(u, \theta, \tau) = \sum_{i,j=1}^n \int_{\Omega} u_i(x) \frac{\partial \theta(x)}{\partial x_j} \tau(x) g dx.$$

We denote $B(z) = B(z, z)$, its associated bilinear operator $B: V \times V \rightarrow V'$ by

$$b(z, \hat{z}, \tilde{z}) = \langle B(z, \hat{z}), \tilde{z} \rangle = b_g(u, \hat{u}, \tilde{u}) + \gamma \tilde{b}_g(\theta, \hat{\theta}, \tilde{\theta}),$$

where $z = (u, \theta)$, $\tilde{z} = (\tilde{u}, \tilde{\theta})$, $\hat{z} = (\hat{u}, \hat{\theta}) \in H$ and we have [3]

$$b(z, \tilde{z}, \tilde{z}) = \langle B(z, \tilde{z}), \tilde{z} \rangle = 0. \quad (2.3)$$

We denote the operators $C_g u = P_g\left[\frac{1}{g}(\nabla g \cdot \nabla)u\right]$ and $\tilde{C}_g \theta = \tilde{P}_g\left[\frac{1}{g}(\nabla g \cdot \nabla)\theta\right]$ such that

$$\langle C_g u, v \rangle_g = b_g\left(\frac{\nabla g}{g}, u, v\right), \langle \tilde{C}_g \theta, \tau \rangle_g = \tilde{b}_g\left(\frac{\nabla g}{g}, \theta, \tau\right).$$

We define the associated linear operator $R: V \rightarrow V'$ by

$$\langle Rz, \tilde{z} \rangle = \nu b_g\left(\frac{\nabla g}{g}, u, \tilde{u}\right) + \gamma \kappa \tilde{b}_g\left(\frac{\nabla g}{g}, \tilde{\theta}, \theta\right) - (\xi \theta, \tilde{u})$$

where $z = (u, \theta)$, $\tilde{z} = (\tilde{u}, \tilde{\theta}) \in H$, and

$$\begin{aligned} \langle Rz, z \rangle &= \nu b_g\left(\frac{\nabla g}{g}, u, u\right) + \gamma \kappa \tilde{b}_g\left(\frac{\nabla g}{g}, \theta, \theta\right) - (\xi \theta, u) \\ &\leq \nu \left| b_g\left(\frac{\nabla g}{g}, u, u\right) \right| + \gamma \kappa \left| \tilde{b}_g\left(\frac{\nabla g}{g}, \theta, \theta\right) \right| + |(\xi \theta, u)| \\ &\leq \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} (\nu \|u\|_g^2 + \gamma \kappa \|\theta\|_g^2) + \frac{|\xi|_{\infty}}{\lambda_1} \|\theta\|_g \|u\|_g \\ &\leq (\text{from 2.1}) \\ &\leq \frac{|\nabla g|_{\infty}}{m_0 \lambda_1^{1/2}} (\nu \|u\|_g^2 + \gamma \kappa \|\theta\|_g^2) + (\gamma \kappa)^{1/2} \|\theta\|_g \nu^{1/2} \|u\|_g \\ &\leq (\text{by the Hölder inequality}) \end{aligned}$$

$$\leq \left(\frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} + \frac{1}{2} \right) \left(\nu \|u\|_g^2 + \gamma \kappa \|\theta\|_g^2 \right). \quad (2.4)$$

Besides, $\tilde{D}_g \theta = \tilde{P}_g \left[\frac{\nabla g}{g} \theta \right]$ such that

$$\langle \tilde{D}_g \theta, \tau \rangle_g = -\tilde{b}_g \left(\frac{\nabla g}{g}, \theta, \tau \right) - \tilde{b}_g \left(\frac{\nabla g}{g}, \tau, \theta \right).$$

The system of equations (1.1) is rewritten as an abstract evolution equation

$$\begin{cases} \frac{du}{dt} + B_g(u, u) + \nu A_g u + \nu C_g u = \xi \theta + f_1, \\ \frac{d\theta}{dt} + \tilde{B}_g(u, \theta) + \kappa \tilde{A}_g \theta - \kappa \tilde{C}_g \theta - \kappa \tilde{D}_g \theta = f_2, \\ u(0) = u_0, \\ \theta(0) = \theta_0 \end{cases} \quad (2.5)$$

3. EXISTENCE OF A GLOBAL ATTRACTOR

From now on, we assume that $f_1 \in H_g$ and $f_2 \in L^2(\Omega, g)$ are time-independent. Then Theorem (3.1) in [10] allows us to define a continuous semigroup

$$S(t) : H_g \times L^2(\Omega, g) \rightarrow H_g \times L^2(\Omega, g)$$

associated to problem (1.1) by the formula $S(t)z_0 = z(t) = (u(t), \theta(t))$,

where $z(\cdot)$ is the unique global weak solution of (1.1) with the initial datum $z_0 \in H = H_g \times L^2(\Omega, g)$.

The aim of this section is to prove the following theorem.

Theorem 3.1. Assume that **(F)**, **(G)** hold. Then the semigroup $S(t)$ has a global attractor \mathcal{A}_g in the space H .

Proof. To prove this theorem, by the classical abstract results on existence of global attractors, we need to show that the semigroup $S(t)$ has a bounded absorbing set B_0 in H and $S(t)$ is asymptotically compact in H .

An absorbing set in $H_g \times L^2(\Omega, g)$.

Multiplying the first equation of (1.1) by u and the second equation by θ , we get

$$\frac{1}{2} \frac{d}{dt} \|u\|_g^2 + \nu \|u\|_g^2 + \nu b_g \left(\frac{\nabla g}{g}, u, u \right) = (\xi \theta, u)_g + (f_1, u)_g,$$

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_g^2 + \kappa \|\theta\|_g^2 + \kappa \tilde{b}_g \left(\frac{\nabla g}{g}, \theta, \theta \right) = (f_2, \theta)_g.$$

By using properties of the trilinear form b_g and \tilde{b}_g , the Cauchy-Schwarz and the Young inequalities, we see that

$$\frac{d}{dt} |u|_g^2 + \nu \|u\|_g^2 \leq \frac{4M_0 |\xi|_\infty^2}{cm_0 \nu} |\theta|_g^2 + \frac{4}{\nu \lambda_1} |f_1|_g^2 + \frac{2\nu |\nabla g|_\infty^2}{m_0^2} |u|_g^2,$$

$$\frac{d}{dt} |\theta|_g^2 + \kappa \|\theta\|_g^2 \leq \frac{2}{\kappa \lambda_1} |f_2|_g^2 + \frac{2\kappa |\nabla g|_\infty^2}{m_0^2} |\theta|_g^2,$$

so that (1.2), $\|u\|_g^2 \geq \lambda_1 |u|_g^2$ and force

$$\nu' = \nu \left(1 - \frac{2M_0 |\nabla g|_\infty^2}{cm_0^3}\right), \kappa' = \kappa \left(1 - \frac{2M_0 |\nabla g|_\infty^2}{cm_0^3}\right), c' = \frac{4M_0 |\xi|_\infty^2}{cm_0}.$$

We get the inequalities

$$\frac{d}{dt} |u|_g^2 + \nu' \|u\|_g^2 \leq \frac{c'}{\nu} |\theta|_g^2 + \frac{4}{\nu \lambda_1} |f_1|_g^2, \quad (3.1)$$

$$\frac{d}{dt} |\theta|_g^2 + \kappa' \|\theta\|_g^2 \leq \frac{2}{\kappa \lambda_1} |f_2|_g^2. \quad (3.2)$$

Using $\|u\|_g^2 \geq \lambda_1 |u|_g^2$, we have

$$\frac{d}{dt} |u|_g^2 + \nu' \lambda_1 |u|_g^2 \leq \frac{c'}{\nu} |\theta|_g^2 + \frac{4}{\nu \lambda_1} |f_1|_g^2, \quad (3.3)$$

$$\frac{d}{dt} |\theta|_g^2 + \kappa' \lambda_1 |\theta|_g^2 \leq \frac{2}{\kappa \lambda_1} |f_2|_g^2. \quad (3.4)$$

Applying the Gronwall inequality for (3.4), we obtain

$$|\theta|_g^2 \leq |\theta|_g^2 e^{-\kappa' \lambda_1 t} + \frac{2}{\kappa \kappa' \lambda_1^2} |f_2|_g^2.$$

$$\text{Set } \tilde{R}_{L^2}^2 = \frac{3}{\kappa \kappa' \lambda_1^2} |f_2|_g^2, \text{ where } t \rightarrow \infty \text{ then } |\theta|_g^2 \leq \tilde{R}_{L^2}^2.$$

Applying the Gronwall inequality once again for (3.3) and $|\theta|_g^2 \leq \tilde{R}_{L^2}^2$, we get

$$|u|_g^2 \leq |u|_g^2 e^{-\nu' \lambda_1 t} + \frac{c' \lambda_1 \tilde{R}_{L^2}^2 + 4 |f_1|_g^2}{\nu \nu' \lambda_1^2}.$$

$$\text{Set } R_H^2 = 2 \left(\frac{c' \lambda_1 \tilde{R}_{L^2}^2 + 4 |f_1|_g^2}{\nu \nu' \lambda_1^2} \right), \text{ where } t \rightarrow \infty \text{ then } |u|_g^2 \leq R_H^2.$$

An absorbing set in $V_g \times W_g$.

From (1.1), multiplying the first equation by $A_g u$ and the second by $\tilde{A}_g \theta$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_g^2 + \nu |A_g u|_g^2 + \nu b_g \left(\frac{\nabla g}{g}, u, A_g u \right) + b_g(u, u, A_g u) = (\xi \theta, A_g u)_g + (f_1, A_g u)_g,$$

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_g^2 + \kappa |\tilde{A}_g \theta|_g^2 + \kappa \tilde{b}_g \left(\frac{\nabla g}{g}, \tilde{A}_g \theta, \theta \right) + \tilde{b}_g(u, \theta, A_g \theta) = (f_2, \tilde{A}_g \theta)_g.$$

By using properties of the trilinear form b_g an \tilde{b}_g , the Cauchy-Schwarz and Young inequalities, we see that

$$\frac{d}{dt} \|u\|_g^2 + \nu |A_g u|^2 \leq \frac{C}{\nu^3} |u|_g^2 \|u\|_g^4 + \left(\frac{4|\nabla g|_\infty^2}{m_0^2 \nu} |u|_g^2 + \frac{4|\xi|_\infty^2}{\nu} |\theta|_g^2 + \frac{4}{\nu} |f_1|_g^2 \right),$$

$$\frac{d}{dt} \|\theta\|_g^2 + \kappa |\tilde{A}_g \theta|_g^2 \leq \frac{C}{\kappa^3} |u|_g^2 \|u\|_g^2 \|\theta\|_g^2 + \left(\frac{4|\nabla|_\infty^2}{\kappa m_0^2} |\theta|_g^2 + \frac{4}{\kappa} |f_2|_g^2 \right).$$

Using $|A_g u|^2 \geq \lambda_1 \|u\|_g^2$, we have

$$\frac{d}{dt} \|u\|_g^2 + \nu \lambda_1 \|u\|_g^2 \leq \frac{C}{\nu^3} |u|_g^2 \|u\|_g^4 + K_1, \quad (3.5)$$

$$\frac{d}{dt} \|\theta\|_g^2 + \kappa \lambda_1 \|\theta\|_g^2 \leq \frac{C}{\kappa^3} |u|_g^2 \|u\|_g^2 \|\theta\|_g^2 + K_2, \quad (3.6)$$

where

$$K_1 = \frac{4|\nabla g|_\infty^2}{m_0^2 \nu} |u|_g^2 + \frac{4|\xi|_\infty^2}{\nu} |\theta|_g^2 + \frac{4}{\nu} |f_1|_g^2, \quad K_2 = \frac{4|\nabla|_\infty^2}{\kappa m_0^2} |\theta|_g^2 + \frac{4}{\kappa} |f_2|_g^2.$$

Integrating (3.1) from t to $t+1$, we obtain

$$\begin{aligned} |u(t+1)|_g^2 + \nu \int_t^{t+1} \|u(s)\|_g^2 ds &\leq \frac{c'}{\nu} \tilde{R}_{L^2}^2 + |u(t)|_g^2 + \frac{4}{\nu} |f_1|_g^2 \\ &\leq \frac{c'}{\nu} \tilde{R}_{L^2}^2 + R_H^2 + \frac{4}{\nu} |f_1|_g^2, \end{aligned}$$

$$\text{we have } \nu' \int_t^{t+1} \|u(s)\|_g^2 ds \leq \frac{c'}{\nu} \tilde{R}_{L^2}^2 + R_H^2 + \frac{4}{\nu} |f_1|_g^2.$$

From (3.5), by the uniform Gronwall inequality with

$$y(t) = \|u\|_g^2, \quad a(t) = \frac{C}{\nu^3} |u|_g^2 \|u\|_g^2, \quad b(t) = K_1,$$

$$\text{we obtain } \|u\|_g^2 \leq R_V^2.$$

Integrating (3.2) from t to $t+1$, we obtain

$$\begin{aligned} |\theta(t+1)|_g^2 + \kappa \int_t^{t+1} \|\theta(s)\|_g^2 ds &\leq |\theta(t)|_g^2 + \frac{2}{\kappa} |f_2|_g^2 \\ &\leq \tilde{R}_{L^2}^2 + \frac{2}{\kappa} |f_2|_g^2, \end{aligned}$$

$$\text{we have } \kappa' \int_t^{t+1} \|\theta(s)\|_g^2 ds \leq \tilde{R}_{L^2}^2 + \frac{2}{\kappa} |f_2|_g^2.$$

From (3.6), by the uniform Gronwall inequality with

$$y(t) = \|\theta\|_g^2, \quad a(t) = \frac{C}{\kappa^3} |u|_g^2 \|u\|_g^2, \quad b(t) = K_2,$$

$$\text{we obtain } \|\theta\|_g^2 \leq \tilde{R}_W^2.$$

The existence of a bounded absorbing set in $V_g \times W_g$ and the compactness of the embeddings $V_g \hookrightarrow H_g$ and $W_g \hookrightarrow L^2(\Omega, g)$ implies the existence of a global attractor \mathcal{A}_g in H .

4. CONCLUSION

We have shown the existence of a global attractor for the g -Bénard problem. The result of this paper is a development, complementing the results in [10]. In particular, the computational techniques we use can be applied to other classes of equation systems such as: Boussinesq and MHD.

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