ON THE EXISTENCE AND PROPERTIES OF SOLUTIONS FOR A 3-ORDER NONLINEAR INTEGRODIFFERENTIAL EQUATION IN THREE VARIABLES

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Title:
Sự tồn tại nghiệm và một số tính chất của nghiệm của một phương trình vi tích phân phi tuyến bậc 3 theo ba biến

Từ khóa:
Phương trình vi tích phân theo ba biến; Định lý điểm bất động Banach; Định lý điểm bất động Schauder; Định lý Ascoli-Arzela.

Keywords:
Integrodifferential equation in three variables; The Banach fixed point theorem; The Schauder fixed point theorem; The Ascoli - Arzela theorem.

Tóm tắt: *Trong bài viết này, chúng tôi chứng minh sự tồn tại nghiệm và một số tính chất của nghiệm của một phương trình vi tích phân phi tuyến bậc 3 theo ba biến, trong một không gian Banach tùy ý. Công cụ chính được sử dụng để nhận được các kết quả là việc áp dụng một cách thích hợp các định lý điểm bất động và dạng tổng quát của định lý Ascoli-Arzela. Trước hết, chúng tôi thiết lập một không gian Banach tương thích cho việc giải phương trình đang xét và chứng minh một điều kiện đủ để các tập con là compact tương đối trong không gian này. Tiếp theo, bằng cách sử dụng định lý điểm bất động Banach, chúng tôi xét sự tồn tại duy nhất nghiệm, tính ổn định và tính bị chặn của nghiệm. Cuối cùng, sử dụng định lý điểm bất động Schauder, chúng tôi thảo luận về sự tồn tại nghiệm và tính compact của tập nghiệm. Ngoài ra, một số ví dụ cũng được nêu để minh họa các kết quả đạt được.*

Abstract: *This paper is devoted to the study of existence, uniqueness and other properties of solutions of a 3-order nonlinear integrodifferential equation (IDE) in three variables in an arbitrary Banach space. The main tools employed in the analysis are based on the applications of fixed point theorems and a generalization of Ascoli-Arzela theorem. At first, we establish an appropriate Banach space for IDE and prove a sufficient condition for relatively compact subsets in this space. Next, by using Banach's fixed point theorem, we consider the unique existence, stability and boundedness of the solution. Finally, by using Schauder's fixed point theorem, we discuss the existence and compactness of the set of solutions. Furthermore, in order to verify the efficiency of the applied method, examples are given.*

AMS Subject classification: 45G10, 47H10, 47N20, 65J15.

1 Introduction

In this paper, we consider the 3-order nonlinear IDE in three variables of the form

$$u(x) = g(x) + \int_\Omega K(x, y; u(y), D^{[1]}u(y), D^{[2]}u(y), D^{[3]}u(y))dy, \tag{1.1}$$

where $x \in \Omega = [0,1]^3 \subset \mathbb{R}^3$, $g : \Omega \to E$; $K : \Omega\times\Omega\times E^4 \to E$ are given functions, $(E, \|\cdot\|_E)$ is a Banach space, $u = u(x)$ is the unknown function and $D^{[1]}u = D_1 u = \dfrac{\partial u}{\partial x_1}$, $D^{[2]}u = D_2 D_1 u = \dfrac{\partial^2 u}{\partial x_1 \partial x_2}$, $D^{[3]}u = D_3 D_2 D_1 u = \dfrac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3}$.

Nonlinear integral equations (IEs) and integrodifferential equations (IDEs) arise in mathematical, applied and engineering sciences. IEs and IDEs naturally describe many dynamical systems, including population dynamics, nuclear reactor physics, and visco-elastic fluids, see [5], [9] and the references therein. The solvability of such equations and some basic properties of solutions have been extensively interested by many authors, and one of the most efficient tools for proving the existence of solutions to these equations is fixed point method, see [1] - [16] and the references therein.

Many papers have been devoted to the study of some types of Eq. (1.1) and its special versions, in one variable, two variables or N variables, by using many techniques and methods, one of which is using the fixed point theorems, see [2], [4], [13], [14] and the references therein.

In [4], A.M. Bica et al. considered the following Fredholm IDE

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s))ds,\ t \in [a, b], \tag{1.2}$$

where X is a Banach space, $f : [a, b]^2 \times X^2 \to X$ is continuous and $g \in C^1([a, b]; X)$. To obtain the existence, uniqueness and global approximation of the solution of Eq. (1.2), the authors used the concept of generalized metric and Perov's fixed point theorem. In [14], B.G. Pachpatte studied the following Fredholm type integral equation in two variables

$$u(x, y) = f(x, y) + \int_0^a \int_0^b g(x, y, s, t, u(s, t), D_1u(s, t), D_2u(s, t))\, dtds. \tag{1.3}$$

Based on the application of the Banach fixed point theorem coupled with a Bielecki-type norm and a certain integral inequality with explicit estimates, B.G. Pachpatte proved the existence, the uniqueness and some basic properties of solutions of Eq. (1.3). Afterward, A. Aghajani et al. [2] studied the Fredholm type IDE in two variables of the form

$$u(x, y) = f(x, y) + \int_a^b \int_c^d g(x, y, s, t, u(s, t), D_1u(s, t), D_2u(s, t))\, dtds, \tag{1.4}$$

where g, f are given real valued functions, u is the unknown function, $D_iu(x_1, x_2) = \frac{\partial u}{\partial x_i}(x_1, x_2)$, $i = 1, 2$. By using the concept of generalized metric and Perov's fixed point theorem, the authors proved some results on the existence, uniqueness and estimation of the solutions of Eq. (1.4).

Recently, in [6], [8], [13], existence and compactness of the set of solutions for n-order nonlinear IDEs in many variables in a Banach space have proved by applying the fixed point theorems of Banach, Schauder and Krasnosel'skii type.

Because of mathematical context, motivated by the above mentioned works, we continue to consider Eq. (1.1). This paper is organized as follows. Section 2 presents theoretical framework and methods, where we contruct an appropriate Banach space to solve Eq. (1.1), $X_1 = \{u \in C(\Omega; E) : D^{[1]}u,\ D^{[2]}u,\ D^{[3]}u \in C(\Omega; E)\}$, and establish a sufficient condition for relatively compact subsets of X_1, this condition can be seen as a general form of Ascoli-Arzela theorem. Section 3 is devoted to the results and discussion. Under suitable assumptions on the given functions, by using the fixed point theorems of Banach and Schauder, Subsection 3.1 proves the unique existence, stability and boundedness of the solution and Subsection 3.2 proves existence and compactness of the set of solutions. Furthermore, illustrated examples are given in Subsection 3.3. Finally, we present conclusion in Section 4.

2 Theoretical framework and Methods

In this paper, we use appropriate tools in functional analysis and classical analysis to study the existence and properties of solutions of Eq. (1.1). The main tools are the Banach's fixed point theorem and Schauder's fixed point theorem [16] coupled with the definition of a suitable Banach space (Lemma 2.1) and a sufficient condition for relatively compact subsets in this space (Lemma 2.2).

For our purpose, we need the following preliminaries, which consist of definitions and key lemmas mentioned as above.

Let $X = C(\Omega; E)$ be the space of all continuous functions from Ω into E equipped with the usual norm $\|u\|_X = \sup_{x\in\Omega} \|u(x)\|_E$, $u \in X$. First, we define the space X_1 as follows

$$X_1 = \{u \in X : D^{[1]}u,\ D^{[2]}u,\ D^{[3]}u \in X\}. \tag{2.1}$$

By a solution of Eq.(1.1), we mean a continuous function $u : \Omega \to E$ such that $u \in X_1$ and u satisfies Eq. (1.1).

We note more that, the space X_1 chosen as above is efficient to solve Eq. (1.1), by the fact that

$$C^3(\Omega; E) \subsetneqq X_1 \cap C^2(\Omega; E) \subsetneqq X_1 \subsetneqq C(\Omega; E). \tag{2.2}$$

Lemma 2.1. *X_1 is a Banach space with the norm defined by*

$$\|u\|_{X_1} = \|u\|_X + \sum_{i=1}^{3} \left\|D^{[i]}u\right\|_X,\ u \in X_1. \tag{2.3}$$

Proof of Lemma 2.1. Let $\{u_p\} \subset X_1$ be a Cauchy sequence in X_1, it means that

$$\begin{aligned}&\|u_p - u_q\|_{X_1} = \|u_p - u_q\|_X \\ &+ \sum_{i=1}^{3} \left\| D^{[i]}u_p - D^{[i]}u_q \right\|_X \to 0 \text{ as } p,\ q \to \infty.\end{aligned}$$

Then $\{u_p\}$ and $\{D^{[i]}u_p\}$ $(i = 1, 2, 3)$ are also Cauchy sequences in X. Since X is complete, there exist u, v_1, v_2, $v_3 \in X$ such that

$$\begin{aligned}&\|u_p - u\|_X \to 0, \\ &\left\| D^{[i]}u_p - v_i \right\|_X \to 0,\ \text{as } p \to \infty,\ i = 1, 2, 3.\end{aligned} \tag{2.4}$$

We shall show that $D^{[i]}u = v_i$, $i = 1, 2, 3$.

By the fact that $\|u_p - u\|_X \to 0$, and for all $x = (x_1, x_2, x_3) \in \Omega$,

$$u_p(x) - u_p(0, x_2, x_3) = \int_0^{x_1} D^{[1]}u_p(s, x_2, x_3)ds, \tag{2.5}$$

we obtain

$$u_p(x) - u_p(0, x_2, x_3) \to u(x) - u(0, x_2, x_3) \text{ in } E,\ \forall x \in \Omega. \tag{2.6}$$

We also have $\left\| D^{[1]}u_p - v_1 \right\|_X \to 0$, so

$$\int_0^{x_1} D^{[1]}u_p(s, x_2, x_3)ds \to \int_0^{x_1} v_1(s, x_2, x_3)ds,\ \forall x \in \Omega, \tag{2.7}$$

because of

$$\begin{aligned}&\left\| \int_0^{x_1} D^{[1]}u_p(s, x_2, x_3)ds - \int_0^{x_1} v_1(s, x_2, x_3)ds \right\|_E \\ &\le \int_0^{x_1} \left\| D^{[1]}u_p(s, x_2, x_3) - v_1(s, x_2, x_3) \right\|_E ds \\ &\le \left\| D^{[1]}u_p - v_1 \right\|_X \to 0.\end{aligned}$$

Combining (2.5)-(2.7), it leads to

$$u(x) - u(0, x_2, x_3) = \int_0^{x_1} v_1(s, x_2, x_3)ds,\ \forall x \in \Omega. \tag{2.8}$$

Thus $D^{[1]}u = v_1 \in X$.

By the same argument, because

$$D^{[1]}u_p(x) - D^{[1]}u_p(x_1, 0, x_3) = \int_0^{x_2} D^{[2]}u_p(x_1, t, x_3)dt,$$

$\forall x \in \Omega$, and $\left\| D^{[2]}u_p - v_2 \right\|_X \to 0$, we get

$D^{[1]}u(x) - D^{[1]}u(x_1, 0, x_3) = \int_0^{x_2} v_2(x_1, t, x_3)dt$, $\forall x \in \Omega$.

Then, $D^{[2]}u = v_2 \in X$. It is similar to $D^{[3]}u$, hence $D^{[3]}u = v_3 \in X$. Therefore, $u_p \to u$ in X_1. □

Next, we establish a sufficient condition for relatively compact subsets of the Banach space X_1. This condition is proposed and proved by applying the Ascoli-Arzela theorem [16], so it can be seen as a general form of Ascoli-Arzela theorem.

Lemma 2.2. *Let $\mathcal{F} \subset X_1$. Then $\mathcal{F}$ is relatively compact in X_1 if and only if the following conditions are satisfied*

$$\begin{aligned}&\text{(i) } \forall x \in \Omega,\ \mathcal{F}(x) = \{u(x) : u \in \mathcal{F}\}, \\ &D^{[i]}\mathcal{F}(x) = \{D^{[i]}u(x) : u \in \mathcal{F}\},\ i = 1, 2, 3, \\ &\textit{are relatively compact subsets of } E; \\ &\text{(ii) } \forall \varepsilon > 0, \exists \delta > 0 : \forall x,\ \bar{x} \in \Omega, \\ &|x - \bar{x}| < \delta \Longrightarrow \sup_{u \in \mathcal{F}} [u(x) - u(\bar{x})]_* < \varepsilon,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}&[u(x) - u(\bar{x})]_* = \|u(x) - u(\bar{x})\|_E \\ &\qquad + \sum_{i=1}^{3} \left\| D^{[i]}u(x) - D^{[i]}u(\bar{x}) \right\|_E.\end{aligned}$$

Proof of Lemma 2.2. **(a)** Let $\mathcal{F}$ be relatively compact in X_1. We show that (2.9) (i) (ii) hold.

We begin by considering $\mathcal{F}(x), \forall x \in \Omega$. To prove that $\mathcal{F}(x)$ is relatively compact in E, let $\{u_p(x)\}$ be a sequence in $\mathcal{F}(x)$, we need to show that $\{u_p(x)\}$ contains a convergent subsequence in E. By $\overline{\mathcal{F}}$ is compact in X_1, $\{u_p\} \subset \mathcal{F}$ contains a convergent subsequence $\{u_{p_k}\}$ in X_1. Then, there exists $u \in X_1$ such that

$$\|u_{p_k} - u\|_{X_1} \to 0 \text{ as } k \to \infty.$$

Because

$$\|u_{p_k}(x) - u(x)\|_E \le \|u_{p_k} - u\|_X \le \|u_{p_k} - u\|_{X_1} \to 0,$$

we have $u_{p_k}(x) \to u(x)$ in E. Thus, $\forall x \in \Omega$, $\mathcal{F}(x)$ is relatively compact in E.

By the same argument, by the fact that

$$\begin{aligned}\left\| D^{[1]}u_{p_k}(x) - D^{[1]}u(x) \right\|_E &\le \left\| D^{[1]}u_{p_k} - D^{[1]}u \right\|_X \\ &\le \|u_{p_k} - u\|_{X_1} \to 0,\end{aligned}$$

$D^{[1]}\mathcal{F}(x)$ is relatively compact in E.

Similarly, $D^{[2]}\mathcal{F}(x)$, $D^{[3]}\mathcal{F}(x)$ are relatively compact in E. Then, (2.9) (i) holds.

Next, for every $\varepsilon > 0$, we consider a collection of open balls $B(u, \frac{\varepsilon}{3})$ in X_1 centered at $u \in \mathcal{F}$ with radius $\frac{\varepsilon}{3}$, as below

$$B(u, \frac{\varepsilon}{3}) = \{\bar{u} \in X_1 : \|u - \bar{u}\|_{X_1} < \frac{\varepsilon}{3}\},\ u \in \mathcal{F}.$$

It is clear that $\overline{\mathcal{F}} \subset \bigcup_{u \in \mathcal{F}} B(u, \frac{\varepsilon}{3})$. Because $\overline{\mathcal{F}}$ is compact in X_1, the open cover $\bigcup_{u \in \mathcal{F}} B(u, \frac{\varepsilon}{3})$ of $\overline{\mathcal{F}}$ contains a finite subcover and then, there are $u_1, \cdots, u_q \in \mathcal{F}$ such that $\overline{\mathcal{F}} \subset \bigcup_{j=1}^{q} B(u_j, \frac{\varepsilon}{3})$.

By the functions $u_j, D^{[1]}u_j, \cdots, D^{[3]}u_j$, $j=\overline{1,q}$ are uniformly continuous on Ω, there exists $\delta > 0$ such that for all $x, \bar{x} \in \Omega$,

$$|x-\bar{x}|<\delta \Longrightarrow [u_j(x)-u_j(\bar{x})]_* < \frac{\varepsilon}{3}, \forall j=\overline{1,q}.$$

For all $u \in \mathcal{F}$, by $u \in B(u_{j_0}, \frac{\varepsilon}{3})$ for some $j_0=\overline{1,q}$, it implies that, for all $x, \bar{x} \in \Omega$, if $|x-\bar{x}|<\delta$ then

$$\begin{aligned}[u(x)-u(\bar{x})]_* &\le [u(x)-u_{j_0}(x)]_* + [u_{j_0}(x)-u_{j_0}(\bar{x})]_* \\ &\quad + [u_{j_0}(\bar{x})-u(\bar{x})]_* \\ &\le 2\left\|u-u_{j_0}\right\|_{X_1} + [u_{j_0}(x)-u_{j_0}(\bar{x})]_* \\ &< \frac{2\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

It follows that(2.9) (ii) holds.

(b) Conversely, let (2.9) be correct. We need to prove that $\mathcal{F}$ is relatively compact in X_1.

Let $\{u_p\}$ be a sequence in $\mathcal{F}$, we have to show that $\{u_p\}$ contains a convergent subsequence.

Put $\mathcal{F}_0 = \{u_p : p \in \mathbb{N}\}$. By (2.9), $\mathcal{F}_0(x) = \{u_p(x) : p \in \mathbb{N}\}$ is a relatively compact subset of E, for all $x \in \Omega$ and $\mathcal{F}_0$ is equicontinuous in X. Applying the Ascoli-Arzela theorem to $\mathcal{F}_0$, it is relatively compact in X, so there exists a subsequence $\{u_{p_k}\}$ of $\{u_p\}$ and $u \in X$ such that

$$\left\|u_{p_k}-u\right\|_X \to 0 \text{ as } k\to\infty.$$

Similarly, $\mathcal{F}_1 = \{D^{[1]}u_{p_k} : k \in \mathbb{N}\}$ is also relatively compact in X. We obtain the existence of a subsequence of $\{D^{[1]}u_{p_k}\}$, denoted by the same symbol, and $v_1 \in X$ such that

$$\left\|D^{[1]}u_{p_k}-v_1\right\|_X \to 0 \text{ as } k\to\infty.$$

Since

$$u_{p_k}(x)-u_{p_k}(0,x_2,x_3)=\int_0^{x_1} D^{[1]}u_{p_k}(s,x_2,x_3)ds,$$

$\forall x \in \Omega$, furthermore, $\left\|u_{p_k}-u\right\|_X \to 0$ and

$$\left\|D^{[1]}u_{p_k}-v_1\right\|_X \to 0,$$

we obtain

$$u(x)-u(0,x_2,x_3)=\int_0^{x_1} v_1(s,x_2,x_3)ds \ \forall x \in \Omega.$$

It gives $D^{[1]}u = v_1 \in X$.

By the same argument, $\mathcal{F}_2 = \{D^{[2]}u_{p_k} : k \in \mathbb{N}\}$ is relatively compact in X, there exists a subsequence of $\{D^{[2]}u_{p_k}\}$, denoted by the same symbol, and $v_2 \in X$ such that

$$\left\|D^{[2]}u_{p_k}-v_2\right\|_X \to 0 \text{ as } k\to\infty.$$

For all $x \in \Omega$, by

$$D^{[1]}u_{p_k}(x)-D^{[1]}u_{p_k}(x_1,0,x_3)=\int_0^{x_2} D^{[2]}u_{p_k}(x_1,t,x_3)dt,$$

$$\left\|D^{[1]}u_{p_k}-D^{[1]}u\right\|_X \to 0$$

$$\left\|D^{[2]}u_{p_k}-v_2\right\|_X \to 0,$$

we get

$$D^{[1]}u(x)-D^{[1]}u(x_1,0,x_3)=\int_0^{x_2} v_2(x_1,t,x_3)dt \ \forall x \in \Omega,$$

then $D^{[2]}u = v_2 \in X$.

Similarly, by $\mathcal{F}_3 = \{D^{[3]}u_{p_k} : k \in \mathbb{N}\}$ is relatively compact in X, there exists of a subsequence of $\{D^{[3]}u_{p_k}\}$, denoted by the same symbol, and $v_3 \in X$ such that

$$\left\|D^{[3]}u_{p_k}-v_3\right\|_X \to 0 \text{ as } k\to\infty,$$

then $D^{[3]}u = v_3$. Therefore, $u_{p_k} \to u$ in X_1. □

3 Results and discussion

3.1 The unique existence, stability and boundedness of the solution

We make the following assumptions.

(A_1) $g \in X_1$;

(A_2) $K \in C(\Omega\times\Omega\times E^4;E)$ such that $D^{[1]}K, \cdots, D^{[3]}K \in C(\Omega\times\Omega\times E^4;E)$,

and there exist nonnegative functions $k_0, k_1, k_2, k_3 : \Omega\times\Omega \to \mathbb{R}$ satisfying

$$\alpha_* = \sum_{i=0}^{3} \sup_{x\in\Omega}\int_\Omega k_i(x,y)dy < 1,$$

and

(i) $\left\|K(x,y;u_1,\cdots,u_4)-K(x,y;v_1,\cdots,v_4)\right\|_E \le k_0(x,y)\sum_{i=1}^4 \left\|u_i-v_i\right\|_E$,

(ii) $\left\|D^{[1]}K(x,y;u_1,\cdots,u_4)-D^{[1]}K(x,y;v_1,\cdots,v_4)\right\|_E \le k_1(x,y)\sum_{i=1}^4 \left\|u_i-v_i\right\|_E$,

(iii) $\left\|D^{[2]}K(x,y;u_1,\cdots,u_4)-D^{[2]}K(x,y;v_1,\cdots,v_4)\right\|_E \le k_2(x,y)\sum_{i=1}^4 \left\|u_i-v_i\right\|_E$,

(iv) $\left\|D^{[3]}K(x,y;u_1,\cdots,u_4)-D^{[3]}K(x,y;v_1,\cdots,v_4)\right\|_E \le k_3(x,y)\sum_{i=1}^4 \left\|u_i-v_i\right\|_E$,

$\forall (x,y) \in \Omega\times\Omega$, $\forall (u_1,\cdots,u_4), (v_1,\cdots,v_4) \in E^4$.

Theorem 3.1. *Let the assumptions $(A_1),(A_2)$ hold. Then, we have*

(i) *Eq. (1.1) has a unique solution u in X_1.*

(ii) *The solution u is stable with respect to g in X_1, i.e. if u, $\tilde{u}$ are two solutions of Eq. (1.1) corresponding to two functions g, $\tilde{g}$ in X_1, and if $\|g-\tilde{g}\|_{X_1} \to 0$ then $\|u-\tilde{u}\|_{X_1} \to 0$.*

(iii) *The solution u is bounded with the estimate*

$$\|u\|_{X_1} \leq \frac{1}{1-\alpha_*}\left(\|g\|_{X_1}+\mu\right),$$

where

$$\mu = \sup_{x\in\Omega}\int_\Omega \|K(x,y;0,0,\cdots,0)\|_E \; dy + \sum_{i=1}^3 \sup_{x\in\Omega}\int_\Omega \left\|D^{[i]}K(x,y;0,0,\cdots,0)\right\|_E dy.$$

Proof of Theorem 3.1.

(i) For every $u\in X_1$, we put

$$(Uu)(x) = g(x) + \int_\Omega K(x,y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y))dy, \forall x\in\Omega. \quad (3.1)$$

It is clear to see that $Uu\in X_1$, $\forall u\in X_1$. On the other hand, for every u, $v\in X_1$, for all $x\in\Omega$, using (A_2,i), we have

$$\|(Uu)(x)-(Uv)(x)\|_E \leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy,$$

so

$$\|Uu-Uv\|_X \leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy. \quad (3.2)$$

Similarly, we also have

$$\begin{aligned}
\left\|D^{[1]}(Uu)-D^{[1]}(Uv)\right\|_X &\leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy,\\
\left\|D^{[2]}(Uu)-D^{[2]}(Uv)\right\|_X &\leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_2(x,y)dy,\\
\left\|D^{[3]}(Uu)-D^{[3]}(Uv)\right\|_X &\leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_3(x,y)dy.
\end{aligned} \quad (3.3)$$

It implies that

$$\|Uu-Uv\|_{X_1} \leq \alpha_*\|u-v\|_{X_1}, \ \forall u,v\in X_1, \quad (3.4)$$

with $0\leq\alpha_*<1$. Thus, $U: X_1\to X_1$ is a contraction map. Applying Banach's fixed point theorem, there exists a unique function $u\in X_1$ such that $u=Uu$. It means that Eq. (1.1) has a unique solution $u\in X_1$.

(ii) This solution is stable with respect to g in X_1. Indeed, let u, $\widetilde{u}$ be two solutions of Eq. (1.1) corresponding to two functions g, $\tilde{g}$ in X_1, then

$$u-\tilde{u} = Uu-U\tilde{u} = g-\tilde{g}+\hat{U}u-\hat{U}\tilde{u},$$

where

$$(\hat{U}u)(x) = \int_\Omega K(x,y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y))dy, \forall x\in\Omega. \quad (3.5)$$

By the same argument as above, we get

$$\left\|\hat{U}u-\hat{U}\tilde{u}\right\|_{X_1} \leq \alpha_*\|u-\tilde{u}\|_{X_1}.$$

It leads to

$$\|u-\tilde{u}\|_{X_1} \leq \|g-\tilde{g}\|_{X_1}+\alpha_*\|u-\tilde{u}\|_{X_1},$$

so

$$\|u-\tilde{u}\|_{X_1} \leq \frac{1}{1-\alpha_*}\|g-\tilde{g}\|_{X_1},$$

obviously, if $\|g-\tilde{g}\|_{X_1}\to 0$ then $\|u-\tilde{u}\|_{X_1}\to 0$.

(iii) For all $x\in\Omega$, we note that

$$\begin{aligned}
u(x) = (Uu)(x) &= g(x)+(\hat{U}u)(x)\\
&= g(x)+(\hat{U}u)(x)-(\hat{U}0)(x)+(\hat{U}0)(x),
\end{aligned}$$

hence

$$\begin{aligned}
\|u\|_{X_1} &\leq \|g\|_{X_1}+\left\|\hat{U}u-\hat{U}0\right\|_{X_1}+\left\|\hat{U}0\right\|_{X_1}\\
&\leq \|g\|_{X_1}+\alpha_*\|u\|_{X_1}+\left\|\hat{U}0\right\|_{X_1}.
\end{aligned}$$

On the other hand, for all $x\in\Omega$,

$$(\hat{U}u)(x) = \int_\Omega K(x,y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y))dy,$$

then

$$(\hat{U}0)(x) = \int_\Omega K(x,y;0,0,\cdots,0)dy, \ \forall x\in\Omega,$$

it implies that

$$\begin{aligned}
\left\|(\hat{U}0)(x)\right\|_E &\leq \int_\Omega \|K(x,y;0,0,\cdots,0)\|_E dy\\
&\leq \sup_{x\in\Omega}\int_\Omega \|K(x,y;0,0,\cdots,0)\|_E dy,
\end{aligned}$$

then

$$\left\|\hat{U}0\right\|_X \leq \sup_{x\in\Omega}\int_\Omega \|K(x,y;0,0,\cdots,0)\|_E \; dy.$$

Similarly, we also have

$$\left\|D^{[i]}\hat{U}0\right\|_X \leq \sup_{x\in\Omega}\int_\Omega \left\|D^{[i]}K(x,y;0,0,\cdots,0)\right\|_E \; dy, i=1,2,3.$$

Therefore

$$\left\|\hat{U}0\right\|_{X_1} \leq \sup_{x\in\Omega}\int_\Omega \|K(x,y;0,0,\cdots,0)\|_E \; dy + \sum_{i=1}^3 \sup_{x\in\Omega}\int_\Omega \left\|D^{[i]}K(x,y;0,0,\cdots,0)\right\|_E dy \equiv \mu.$$

Consequently

$$\|u\|_{X_1} \leq \frac{1}{1-\alpha_*}\left(\|g\|_{X_1} + \mu\right).$$

Theorem 3.1 is proved. □

3.2 The existence and compactness of the set of solutions

We make the following assumptions.

(A_1) $g \in X_1$;

$(\hat{A}_2)$ $K \in C(\Omega \times \Omega \times E^4; E)$ such that $D^{[1]}K, \cdots, D^{[3]}K \in C(\Omega \times \Omega \times E^4; E)$, and there exist nonnegative functions

$$\bar{h}_0, \cdots, \bar{h}_3 : \Omega \times \Omega \to \mathbb{R}$$

with the following properties

$$\beta_* = \sum\nolimits_{i=0}^{3} \sup_{x\in\Omega} \int_\Omega \bar{h}_i(x,y)dy < 1,$$

and

(i) $\|K(x,y;u_1,\cdots,u_4)\|_E \leq \bar{h}_0(x,y)\left(1 + \sum_{i=1}^4 \|u_i\|_E\right)$,

(ii) $\left\|D^{[1]}K(x,y;u_1,\cdots,u_4)\right\|_E \leq \bar{h}_1(x,y)\left(1 + \sum_{i=1}^4 \|u_i\|_E\right)$,

(iii) $\left\|D^{[2]}K(x,y;u_1,\cdots,u_4)\right\|_E \leq \bar{h}_2(x,y)\left(1 + \sum_{i=1}^4 \|u_i\|_E\right)$,

(iv) $\left\|D^{[3]}K(x,y;u_1,\cdots,u_4)\right\|_E \leq \bar{h}_3(x,y)\left(1 + \sum_{i=1}^4 \|u_i\|_E\right)$,

$\forall (x,y) \in \Omega \times \Omega$, $\forall (u_1,\cdots,u_4) \in E^4$;

$(\hat{A}_3)$ $K, D^{[1]}K, \cdots, D^{[3]}K : \Omega \times \Omega \times E^4 \to E$ are completely continuous such that for any bounded subset J of E^4, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, \bar{x} \in \Omega,\ |x - \bar{x}| < \delta \Longrightarrow \|K(x,y;u_1,\cdots,u_4) - K(\bar{x},y;u_1,\cdots,u_4)\|_E + \sum\nolimits_{i=1}^{3} \left\|D^{[i]}K(x,y;u_1,\cdots,u_4) - D^{[i]}K(\bar{x},y;u_1,\cdots,u_4)\right\|_E < \varepsilon,$$

$\forall y \in \Omega$, $\forall (u_1,\cdots,u_4) \in J$.

Theorem 3.2. *Let the assumptions* (A_1), $(\hat{A}_2)$, $(\hat{A}_3)$ *hold. Then Eq.* (1.1) *has a solution in* X_1.

Furthermore, the set of solutions of Eq. (1.1) *is compact.*

Proof of Theorem 3.2. Considering the operator $U : X_1 \to X_1$ defined as in Subsection 3.1, that is

$$(Uu)(x) = g(x) + \int_\Omega K(x,y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y))dy,\ \forall x \in \Omega. \tag{3.6}$$

For $\rho > 0$, we consider a closed ball in X_1 defined by

$$B_\rho = \{u \in X_1 : \|u\|_{X_1} \leq \rho\}. \tag{3.7}$$

We will show that there exists $\rho > 0$ such that

(i) $Uu \in B_\rho$, for every $u \in B_\rho$.

Furthermore

(ii) $U : B_\rho \to B_\rho$ is continuous,

(iii) $\mathcal{F} = U(B_\rho)$ is relatively compact in X_1.

Proof of (i). Let $\rho > 0$. For every $u \in B_\rho$, for all $x \in \Omega$, we have

$$\begin{aligned}
&\|(Uu)(x)\|_E \\
&\leq \|g\|_X \\
&+ \int_\Omega h_0(x,y)\left(1 + \|u(y)\|_E + \sum\nolimits_{i=1}^{3}\left\|D^{[i]}u(y)\right\|_E\right)dy \\
&\leq \|g\|_X + \left(1 + \|u\|_{X_1}\right)\int_\Omega \bar{h}_0(x,y)dy \\
&\leq \|g\|_X + (1+\rho)\sup_{x\in\Omega}\int_\Omega \bar{h}_0(x,y)dy.
\end{aligned} \tag{3.8}$$

Hence

$$\|Uu\|_X \leq \|g\|_X + (1+\rho)\sup_{x\in\Omega}\int_\Omega \bar{h}_0(x,y)dy. \tag{3.9}$$

Similarly, we get

$$\begin{aligned}
\left\|D^{[1]}(Uu)\right\|_X &\leq \left\|D^{[1]}g\right\|_X + (1+\rho)\sup_{x\in\Omega}\int_\Omega \bar{h}_1(x,y)dy, \\
\left\|D^{[2]}(Uu)\right\|_X &\leq \left\|D^{[2]}g\right\|_X + (1+\rho)\sup_{x\in\Omega}\int_\Omega \bar{h}_2(x,y)dy, \\
\left\|D^{[3]}(Uu)\right\|_X &\leq \left\|D^{[3]}g\right\|_X + (1+\rho)\sup_{x\in\Omega}\int_\Omega \bar{h}_3(x,y)dy.
\end{aligned} \tag{3.10}$$

Therefore

$$\|Uv\|_{X_1} \leq \|g\|_{X_1} + (1+\rho)\beta_*. \tag{3.11}$$

Choose $\rho \geq \dfrac{\|g\|_{X_1} + \beta_*}{1-\beta_*}$, then $Uu \in B_\rho$, for every $u \in B_\rho$.

To prove (ii), let $\{u_m\} \subset B_\rho$, $u_0 \in B_\rho$, $\|u_m - u_0\|_{X_1} \to 0$, as $m \to \infty$, we have to prove that the following convergences are fulfilled

$$\begin{aligned}
\|Uu_m - Uu_0\|_X &\to 0, \\
\left\|D^{[1]}(Uu_m) - D^{[1]}(Uu_0)\right\|_X &\to 0, \\
\left\|D^{[2]}(Uu_m) - D^{[2]}(Uu_0)\right\|_X &\to 0, \\
\left\|D^{[3]}(Uu_m) - D^{[3]}(Uu_0)\right\|_X &\to 0.
\end{aligned} \tag{3.12}$$

Remark that

$$\begin{aligned}&\|(Uu_m)(x)-(Uu_0)(x)\|_E\\&\leq \int_\Omega \left\|K(x,y;u_m(y),D^{[1]}u_m(y),\cdots,D^{[3]}u_m(y))-K(x,y;u_0(y),D^{[1]}u_0(y),\cdots,D^{[3]}u_0(y))\right\|_E dy.\end{aligned} \tag{3.13}$$

Consider the following sets

$$\begin{aligned}&S_0=\{u_m(y):y\in\Omega,\ m=0,1,2,\cdots\},\\&S_i=\{D^{[i]}u_m(y):y\in\Omega,\ m=0,1,2,\cdots\},\ i=1,2,3.\end{aligned} \tag{3.14}$$

We shall prove that S_0, S_i, $i=1,2,3$, are compact in E based on the property $\|u_m-u_0\|_{X_1}\to 0$.

Indeed, let $\{u_{m_j}(y_j)\}_j$ be a sequence in S_0.

We can assume that $\lim\limits_{j\to\infty} y_j = y_0$ and $\lim\limits_{j\to\infty}\left\|u_{m_j}-u_0\right\|_{X_1}=0$. We have

$$\begin{aligned}&\left\|u_{m_j}(y_j)-u_0(y_0)\right\|_E\\&\leq \left\|u_{m_j}(y_j)-u_0(y_j)\right\|_E+\|u_0(y_j)-u_0(y_0)\|_E\\&\leq \left\|u_{m_j}-u_0\right\|_{X_1}+\|u_0(y_j)-u_0(y_0)\|_E\to 0 \text{ as } j\to\infty,\end{aligned} \tag{3.15}$$

thus, $\lim\limits_{j\to\infty} u_{m_j}(y_j)=u_0(y_0)$ in E. This means that S_0 is compact in E.

Similarly S_i, $i=1,2,3$, are also compact in E.

For $\varepsilon>0$ given, K is uniformly continuous on $\Omega\times\Omega\times S_0\times\cdots\times S_3$, there exists $\delta>0$ such that

$$\begin{aligned}&\forall(u_1,\cdots,u_4),\ (v_1,\cdots,v_4)\in S_0\times\cdots\times S_3,\\&\sum_{i=1}^4\|u_i-v_i\|_E<\delta\\&\Longrightarrow \|K(x,y;u_1,\cdots,u_4)-K(x,y;v_1,\cdots,v_4)\|_E<\varepsilon,\\&\forall(x,y)\in\Omega\times\Omega.\end{aligned}$$

We have $\|u_m-u_0\|_X\to 0$ and $\left\|D^{[i]}u_m-D^{[i]}u_0\right\|_X\to 0$, therefore, with $\delta>0$ as above, there exists $m_0\in\mathbb{N}$ such that, $\forall m\in\mathbb{N}$, if $m\geq m_0$ then it gives

$$\|u_m-u_0\|_X+\sum\nolimits_{i=1}^3\left\|D^{[i]}u_m-D^{[i]}u_0\right\|_X<\delta.$$

It implies that there exists $m_0\in\mathbb{N}$ as above such that $\forall m\in\mathbb{N}$, if $m\geq m_0$ then the following inequality is fulfilled, for all $(x,y)\in\Omega\times\Omega$,

$$\left\|K(x,y;u_m(y),D^{[1]}u_m(y),\cdots,D^{[3]}u_m(y))-K(x,y;u_0(y),D^{[1]}u_0(y),\cdots,D^{[3]}u_0(y))\right\|_E<\varepsilon.$$

Consequently

$$\|(Uu_m)(x)-(Uu_0)(x)\|_E<\varepsilon\ \forall x\in\Omega,\ \forall m\geq m_0.$$

It means that

$$\|Uu_m-Uu_0\|_X<\varepsilon\ \forall m\geq m_0, \tag{3.16}$$

i.e., $\|Uu_m-Uu_0\|_X\to 0$ as $m\to\infty$.

By the same argument, we get

$$\begin{aligned}&\left\|D^{[1]}(Uu_m)-D^{[1]}(Uu_0)\right\|_X\to 0,\\&\left\|D^{[2]}(Uu_m)-D^{[2]}(Uu_0)\right\|_X\to 0,\\&\left\|D^{[3]}(Uu_m)-D^{[3]}(Uu_0)\right\|_X\to 0,\ \text{as } m\to\infty.\end{aligned} \tag{3.17}$$

The continuity of U is proved.

To prove (iii), we use Lemma 2.2. The first condition of Lemma 2.2 holds for U, i.e., the sets

$$\begin{aligned}U(B_\rho)(x)=&\ \{Uu(x):u\in B_\rho\},\\D^{[1]}U(B_\rho)(x)=&\ \{D^{[1]}(Uu)(x):u\in B_\rho\},\\D^{[2]}U(B_\rho)(x)=&\ \{D^{[2]}(Uu)(x):u\in B_\rho\},\\\text{and } D^{[3]}U(B_\rho)(x)=&\ \{D^{[3]}(Uu)(x):u\in B_\rho\},\end{aligned} \tag{3.18}$$

are relatively compact in E. Indeed, we put

$$\begin{aligned}&R_0=\{u(y):y\in\Omega,\ u\in B_\rho\},\\&R_i=\{D^{[i]}u(y):y\in\Omega,\ u\in B_\rho\},\ i=1,2,3.\end{aligned} \tag{3.19}$$

Then R_0, R_i, $i=1,2,3$, are bounded in E.

By $K:\Omega\times\Omega\times E^4\to E$ is completely continuous, $K(\Omega\times\Omega\times R_0\times\cdots\times R_3)$ is relatively compact in E. Then, $\overline{K(\Omega\times\Omega\times R_0\times\cdots\times R_3)}$ is compact in E. So is $\overline{conv}\ K(\Omega\times\Omega\times R_0\times\cdots\times R_3)$, where $\overline{conv}K(\Omega\times\Omega\times R_0\times\cdots\times R_3)$ is the convex closure of $K(\Omega\times\Omega\times R_0\times\cdots\times R_3)$.

For every $x\in\Omega$, for all $u\in B_\rho$, it implies from

$$\begin{aligned}&K(x,y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y))\\&\in K(\Omega\times\Omega\times R_0\times\cdots\times R_3),\ \forall y\in\Omega,\end{aligned} \tag{3.20}$$

that

$$\begin{aligned}\overline{U(B_\rho)(x)}\subset&\ |\Omega|\,\overline{conv}K(\Omega\times\Omega\times R_0\times\cdots\times R_3)\\=&\ \overline{conv}K(\Omega\times\Omega\times R_0\times\cdots\times R_3),\end{aligned} \tag{3.21}$$

then the set $U(B_\rho)(x)$ is relatively compact in E. Similarly,

$$\begin{aligned}\overline{D^{[1]}U(B_\rho)(x)}\subset&\ \overline{conv}D^{[1]}K(\Omega\times\Omega\times R_0\times\cdots\times R_3),\\\overline{D^{[2]}U(B_\rho)(x)}\subset&\ \overline{conv}D^{[2]}K(\Omega\times\Omega\times R_0\times\cdots\times R_3),\\\overline{D^{[3]}U(B_\rho)(x)}\subset&\ \overline{conv}D^{[3]}K(\Omega\times\Omega\times R_0\times\cdots\times R_3).\end{aligned} \tag{3.22}$$

Hence, the sets $D^{[1]}U(B_\rho)(x),\cdots,D^{[3]}U(B_\rho)(x)$ are relatively compact in E.

The second condition of Lemma 2.2 also holds. Indeed, let $\varepsilon>0$ be given.

By $(\hat{A}_3)$, there exists $\delta_1 > 0$ such that $\forall x, \bar{x} \in \Omega$, if $|x - \bar{x}| < \delta_1$ then

$$\|K(x,y;u_1,\cdots,u_4) - K(\bar{x},y;u_1,\cdots,u_4)\|_E + \sum_{i=1}^{3} \left\| D^{[i]}K(x,y;u_1,\cdots,u_4) - D^{[i]}K(\bar{x},y;u_1,\cdots,u_4) \right\|_E < \varepsilon,$$

$\forall y \in \Omega$, $\forall (u_1,\cdots,u_4) \in R_0 \times \cdots \times R_3$.

It leads to

$$\begin{aligned}&[(Uu)(x) - (Uu)(\bar{x})]_* \\ &\leq \int_\Omega \left[K(x,y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y)) - K(\bar{x},y;u(y),D^{[1]}u(y),\cdots,D^{[3]}u(y))\right]_* dy < \varepsilon.\end{aligned} \tag{3.23}$$

By Lemma 2.2, $\mathcal{F} = U(B_\rho)$ is relatively compact in X_1.

Applying the Schauder's fixed point theorem, Eq. (1.1) has a solution in X_1.

It remains to prove that the set S of solutions for Eq. (1.1) is compact in X_1, where

$$S = \{u \in B_\rho : u = Uu\}. \tag{3.24}$$

By the compactness of the operator $U : B_\rho \to B_\rho$ and $S = U(S)$, we only show that S is closed.

Let $\{u_m\} \subset S$, $u \in X_1$, $\|u_m - u\|_{X_1} \to 0$. Then, by continuity of U, we obtain that

$$\begin{aligned}\|u - Uu\|_{X_1} &\leq \|u - u_m\|_{X_1} + \|u_m - Uu\|_{X_1} \\ &= \|u - u_m\|_{X_1} + \|Uu_m - Uu\|_{X_1} \to 0,\end{aligned} \tag{3.25}$$

consequently, $u = Uu \in S$. Theorem 3.2 is proved. □

3.3 Examples

In this subsection, we shall illustrate the results obtained as above by the following examples.

Let $E = C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $v : [0,1] \to \mathbb{R}$ equipped with the norm

$$\|v\|_E = \sup_{0 \leq t \leq 1} |v(t)|, \; v \in E, \tag{3.26}$$

and let $X = C(\Omega;E)$ be the space of all continuous functions from $\Omega = [0,1]^3$ into E equipped with the norm

$$\|u\|_X = \sup_{x \in \Omega} \|u(x)\|_E, u \in X. \tag{3.27}$$

Putting

$$X_1 = \{u \in X : D^{[1]}u, \; D^{[2]}u, \; D^{[3]}u \in X\}. \tag{3.28}$$

Then, $\forall u \in X_1$ and $x \in \Omega$, $u(x)$ is an element of E and we denote by

$$u(x)(t) = u(x;t), \; 0 \leq t \leq 1. \tag{3.29}$$

Example 1. We give an example to show that $C^3(\Omega;E) \subsetneqq X_1 \cap C^2(\Omega;E) \subsetneqq X_1$. Obviously, $C^3(\Omega;E) \subset X_1 \cap C^2(\Omega;E) \subset X_1$.

We show that there exists a function $u : \Omega \to E$ such that $u \in X_1 \cap C^2(\Omega;E)$ and $u \notin C^3(\Omega;E)$.

Fixed $\varphi \in E = C([0,1];\mathbb{R})$. Put

$$u(x_1,x_2,x_3) = x_1 x_2^{\frac{5}{2}} \varphi \in E,$$
$$u(x)(t) = u(x;t) = x_1 x_2^{\frac{5}{2}} \varphi(t).$$

Then, we have

$$\begin{aligned}D^{[1]}u(x) &= D_1 u(x) = x_2^{\frac{5}{2}} \varphi, \\ D^{[2]}u(x) &= D_2 D_1 u(x) = \frac{5}{2} x_2^{\frac{3}{2}} \varphi, \\ D^{[3]}u(x) &= D_3 D_2 D_1 u(x) = 0.\end{aligned}$$

It is clear to see that $D^{[1]}u$, $D^{[2]}u$, $D^{[3]}u \in X$, so $u \in X_1$. On the other hand, we also have

$$D_1 u(x) = x_2^{\frac{5}{2}} \varphi, \; D_2 u(x) = \frac{5}{2} x_1 x_2^{\frac{3}{2}} \varphi, \; D_3 u(x) = 0,$$
$$D_1^2 u(x) = D_3^2 u(x) = 0,$$
$$D_1 D_3 u(x) = D_3 D_1 u(x) = D_2 D_3 u(x) = D_3 D_2 u(x) = 0,$$
$$D_2^2 u(x) = \frac{15}{4} x_1 x_2^{\frac{1}{2}} \varphi,$$
$$D_1 D_2 u(x) = D_2 D_1 u(x) = \frac{5}{2} x_2^{\frac{3}{2}} \varphi,$$

it leads to $u \in C^2(\Omega;E)$.

However, $u \notin C^3(\Omega;E)$, since $\nexists D_2^3 u(x_1,0,x_3)$, by the fact that

$$D_2^3 u(x) = \frac{15}{8} \frac{x_1}{\sqrt{x_2}} \varphi, \; (x_1,x_2,x_3) \in [0,1] \times (0,1] \times [0,1].$$

We now show that there exists $v : \Omega \to E$ such that $v \in X_1$ and $v \notin C^2(\Omega;E)$.

Fixed $\widetilde{\varphi} \in E = C([0,1];\mathbb{R})$. Put

$$v(x_1,x_2,x_3) = x_1 x_2^{\frac{3}{2}} \; \widetilde{\varphi} \in E,$$
$$v(x)(t) = v(x;t) = x_1 x_2^{\frac{3}{2}} \; \widetilde{\varphi}(t).$$

Then, we have

$$\begin{aligned}D^{[1]}v(x) &= D_1 v(x) = x_2^{\frac{3}{2}} \; \widetilde{\varphi}, \\ D^{[2]}v(x) &= D_2 D_1 v(x) = \frac{3}{2} x_2^{\frac{1}{2}} \; \widetilde{\varphi}, \\ D^{[3]}v(x) &= D_3 D_2 D_1 v(x) = 0.\end{aligned}$$

Hence $D^{[1]}v$, $D^{[2]}v$, $D^{[3]}v \in X$, so $v \in X_1$. But

$$D_2^2 v(x) = \frac{3}{4}\frac{x_1}{\sqrt{x_2}}\,\widetilde{\varphi},\ x = (x_1, x_2, x_3) \in [0,1]\times(0,1]\times[0,1],$$

therefore $\nexists D_2^2 v(x_1, 0, x_3)$, then $v \notin C^2(\Omega; E)$. $\square$

Example 2. We illustrate the result obtained in Subsection 3.1 by considering Eq. (1.1) with the following functions $K : \Omega \times \Omega \times E^4 \to E$, $g : \Omega \to E$.

The function K :

$$\begin{aligned} &K : \Omega \times \Omega \times E^4 \to E \\ &(x, y; u_1, \cdots, u_4) \longmapsto K(x, y; u_1, \cdots, u_4), \end{aligned} \tag{3.30}$$

$$K(x, y; u_1, \cdots, u_4)(t) = k(x; t)K_1(y; u_1, \cdots, u_4)(t),$$

where

$$K_1(y; u_1, \cdots, u_4)(t) = (y_1y_2y_3)^{\alpha_1} \cos\left(\frac{2\pi u_1(t)}{\theta_0(y; t)}\right) + \sum_{i=2}^{4} (y_1y_2y_3)^{\alpha_i} \sin\left(\frac{\pi u_i(t)}{D^{[i-1]}\theta_0(y; t)}\right),$$

$0 \le t \le 1$, $(x, y; u_1, \cdots, u_4) \in \Omega \times \Omega \times E^4$,

$$\begin{cases} k,\, \theta_0 : \Omega \to E, \\ k(x; t) = e^{-t}\left(x_1^{\bar{\gamma}_1} x_2^{\bar{\gamma}_2} x_3^{\bar{\gamma}_3} + e^{x_1+x_2+x_3}\right), 0 \le t \le 1, \\ \theta_0(x; t) = e^{-t}\left(x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} + e^{x_1+x_2+x_3}\right), 0 \le t \le 1, \end{cases} \tag{3.31}$$

α_i, γ_j, $\bar{\gamma}_j$, $i = \overline{1,4}; j = \overline{1,3}$) are positive constants.

The function g :

$$\begin{aligned} &g : \Omega \to E, \\ &g(x; t) = \theta_0(x; t) - \frac{1}{(1+\alpha_1)^3} k(x; t), \\ &0 \le t \le 1, x \in \Omega. \end{aligned} \tag{3.32}$$

The above positive constants α_1, α_2, γ_1, γ_2, γ_3, $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_3$, with γ_j, $\bar{\gamma}_j \in (1, 2)$, $(j = 1, 2, 3)$ and α_i, $(i = \overline{1,4})$ are big enough, satisfying

$$2\pi e\left(1 + \bar{\gamma}_1 + \bar{\gamma}_1\bar{\gamma}_2 + \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3 + 4e^3\right) \times \left[\frac{1}{(1+\alpha_1)^3} + \frac{1}{2}\sum_{i=2}^{4}\frac{1}{(1+\alpha_i)^3}\right] < 1. \tag{3.33}$$

We can check to see that the assumptions (A_1), (A_2) of Theorem 3.1 hold, then we obtain the result in this theorem. We also can check to see that, $\theta_0 \in X_1$ is a unique solution of Eq. (1.1). This solution is bounded with the estimate as follows

$$\|\theta_0\|_{X_1} \le \left(1 + \gamma_1 + \gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_3 + 4e^3\right).$$

Indeed, it is clear to see that (A_1) holds by θ_0, $k \in X_1$. After implementing detailed calculations, we see that (A_2) also holds because of the following estimates obtained.

$\forall (x, y; u_1, \cdots, u_4), (\bar{x}, \bar{y}; \bar{u}_1, \cdots, \bar{u}_4) \in \Omega\times\Omega\times E^4$,

$$\begin{aligned} &\|K(x, y; u_1, \cdots, u_4) - K(\bar{x}, \bar{y}; \bar{u}_1, \cdots, \bar{u}_4)\|_E \\ &\le 4\|k(x) - k(\bar{x})\|_E \\ &+ \|k(\bar{x})\|_E \sum_{i=1}^{4} |(y_1y_2y_3)^{\alpha_i} - (\bar{y}_1\bar{y}_2\bar{y}_3)^{\alpha_i}| \\ &+ 2\pi e^2\|k(\bar{x})\|_E \|\theta_0\|_{X_1} \sum_{i=1}^{4} \|u_i - \bar{u}_i\|_E \\ &+ 2\pi e^2\|k(\bar{x})\|_E \sum_{i=1}^{4} \|\bar{u}_i\|_E \left\|D^{[i-1]}\theta_0(\bar{y}) - D^{[i-1]}\theta_0(y)\right\|_E, \end{aligned}$$

so, the continuity of K is proved. Similarly, $D^{[1]}K$, $D^{[2]}K$, $D^{[3]}K : \Omega \times \Omega \times E^4 \to E$ are continuous.

$\forall (x, y; u_1, \cdots, u_4), (x, y; \bar{u}_1, \cdots, \bar{u}_4) \in \Omega\times\Omega\times E^4$,

$$\begin{aligned} &\|K(x, y; u_1, \cdots, u_4) - K(x, y; \bar{u}_1, \cdots, \bar{u}_4)\|_E \\ &\le k_0(x, y) \sum_{i=1}^{4} \|u_i - \bar{u}_i\|_E, \end{aligned}$$

$$\begin{aligned} &\left\|D^{[1]}K(x, y; u_1, \cdots, u_4) - D^{[1]}K(x, y; \bar{u}_1, \cdots, \bar{u}_4)\right\|_E \\ &\le k_1(x, y) \sum_{i=1}^{4} \|u_i - \bar{u}_i\|_E, \\ &\left\|D^{[2]}K(x, y; u_1, \cdots, u_4) - D^{[2]}K(x, y; \bar{u}_1, \cdots, \bar{u}_4)\right\|_E \\ &\le k_2(x, y) \sum_{i=1}^{4} \|u_i - \bar{u}_i\|_E, \\ &\left\|D^{[3]}K(x, y; u_1, \cdots, u_4) - D^{[3]}K(x, y; \bar{u}_1, \cdots, \bar{u}_4)\right\|_E \\ &\le k_3(x, y) \sum_{i=1}^{4} \|u_i - \bar{u}_i\|_E, \end{aligned}$$

in which

$$\begin{aligned} k_0(x, y) &= 2\pi e\|k(x)\|_E \left[(y_1y_2y_3)^{\alpha_1} + \frac{1}{2}\sum_{i=2}^{4}(y_1y_2y_3)^{\alpha_i}\right], \\ k_1(x, y) &= 2\pi e\left\|D^{[1]}k(x)\right\|_E \left[(y_1y_2y_3)^{\alpha_1} + \frac{1}{2}\sum_{i=2}^{4}(y_1y_2y_3)^{\alpha_i}\right], \\ k_2(x, y) &= 2\pi e\left\|D^{[2]}k(x)\right\|_E \left[(y_1y_2y_3)^{\alpha_1} + \frac{1}{2}\sum_{i=2}^{4}(y_1y_2y_3)^{\alpha_i}\right], \\ k_3(x, y) &= 2\pi e\left\|D^{[3]}k(x)\right\|_E \left[(y_1y_2y_3)^{\alpha_1} + \frac{1}{2}\sum_{i=2}^{4}(y_1y_2y_3)^{\alpha_i}\right]. \end{aligned}$$

$$\int_\Omega k_0(x,y)dy \le 2\pi e\left(1+e^3\right)\tilde{D}(\alpha_1,\cdots,\alpha_4),$$
$$\int_\Omega k_1(x,y)dy \le 2\pi e\left(\bar{\gamma}_1+e^3\right)\tilde{D}(\alpha_1,\cdots,\alpha_4),$$
$$\int_\Omega k_2(x,y)dy \le 2\pi e\left(\bar{\gamma}_1\bar{\gamma}_2+e^3\right)\tilde{D}(\alpha_1,\cdots,\alpha_4),$$
$$\int_\Omega k_3(x,y)dy \le 2\pi e\left(\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3+e^3\right)\tilde{D}(\alpha_1,\cdots,\alpha_4),$$
$$\tilde{D}(\alpha_1,\cdots,\alpha_4)=\frac{1}{(1+\alpha_1)^3}+\frac{1}{2}\sum_{i=2}^4\frac{1}{(1+\alpha_i)^3},$$

it leads to

$$\alpha^*=\sum_{i=0}^3\sup_{x\in\Omega}\int_\Omega k_i(x,y)dy$$
$$\le 2\pi e\left(1+\bar{\gamma}_1+\bar{\gamma}_1\bar{\gamma}_2+\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3+4e^3\right)\tilde{D}(\alpha_1,\cdots,\alpha_4),$$

thus $\alpha^*<1$. Example 2 is finished. □

Example 3. In order to validate our results in Subsection 3.2, we consider Eq. (1.1) with the functions $K:\Omega\times\Omega\times E^4\to E$, $g:\Omega\to E$ defined as follows.

The function K :

$$\begin{aligned}&K:\Omega\times\Omega\times E^4\to E\\&(x,y;u_1,\cdots,u_4)\longmapsto K(x,y;u_1,\cdots,u_4),\end{aligned}\tag{3.34}$$

where

$$K(x,y;u_1,\cdots,u_4)(t)=h(x;t)\bar{K}_1(y;u_1,\cdots,u_4)(t),$$
$$\bar{K}_1(y;u_1,\cdots,u_4)(t)=(y_1y_2y_3)^{\bar{\alpha}_1}\int_0^t\left|\frac{u_1(s)}{\zeta_0(y;s)}\right|^{\frac{1}{2}}ds$$
$$+\sum_{i=2}^4(y_1y_2y_3)^{\bar{\alpha}_i}\int_0^t\left(\frac{u_i(s)}{D^{[i-1]}\zeta_0(y;s)}\right)^{\frac{1}{5}}ds,$$
$0\le t\le 1$, $(x,y;u_1,\cdots,u_4)\in\Omega\times\Omega\times E^4$,

$$\begin{aligned}&h,\zeta_0:\Omega\to E,\\&h(x;t)=e^{-t}\left(x_1^{\bar{\gamma}_1}x_2^{\bar{\gamma}_2}x_3^{\bar{\gamma}_3}+e^{x_1+x_2+x_3}\right),\\&\zeta_0(x;t)=e^{-t}\left(x_1^{\gamma_1}x_2^{\gamma_2}x_3^{\gamma_3}+e^{x_1+x_2+x_3}\right),\\&0\le t\le 1, x\in\Omega,\end{aligned}\tag{3.35}$$

$\bar{\alpha}_i$, γ_j, $\bar{\gamma}_j$, $i=\overline{1,4}$; $j=\overline{1,3}$ are positive constants.

The function g :

$$\begin{aligned}&g:\Omega\to E,\\&g(x;t)=\zeta_0(x;t)-th(x;t)\sum_{i=1}^4\frac{1}{(1+\bar{\alpha}_i)^3},\\&0\le t\le 1, x\in\Omega.\end{aligned}\tag{3.36}$$

The above positive constants $\bar{\alpha}_i$, γ_j, $\bar{\gamma}_j$, such that γ_j, $\bar{\gamma}_j\in(1,2)$ $(j=\overline{1,3})$ and $\bar{\alpha}_i$ $(i=\overline{1,4})$ are big enough satisfying the following condition

$$\begin{aligned}&4\left(1+\bar{\gamma}_1+\bar{\gamma}_1\bar{\gamma}_2+\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3+4e^3\right)\\&\times\left[\frac{e^{\frac{1}{2}}}{(1+\bar{\alpha}_1)^3}+\sum_{i=2}^4\frac{e^{\frac{1}{5}}}{(1+\bar{\alpha}_i)^3}\right]<1.\end{aligned}\tag{3.37}$$

We also can check to see that the assumptions (A_1), $(\hat{A}_2)$, $(\hat{A}_3)$ of Theorem 3.2 are correct, then Eq. (1.1) has a solution in X_1. Furthermore, the set of solutions of Eq. (1.1) is compact. We also can see that $\zeta_0\in X_1$ is a solution of Eq. (1.1).

Indeed, (A_1) holds by ζ_0, $h\in X_1$. Assumption $(\hat{A}_2)$ also holds, by the fact that

$\forall(x,y;u_1,\cdots,u_4)$, $(x,y;\bar{u}_1,\cdots,\bar{u}_4)\in\Omega^2\times E^4$,

$$\|K(x,y;u_1,\cdots,u_4)-K(\bar{x},\bar{y};\bar{u}_1,\cdots,\bar{u}_4)\|_E$$
$$\le\left(e^{\frac{1}{2}}\|u_1\|_E^{\frac{1}{2}}+e^{\frac{1}{5}}\sum_{i=2}^4\|u_i\|_E^{\frac{1}{5}}\right)(\|h(x)-h(\bar{x})\|_E)$$
$$+\left(e^{\frac{1}{2}}\|u_1\|_E^{\frac{1}{2}}+e^{\frac{1}{5}}\sum_{i=2}^4\|u_i\|_E^{\frac{1}{5}}\right)$$
$$\times\left(\|h(\bar{x})\|_E\sum_{i=1}^4\left|(y_1y_2y_3)^{\bar{\alpha}_i}-(\bar{y}_1\bar{y}_2\bar{y}_3)^{\bar{\alpha}_i}\right|\right)$$
$$+\|h(\bar{x})\|_E\int_0^1\left|\left|\frac{u_1(s)}{\zeta_0(y;s)}\right|^{\frac{1}{2}}-\left|\frac{\bar{u}_1(s)}{\zeta_0(\bar{y};s)}\right|^{\frac{1}{2}}\right|ds$$
$$+\|h(\bar{x})\|_E$$
$$\times\sum_{i=2}^4\int_0^1\left|\left(\frac{u_i(s)}{D^{[i-1]}\zeta_0(y;s)}\right)^{\frac{1}{5}}-\left(\frac{\bar{u}_i(s)}{D^{[i-1]}\zeta_0(\bar{y};s)}\right)^{\frac{1}{5}}\right|ds$$
$$\equiv R_1+R_2+R_3.$$

$$R_1=\left(e^{\frac{1}{2}}\|u_1\|_E^{\frac{1}{2}}+e^{\frac{1}{5}}\sum_{i=2}^4\|u_i\|_E^{\frac{1}{5}}\right)(\|h(x)-h(\bar{x})\|_E)$$
$$+\left(e^{\frac{1}{2}}\|u_1\|_E^{\frac{1}{2}}+e^{\frac{1}{5}}\sum_{i=2}^4\|u_i\|_E^{\frac{1}{5}}\right)$$
$$\times\|h(\bar{x})\|_E\sum_{i=1}^4\left|(y_1y_2y_3)^{\bar{\alpha}_i}-(\bar{y}_1\bar{y}_2\bar{y}_3)^{\bar{\alpha}_i}\right|,$$

$R_1\to 0$, as $|x-\bar{x}|+|\bar{y}-y|\to 0$.

$R_2=\|h(\bar{x})\|_E\int_0^1\left|\left|\frac{u_1(s)}{\zeta_0(y;s)}\right|^{\frac{1}{2}}-\left|\frac{\bar{u}_1(s)}{\zeta_0(\bar{y};s)}\right|^{\frac{1}{2}}\right|ds.$

By applying the following inequalies

$||a|^q-|b|^q|\le|a-b|^q$, $\forall a,b\in\mathbb{R}$, $\forall q\in(0,1]$,

$\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$, $\forall a,b\ge 0$,

$$\left|\left|\frac{u_1(s)}{\zeta_0(y;s)}\right|^{\frac{1}{2}}-\left|\frac{\bar{u}_1(s)}{\zeta_0(\bar{y};s)}\right|^{\frac{1}{2}}\right|$$
$$\le\left|\left|\frac{u_1(s)}{\zeta_0(y;s)}\right|-\left|\frac{\bar{u}_1(s)}{\zeta_0(\bar{y};s)}\right|\right|^{\frac{1}{2}}$$
$$\le e^{\frac{1}{2}}\|u_1-\bar{u}_1\|_E^{\frac{1}{2}}+e\|\bar{u}_1\|_E^{\frac{1}{2}}\|\zeta_0(\bar{y})-\zeta_0(y)\|_E^{\frac{1}{2}}.$$

Hence, $R_2 \to 0$ as $|\bar{y}-y| + \|u_1 - \bar{u}_1\|_E \to 0$.

$$R_3 = \|h(\bar{x})\|_E \times \sum_{i=2}^{4} \int_0^1 \left| \left(\frac{u_i(s)}{D^{[i-1]}\zeta_0(y;s)} \right)^{\frac{1}{5}} - \left(\frac{\bar{u}_i(s)}{D^{[i-1]}\zeta_0(\bar{y};s)} \right)^{\frac{1}{5}} \right| ds.$$

By applying the following inequalities

$$\left| |a|^{q-1} a - |b|^{q-1} b \right| \le 2^{1-q} |a-b|^q, \ \forall a,b \in \mathbb{R}, \forall q \in (0,1],$$

$$(a+b)^{\frac{1}{m}} \le a^{\frac{1}{m}} + b^{\frac{1}{m}}, \forall a,b \ge 0, \forall m \ge 1,$$

$$\begin{aligned} &\left| \left(\frac{u_i(s)}{D^{[i-1]}\zeta_0(y;s)} \right)^{\frac{1}{5}} - \left(\frac{\bar{u}_i(s)}{D^{[i-1]}\zeta_0(\bar{y};s)} \right)^{\frac{1}{5}} \right| \\ &\le 2^{\frac{4}{5}} e^{\frac{1}{5}} \|u_i - \bar{u}_i\|_E^{\frac{1}{5}} \\ &+ 2^{\frac{4}{5}} e^{\frac{2}{5}} \|\bar{u}_i\|_E^{\frac{1}{5}} \left\| D^{[i-1]}\zeta_0(\bar{y}) - D^{[i-1]}\zeta_0(y) \right\|_E^{\frac{1}{5}}, \end{aligned}$$

$R_3 \to 0$, as $|\bar{y}-y| + \sum_{i=1}^4 \|u_i - \bar{u}_i\|_E \to 0$.

It leads to

$$\begin{aligned} &\|K(x,y;u_1,\cdots,u_4) - K(\bar{x},\bar{y};\bar{u}_1,\cdots,\bar{u}_4)\|_E \\ &\le R_1 + R_2 + R_3 \to 0, \end{aligned}$$

as $|x-\bar{x}| + |\bar{y}-y| + \sum_{i=1}^4 \|u_i - \bar{u}_i\|_E \to 0$, so K is continuous.

Similarly, $D^{[1]}K$, $D^{[2]}K$, $D^{[3]}K : \Omega^2 \times E^4 \to E$ are continuous.

Assumption $(\hat{A}_2)$ holds, by the fact that, using the inequality $a^q \le 1 + a$, $\forall a \ge 0$, $\forall q \in (0,1]$,

$$\begin{aligned} &\|K(x,y;u_1,\cdots,u_4)\|_E \\ &\le \bar{h}_0(x,y) \left(1 + \sum\nolimits_{i=1}^4 \|u_i\|_E \right), \end{aligned}$$

with

$$\bar{h}_0(x,y) = 4\|h(x)\|_E \times \left(e^{\frac{1}{2}} (y_1y_2y_3)^{\bar{\alpha}_1} + e^{\frac{1}{5}} \sum_{i=2}^{4} (y_1y_2y_3)^{\bar{\alpha}_i} \right).$$

Similarly,

$$\begin{aligned} &\left\| D^{[i]}K(x,y;u_1,\cdots,u_4) \right\|_E \\ &\le \ \bar{h}_i(x,y) \left(1 + \sum\nolimits_{j=1}^4 \|u_j\|_E \right), \ i = 1,2,3, \end{aligned}$$

where

$$\bar{h}_i(x,y) = 4\left\| D^{[i]}h(x) \right\|_E \times \left(e^{\frac{1}{2}} (y_1y_2y_3)^{\bar{\alpha}_1} + e^{\frac{1}{5}} \sum_{i=2}^{4} (y_1y_2y_3)^{\bar{\alpha}_i} \right), i = 1,2,3.$$

Put

$$C(\bar{\alpha}_1,\cdots,\bar{\alpha}_4) = \frac{e^{\frac{1}{2}}}{(1+\bar{\alpha}_1)^3} + \sum\nolimits_{i=2}^4 \frac{e^{\frac{1}{5}}}{(1+\bar{\alpha}_i)^3},$$

then

$$\begin{aligned} &\int_\Omega \bar{h}_0(x,y)dy \le 4\left(1+e^3\right) C(\bar{\alpha}_1,\cdots,\bar{\alpha}_4); \\ &\int_\Omega \bar{h}_1(x,y)dy \le 4\left(\bar{\gamma}_1+e^3\right) C(\bar{\alpha}_1,\cdots,\bar{\alpha}_4); \\ &\int_\Omega \bar{h}_2(x,y)dy \le 2\left(\bar{\gamma}_1\bar{\gamma}_2+e^3\right) C(\bar{\alpha}_1,\cdots,\bar{\alpha}_4); \\ &\int_\Omega \bar{h}_3(x,y)dy \le 4\left(\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3+e^3\right) C(\bar{\alpha}_1,\cdots,\bar{\alpha}_4). \end{aligned}$$

Hence

$$\begin{aligned} \beta_* &= \sum\nolimits_{i=0}^3 \sup_{x\in\Omega} \int_\Omega \bar{h}_i(x,y)dy \\ &\le 4\left(1+\bar{\gamma}_1+\bar{\gamma}_1\bar{\gamma}_2+\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3+4e^3\right) C(\bar{\alpha}_1,\cdots,\bar{\alpha}_4), \end{aligned}$$

so $\beta < 1$.

Assumption $(\hat{A}_3)$ also holds, by the following.

(a) $K : \Omega \times \Omega \times E^4 \to E$ is compact.

By $K \in C(\Omega \times \Omega \times E^4; E)$, it remains to check that $K : \Omega \times \Omega \times E^4 \to E$ is compact. Let B be bounded in $\Omega \times \Omega \times E^4$, since

$$\begin{aligned} &\|K(x,y;u_1,\cdots,u_4)\|_E \\ &\le \bar{h}_0(x,y) \left(1 + \sum\nolimits_{i=1}^4 \|u_i\|_E \right) \\ &\le \sup_{(x,y;u_1,\cdots,u_4)\in B} \bar{h}_0(x,y) \left(1 + \sum\nolimits_{i=1}^4 \|u_i\|_E \right), \end{aligned}$$

for all $(x,y;u_1,\cdots,u_4) \in B$, $K(B)$ is uniformly bounded in E.

For all t, $\bar{t} \in [0,1]$, for all $(x,y;u_1,\cdots,u_4) \in B$,

$$\begin{aligned} &|K(x,y;u_1,\cdots,u_4)(t) - K(x,y;u_1,\cdots,u_4)(\bar{t})| \\ &\le 2\left(1+e^3\right) \left(e^{\frac{1}{2}} \|u_1\|_E^{\frac{1}{2}} + e^{\frac{1}{5}} \sum_{i=2}^{4} \|u_i\|_E^{\frac{1}{5}} \right) |t-\bar{t}|, \end{aligned}$$

so $K(B)$ is equicontinuous, it leads to $K(B)$ is relatively compact in E. We verify that $K : \Omega^2 \times E^4 \to E$ is compact.

(b) By the same argument, we also verify that $D^{[1]}K$, $D^{[2]}K$, $D^{[3]}K : \Omega \times \Omega \times E^4 \to E$ are compact.

(c) Finally, for each bounded subset J of E^4, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &\forall x,\bar{x} \in \Omega, |x-\bar{x}| < \delta \\ &\Longrightarrow \|K(x,y;u_1,\cdots,u_4) - K(\bar{x},y;u_1,\cdots,u_4)\|_E \\ &+ \sum_{i=1}^{3} \left\| D^{[i]}K(x,y;u_1,\cdot,u_4) - D^{[i]}K(\bar{x},y;u_1,\cdot,u_4) \right\|_E \\ &< \varepsilon, \ \forall y \in \Omega, \forall (u_1,\cdots,u_4) \in J. \end{aligned}$$

The above property is true, by the fact that

$$\|K(x,y;u_1,\cdots,u_4)-K(\bar{x},y;u_1,\cdots,u_4)\|_E$$
$$+\sum_{i=1}^{3}\left\|D^{[i]}K(x,y;u_1,\cdot,u_4)-D^{[i]}K(\bar{x},y;u_1,\cdot,u_4)\right\|_E$$
$$\leq\left(e^{\frac{1}{2}}\|u_1\|_E^{\frac{1}{2}}+e^{\frac{1}{5}}\sum_{i=2}^{4}\|u_i\|_E^{\frac{1}{5}}\right)$$
$$\times\left[\|h(x)-h(\bar{x})\|_E+\sum_{i=1}^{3}\left\|D^{[i]}h(x)-D^{[i]}h(\bar{x})\right\|_E\right]$$
$$\leq C\left[\|h(x)-h(\bar{x})\|_E+\sum_{i=1}^{3}\left\|D^{[i]}h(x)-D^{[i]}h(\bar{x})\right\|_E\right],$$

$\forall y\in\Omega$, $\forall(u_1,\cdots,u_4)\in J$, $\forall x,\,\bar{x}\in\Omega$, where we note that $h,\ D^{[1]}h,\ D^{[2]}h,\ D^{[3]}h:\Omega\to E$ are uniformly continuous on Ω. Example 3 is finished. □

4 Conclusion

By using appropriate tools in functional analysis and classical analysis, we have obtained sufficient conditions which ensure existence and some properties of solutions for a 3-order nonlinear IDE in three variables in an arbitrary Banach space. The correct setting of the problem in appropriate function spaces and choosing the correct tool for each equation are the key stages to solve them. The methods and techniques used in this paper can be applied to study problems for some IDEs in more general forms.

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