TẠP CHÍ KHOA HỌC ĐẠI HỌC HẠ LONG
Scientific Journal of Ha Long University
http://uhl.edu.vn/

SỐ LƯỢNG TRUNG BÌNH KHÁCH HÀNG TRONG CÁC HỆ THỐNG HÀNG ĐỢI VỚI THỜI GIAN TRUNG BÌNH HỮU HẠN

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Ngày nhận bài: 27/8/2022 *Ngày nhận bài sửa sau phản biện: 10/11/2022* *Ngày chấp nhận đăng: 13/11/2022*

TÓM TẮT

Trong lý thuyết về hàng đợi, các hệ thống xếp hàng với một số loại ràng buộc về tham số điều khiển của hệ thống luôn được quan tâm đặc biệt. Vấn đề phát sinh với hệ thống này là các hệ thống có nhiều ràng buộc khác nhau thường được yêu cầu khi giải quyết nhiều vấn đề ứng dụng trong các lĩnh vực, chủ đề khác nhau, đôi khi rất xa nhau. Trong bài báo này, chúng tôi đề xuất một mô hình toán học của hệ thống hàng đợi với cách thức xếp hàng đa kênh mở, kèm theo đó thời gian lưu trú tại hàng đợi là hữu hạn đối với các yêu cầu trong hàng đợi và thỏa mãn tính chất "thiếu kiên nhẫn" và "kiên nhẫn" đối với các yêu cầu được trình bày. Với sự trợ giúp của hàm siêu hình học Kummer, một công thức toán học chi tiết của mô hình này đã được thực hiện và các đặc điểm xác suất cơ bản của các hệ thống xếp hàng đã được tính toán.

Từ khóa: *hàng đợi, kiên nhẫn, mô hình, toán học, xác suất, yêu cầu.*

AVERAGE NUMBER OF CLIENTS IN THE QUEUING SYSTEMS WITH LIMITED AVERAGE TIME

ABSTRACT

Queuing systems with some kind of constraint on the system's control parameters have always been of particular interest in queuing theory. The point in this case is that systems with various constraints are frequently in demand when a variety of applied problems in different, sometimes very far from each other subject areas, are solved. In particular, logistics traditionally connected with the theory of mass service, as well as such innovative areas of modern applications as, for example, teletraffic theory, telecommunication theory, and many others can be referred to such subject areas. A mathematical model of an open multi-channel queuing system with a limited average residence time for claims in a queue containing "impatient" and "patient claims" is presented. With the help of the Kummer confluent hypergeometric function, a detailed mathematical formalization of the model was carried out and the basic probability characteristics of queuing systems of this type were computed.

Keywords: *mathematics, model, patient, probability, queuing system, requirement.*

1. INTRODUCTION

Queuing systems (QS) with some kind of constraint on the system's control parameters have always been of particular interest in queuing theory. The point in this case is that systems with various constraints are very often in demand when a variety of applied problems in different, sometimes very far from each other subject areas, are solved. In particular, logistics traditionally connected with the theory of mass service, as well as such innovative areas of modern applications as, for example, teletraffic theory, telecommunication theory, and many others can be referred to such subject areas.

Among problems with constraints, a class of problems with constraints imposed on the average residence time of a claim received by the system, both waiting for the service to start in the queue and in the queuing system as a whole (in the queue and under the service as well) is specified. In other words, some claims in the QS of these types are the so-called "impatient" ones which, after waiting for some time, may leave either the queue or the system, including those being at the service stage (A. Kirpichnikov và cs., 2018; A. P. Kirpichnikov, 2008, 2011). However, models of this type are still the least studied classes among all types of QS.

The primary goal of studying such systems from the standpoint of mathematics is as follows. In the course of computation, even for the most convenient research Markov-type models, sums of an infinite or finite number of summands appear, however, they are not reduced to the sums of corresponding geometric progressions. Thus, to solve problems of this kind, one has to resort to approximate numerical schemes in which each numerical characteristic of the problem is computed separately from others by summing several first terms of the corresponding finite or infinite series. In this case, it is impossible, of course, to obtain a closed analytical solution to the problem, though it is feasible, rather roughly, to evaluate the main characteristics of a queuing system of this type. For a number of applied problems, this procedure is quite sufficient, however, as a rule, it is not possible to study processes occurring in systems of this kind in more detail. In modern conditions, the absence of a closed analytical solution within which all main numerical characteristics of the system would be computed with the required accuracy and at the same time associated with each other, is a significant gap in queuing theory, understood as an applied field of research.

Given the foregoing, a more or less successful attempt of a new approach was to studying the queuing systems with constraints on the average residence time in the queue or in the system as a whole was made. The meaning of these works is as follows. Namely, (A. Kirpichnikov và cs., 2018) first proposed using a first-order Mittag-Leffler function (generalization of the exponential function) applied in the theory of the function of complex variables and integral transformations (A. Kirpichnikov và cs., 2016) for summing series other than geometric progressions. Some of the results obtained in these papers were also presented in (A. P. Kirpichnikov và cs., 2016). It was noted that this model formalization can also be carried out using Kummer confluent hypergeometric function to simplify a number of intermediate calculations and the type of final formulas. This idea was later implemented in the paper (A. Kirpichnikov & Titovtsev, 2016).

Full mathematical formalization of the appropriately assigned problem, including the calculation of the first and second moments of corresponding numerical characteristics of the system, was carried out for the first time in (A. Kirpichnikov và cs., 2018; A. P. Kirpichnikov, 2017; A. Kirpichnikov & Titovtsev, 2016). However, these works propose solutions to the problem with the constraint on the average time only in the case when all the demands entering the queuing system can be called "impatient" demands (claims). Meanwhile, from the point of view of possible applications, the expanded formulation of the problem seems to be more interesting in a certain way.

2. MATHEMATICAL MODEL OF "IMPATIENT" AND "PATIENT" REQUIREMENTS (CLAIMS) IN THE QUEUE

From the point of view of practice, it would be very interesting to consider such a variant of problem statement in which the so-called "impatient" claims can leave a queue only when its length exceeds a predetermined fixed numerical value. Let us denote this value by the letter E in our further computations. After E claims are left before the demand in the service queue, the demands cease to leave the queue, and in any case, wait for the service to start, i.e., they leave the category of "impatient" demands and become "patient" ones.

Let us suppose that we have a multichannel QS with a homogeneous infinite simplest flow of claims and a queue of unlimited length. Let the intensity of the claims stream be equal to λ, and the intensity of service, i.e., the average number of claims served by one device per unit of time, be μ. The flow of service will also be considered the simplest (with the intensity μ). Let us assume now that the time spent by a single claim in a queue is limited to some random time t with an average value of $\bar{t}$. Thus, each claim in the queue is impacted by a kind of exit flow with the intensity $\upsilon = \frac{1}{\bar{t}}$. If this flow is of the simplest nature, then the process proceeding in the QS will be a Markov process. In accordance with the foregoing, we now assume that queued claims cease to leave the queue when E demands waiting for service are left before them. Thus, this QS has claims of two different types at the same time – both "impatient" and "patient" ones. Fig. 1 presents the graph of such a queuing system. Let us carry out the mathematical formalization of the problem.

As we can see from the graph, we have a classical scheme of a death and reproduction process. Applying general expressions for the probabilities of the limit (stationary) states in this scheme, we obtain:

$$p_1 = \frac{\lambda}{\mu} p_0;\ p_2 = \frac{\lambda^2}{2\mu^2} p_0;\ p_3 = \frac{\lambda^3}{3!\mu^3} p_0; \dots$$

$$\dots\ p_m = \frac{\lambda^m}{m!\mu^m} p_0;\ \ p_{m+E} = \frac{\lambda^{m+E}}{m!m^E\mu^{m+E}} p_0;$$

$$p_{m+E+1} = \frac{\lambda^{m+E+1}}{m!m^E\mu^{m+E}(m\mu+\upsilon)} p_0;$$

$$p_{m+E+2} = \frac{\lambda^{m+E+2}}{m!m^E\mu^{m+E}(m\mu+\upsilon)(m\mu+2\upsilon)} p_0 \dots$$

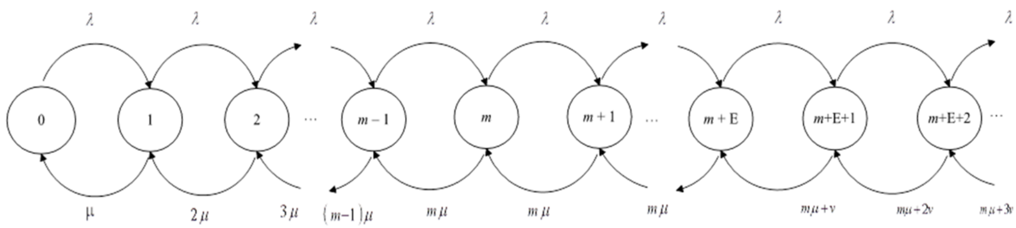

Fig. 1. The graph of the queuing system

In accordance with the above, we now propose that the queuing system under study has m serving channels with the same service intensity. As a result, we have the following formulas for p_k. Here, we can describe p_k as the probability in the interval from 0 to t with k claims and p_0 is the initial probability of the system:

$$p_k = \frac{p^k}{k!} p_0 \ \ at\ k \le m;$$

$$p_k = \frac{p^k}{m!\, m^{k-m}} p_0 \quad at\ m \le k \le m + E;$$

$$p_k = \frac{p^k}{m!\,(m+\beta)(m+2\beta)\Lambda[m+(k-m-E)\beta]} p_0$$

at $k \ge m + E$.

Further, according to the algorithm proposed in the above works, we divide the numerator and denominator of the third of these ratios by β^{k-m-E}. In the notation $\rho = \frac{\lambda}{\mu}$ (the reduced intensity of the claims stream, i.e., the average number of claims entering the system in the average service time of one claim). The general formula is $p_k = \rho^k p_0$, $k = 1, 2, \dots$ Following the normalization condition, we have:

$$p_0 = \frac{1}{1 + \rho + \rho^2 + \cdots}.$$

Obviously, if $\rho \ge 1$ the series in the denominator of this expression diverges, and in this case, the number of queued claims will increase infinitely. The stationary mode (stationary state) of the classical QS is thus established only if $\rho < 1$.

The $\beta = \frac{\upsilon}{\mu}$ (the average number of claims leaving the queue unserved in the average service time of one claim), we obtain:

$$p_k = \frac{\rho^k}{k!} p_0 \quad at\ k \le m;$$

$$p_k = \frac{\rho^k}{m!\, m^{k-m}} p_0 \quad at\ m \le k \le m+E;$$

$$p_k = \frac{\rho^{m+E}}{m!} \frac{\alpha^{k-m-E}}{m^E (\frac{m}{\beta}+1)_{k-m-E}} p_0$$

$$at\ k \ge m+E.$$

Where $(a)_k = a(a+1)(a+2)\dots(a+k-1)$; $(a)_0 = 1$ is the Pochhammer symbol. The value $\alpha = \frac{\rho}{\beta} = \frac{\lambda}{\upsilon}$ obviously shows what average number of claims entering the system during the average time of one "impatient" claim being in the queue. The total sum of probabilities of stationary states of the system is:

$$p_0 + \rho p_0 + \frac{\rho^2}{2!} p_0 + \frac{\rho^3}{3!} p_0 + \cdots + \frac{\rho^m}{m!} p_0 + \frac{\rho^{m+1}}{m!\, m} p_0 + \frac{\rho^{m+2}}{m!\, m^2} p_0 + \cdots + \frac{\rho^{m+E}}{m!\, m^E} p_0 + \cdots$$

$$= \left\{ e_{m-1}(\rho) + \frac{\rho^m}{(m-1)!\,(m-\rho)} \left[1 - \left(\frac{\rho}{m}\right)^E\right] + \frac{\rho^{m+E}}{m!\, m^E} \sum_{k=0}^{\infty} \frac{\alpha^k}{\left(\frac{m}{\beta}+1\right)_k} \right\} p_0.$$

In this relation, $e_m(\rho) = 1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \cdots + \frac{\rho^m}{m!}$ is an incomplete exponential function (incomplete exponent). Moreover, $e_0(\rho) = 1$, it is clear that $e_m(\rho) \to e^\rho$ with $m \to \infty$, the normalization condition for $\sum_{i=0}^{\infty} p_i = 1$ obviously gives in this case:

$$\left\{ e_{m-1}(\rho) + \frac{\rho^m}{(m-1)!\,(m-\rho)} \left[1 - \left(\frac{\rho}{m}\right)^E\right] + \frac{\rho^{m+E}}{m!\, m^E} \sum_{k=0}^{\infty} \frac{\alpha^k}{(m/\beta+1)_k} \right\} p_0$$
$$= \left\{ e_{m-1}(\rho) + \frac{\rho^m}{(m-1)!\,(m-\rho)} \left[1 - \left(\frac{\rho}{m}\right)^E\right] + \frac{\rho^{m+E}}{m!\, m^E} M\left(1; \frac{m}{\beta}+1; \alpha\right) \right\} p_0 = 1.$$

Where $M(a;b;z)$ is the Kummer confluent hypergeometric function defined by the relation:

$$M(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}.$$

Since $(1)_k = k!$. Then for the probability p_0 of complete system downtime, we have:

$$p_0 = \left\{ e_{m-1}(\rho) + \frac{\rho^m}{(m-1)!\,(m-\rho)} \left[1 - \left(\frac{\rho}{m}\right)^E\right] + \frac{\rho^{m+E}}{m!\, m^E} M\left(1; \frac{m}{\beta}+1; \alpha\right) \right\}^{-1}.$$

3. SIMPLIFYING THE FORMULA

The latter expression, in turn, can be further simplified as follows. Noticing that by definition:

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

where Γ is a gamma function, let us rewrite the expression for the Kummer confluent hypergeometric function as:

$$M(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{z^k}{k!}.$$

And the quantity of interest in the way:

$$M(1;b+1;z) = \Gamma(b+1) \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b+1+k)}.$$

Since $\Gamma(k+1) = k!$. Further, to simplify the latter expression, we use the following chain formulas:

$$\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(b+1+k)}$$
$$=\sum_{k=1}^{\infty}\frac{z^{k-1}}{\Gamma(b+k)}=\frac{1}{z}\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(b+k)}$$
$$=\frac{1}{z}\left[\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(b+k)}-\frac{1}{\Gamma(b)}\right].$$

Thus, we obtain:

$$M(1;b+1;z)$$
$$=\frac{\Gamma(b+1)}{z}\left[\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(b+k)}-\frac{1}{\Gamma(b)}\right].$$

As a result, the probability of complete system downtime will take the following form:

$$p_0=\left\{e_{m-1}(\rho)+\frac{\rho^m}{(m-1)!(m-\rho)}\left[1-\left(\frac{\rho}{m}\right)^E\right]+\frac{\rho^{m+E-1}}{(m-1)!m^E}\left[M\left(1;\frac{m}{\beta};\alpha\right)-1\right]\right\}^{-1}. \quad (1)$$

To monitor the correctness of the results obtained due to these computations, let us verify the obtained relation in two limiting cases for which the corresponding solutions are known. In the first limiting case, when $E\to 0$ the relation (1) turns into a previously published dependence (A. Kirpichnikov & Titovtsev, 2016):

$$p_0(E=0)$$
$$=\left\{e_{m-1}(\rho)+\frac{\rho^{m-1}}{(m-1)!}\left[M\left(1;\frac{m}{\beta};\alpha\right)-1\right]\right\}^{-1}.$$

It is interesting to note that the same formula will apparently remain valid for the general model $E\neq 0$, but in the degenerate case when $\rho=m$.

Let us regard the second limiting case when $\beta\to 0$. This obviously requires the consideration of how the Kummer confluent hypergeometric function $M(1;\frac{m}{\beta};\alpha)$ behaves when the values of the parameter β are close to zero. The general formula for the expansion of the function $M(1;\frac{m}{\beta};\alpha)$ in its K. Maclaurin series for small values of the parameter β is obtained in (A. P. Kirpichnikov et al., 2016) and has the following form:

$$M\left(1;\frac{m}{\beta};\alpha\right)\approx 1+$$
$$+\frac{\rho}{m-\rho}\left[1-\frac{\rho}{(m-\rho)^2}\beta+\frac{\rho(m+2\rho)}{(m-\rho)^4}\beta^2\right].$$

Thus, it is easy to verify that:

$$lim\, M\left(1;\frac{m}{\beta};\alpha\right)\Bigg|_{\beta\to 0}=\frac{m}{m-\rho},$$

and then in the second limiting case when $\beta\to 0$, the relation (1) turns into a known dependence

$$p_0(\beta=0)=\left[e_{m-1}(\rho)+\frac{\rho^m}{(m-1)!(m-\rho)}\right]^{-1}$$
$$=\left[e_m(\rho)+\frac{\rho^{m+1}}{m!(m-\rho)}\right]^{-1}$$

of the multi-channel QS models with expectation (according to Kendall's classification – M/M/m model).

4. SIMULATION AND RESULTS

In this work, we used the general-purpose simulation system GPSS World based on the Monte Carlo method (random test method) and Mathcad as tools for simulation and visualization. These tools are widely used for modeling both discrete and continuous systems like A. A. Markov. Such modeling systems have specialized tools that implement additional capabilities for organizing model experiments on a PC. What is especially valuable, they provide an opportunity to consider the time factor in the models, that is, to build dynamic simulation models, which is extremely important when studying systems with queues. In the models written in GPSS World, it is possible to take into account a large number of different factors and at the same time abandon many restrictions and assumptions. Such software tools usually contain built-in high-level programming languages focused on a specific subject area, which makes it possible to create model programs for studying complex systems at the lowest cost, which fully applies to the GPSS World simulation system. Finally, the GPSS World system operates in the Windows operating environment and is maximally focused on the use of modern technologies that provide high interactivity and visual presentation of information. All the above advantages led to the choice in this case in favor of GPSS World.

Here, a request entering the system is rejected, that is, it leaves the system without service, if two conditions are met at the time the transaction enters the system: the number of occupied places in the queue exceeds the specified number E and the average waiting time in the queue exceeds the specified value T.

Firstly, suppose that we have one channel for the service, then $m = 1$ correlatively:

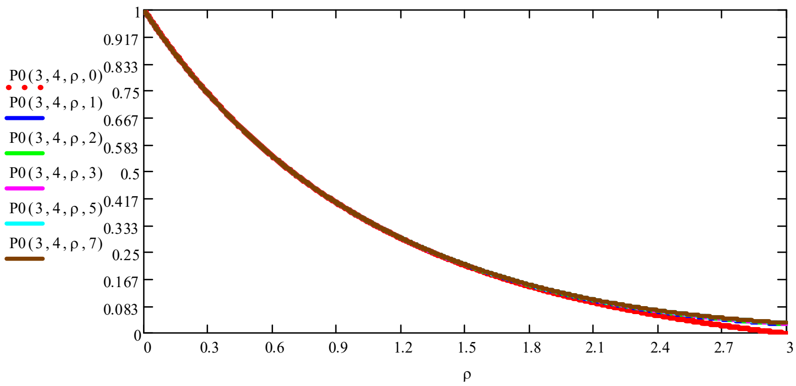

Fig. 2. Probabilities of system states for the case $m = 3$, $E = 4$ when only one channel for the service

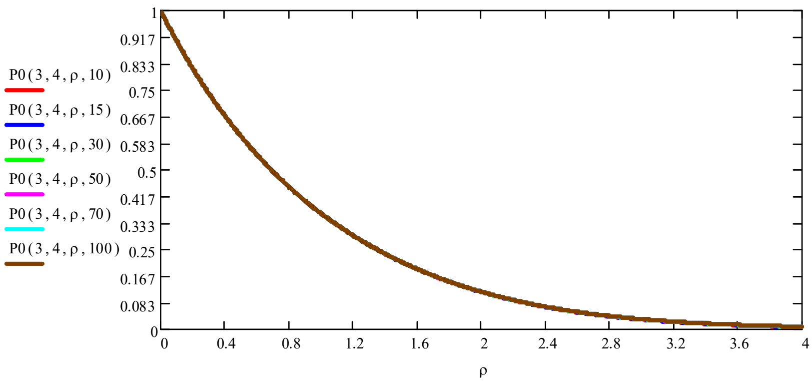

Fig. 3. Probabilities of system states for the case $m = 3$, $E = 4$

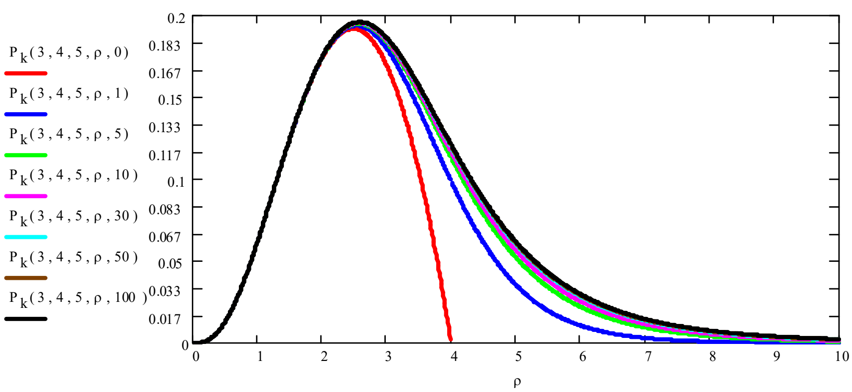

Fig. 4. Probabilities of system states for the case $k = 3$, $m = 4$, $E = 5$

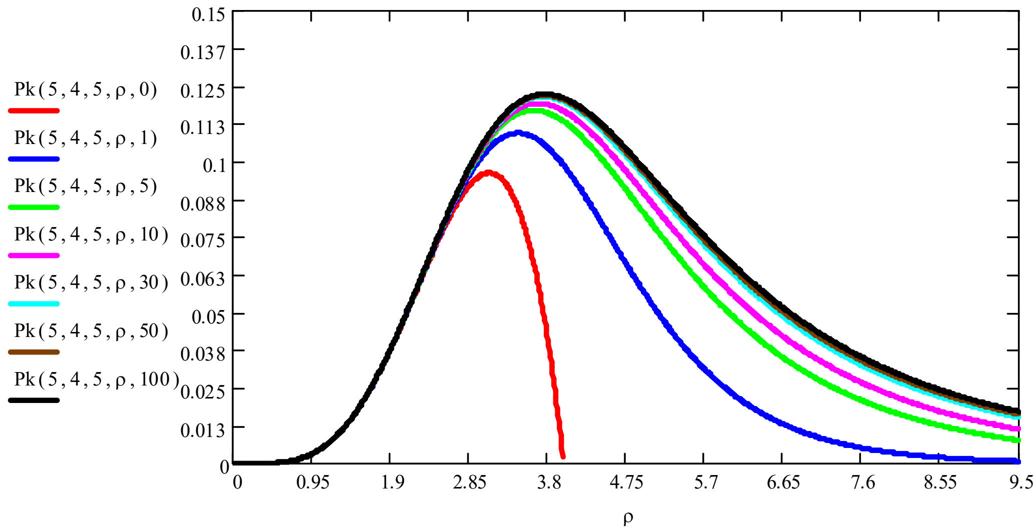

Fig. 5. Probabilities of system states for the case $k = 5, m = 4, E = 5$

Following the graphs, in the intermediate region of states of the system, when $m \leq k \leq m + E$, by absolute probability value, the probabilities of its states are minimal by the ratio to two other regions of states. Or, in other words, the probability of leaving from the queue (impatient) of the service queue among the "patient" claims is less than the probability of service, it is the similarity of being in the queue among the claims that can exit it without waiting for the service to start

5. CONCLUSION

A mathematical model for an open multichannel queuing system with a limited average time of a claim being in the queue for service is proposed. The main difference between this model and previous models of similar systems is the first introduced assumption that the so-called "impatient" customers can leave the queue only when the number of customers in the queue exceeds some predetermined fixed value.

The main probabilistic characteristics of the considered QS are calculated, including the probabilities of stationary states of the system, the probability of its complete downtime and the probability of waiting for the customer to begin service.

Analytical expressions are obtained for the first and second moments of the main discrete and continuous quantities that characterize systems with queues. These values include the number of busy service channels in the serving multichannel device, the number of claims in the queue awaiting the start of service, the time spent by one claim in the queue, as well as the total number of claims in the system and the total residence time of claims in the system (as in queue and under service).

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