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A NOTE ON STABLE SOLUTIONS OF A SUB-ELLIPTIC SYSTEM WITH SINGULAR NONLINEARITY

Vu Thi Hien Anh[1] and Dao Manh Thang[2]
[1]*Faculty of Mathematics, Hanoi National University of Education*
[2]*Hung Vuong High School for Gifted Student, Viet Tri, Phu Tho*

Abstract. In this paper, we study a system of the form

$$\begin{cases}\Delta_\lambda u = v \\ \Delta_\lambda v = -u^{-p}\end{cases} \quad \text{in } \mathbb{R}^N,$$

where $p > 1$ and Δ_λ is a sub-elliptic operator. We obtain a Liouville type theorem for the class of stable positive solutions of the system.
Keywords: Liouville-type theorem, stable positive solutions, Δ_λ-Laplacian, sub-elliptic operators.

1. Introduction

In this paper, we are interested in stable positive solutions of the following problem:

$$\begin{cases}\Delta_\lambda u = v \\ \Delta_\lambda v = -u^{-p}\end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 1$, and Δ_λ is a sub-elliptic operator defined by

$$\Delta_\lambda = \sum_{i=1}^{N} \partial_{x_i} \left(\lambda_i^2 \partial_{x_i}\right).$$

Throughout this paper, we always assume that the operator Δ_λ satisfies the following hypotheses which are first proposed in [1] and then used in many papers [2-7].

(H1) There is a group of dilations $(\delta_t)_{t>0}$

$$\delta_t : \mathbb{R}^N \to \mathbb{R}, (x_1, ..., x_N) \mapsto (t^{\varepsilon_1} x_1, ..., t^{\varepsilon_N} x_N)$$

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Contact Vu Thi Hien Anh, e-mail address: hienanh.k63hnue@gmail.com

with $1=\varepsilon_1 \le \varepsilon_2 \le ... \le \varepsilon_N$, such that λ_i is δ_t-homogeneous of degree $(\varepsilon_i - 1)$, i.e.,

$$\lambda_i(\delta_t(x)) = t^{\varepsilon_i - 1}\lambda_i(x), \text{ for all } x \in \mathbb{R}^N, t>0, i=1,2,...,N.$$

The number

$$Q = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_N \tag{1.2}$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to the group of dilations $(\delta_t)_{t>0}$.

(H2) The functions λ_i satisfy $\lambda_1 = 1$ and $\lambda_i(x) = \lambda_i(x_1, ..., x_{i-1})$, i.e., λ_i depends only on the first $(i-1)$ variables $x_1, x_2, ..., x_{i-1}$, for $i = 2, 3, ..., N$. Moreover, the function λ_i's are continuous on $\mathbb{R}^N$, strictly positive and of class C^2 on $\mathbb{R}^N \setminus \Pi$ where

$$\Pi = \left\{ (x_1, ..., x_N) \in \mathbb{R}^N; \prod_{i=1}^{N} x_i = 0 \right\}.$$

(H3) There exists a constant $\rho \ge 0$ such that

$$0 \le x_k \partial_{x_k} \lambda_i(x), x_k^2 \partial_{x_k}^2 \lambda_i(x) \le \rho \lambda_i(x)$$

for all $k \in \{1, 2, ..., i-1\}$, $i = 1, 2, ..., N$ and $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$.
These hypotheses allow us to use

$$\nabla_\lambda := (\lambda_1 \partial_{x_1}, \lambda_2 \partial_{x_2}, ..., \lambda_N \partial_{x_N})$$

which satisfies $\Delta_\lambda = (\nabla_\lambda)^2$. The norm corresponding to the Δ_λ is defined by

$$|x|_\lambda = \left(\sum_{i=1}^{N} \varepsilon_i \prod_{j \ne i} \lambda_i^2 |x_i|^2 \right)^{\frac{1}{2\gamma}},$$

where $\gamma = 1 + \sum_{i=1}^{N} (\varepsilon_i - 1) \ge 1$.

Let us first consider the case $\lambda_i = 1$ for $i = 1, 2, ..., N$. Then, the problem (1.1) becomes

$$\begin{cases} \Delta u = v \\ \Delta v = -u^{-p} \end{cases} \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Based on the idea in [8] for $N = 3$, Lai and Ye pointed out that the system (1.3) has no positive classical solution provided $0 < p \le 1$ in any dimension, [9]. When $p > 1$, the existence of positive classical solutions of the problem (1.3) and of the biharmonic problem

$$-\Delta^2 u = u^{-p} \tag{1.4}$$

are equivalent, see [9-11]. In the low dimensions, $N = 3, 4$, the problem (1.4) has no C^4-positive solution [11]. In the case $N \ge 5$, the existence and the assymptotic behavior of radial solutions of (1.3) have been studied by many mathematicians [8, 9, 11, 12]. For a special class of solutions, i.e., the class of stable positive solutions, an interesting and open problem posed by Guo and Wei [10] is as follows:

Conjecture A: *Let $p > 1$ and $N \geq 5$. A smooth stable solution to* (1.3) *with growth rate $O(|x|^{\frac{4}{p+1}})$ at ∞ does NOT exist if and only if p satisfies the following condition*

$$p > p_0(N) := \frac{N + 2 - \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}{6 - N + \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}$$

where $H_N = \left(\frac{N(N-4)}{4}\right)^2$. As shown in [10], the growth condition $O(|x|^{\frac{4}{p+1}})$ in this conjecture is natural since the equation (1.4) admits entire radial solutions with growth rate $O(r^2)$. The following result was obtained in [10].

Theorem A. *Let $p > 1$ and $N \geq 5$. The problem* (1.4) *has no classical stable solution $u(x)$ satisfying*

$$u(x) = O(|x|^{\frac{4}{p+1}}),\ \textit{as } |x| \to \infty$$

provided that $p > \max(\bar{p}, p_*(N))$. Here

$$p_*(N) = \begin{cases} \frac{N+2-\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}}{6-N+\sqrt{4+N^2-4\sqrt{N^2+H_N^*}}} & \textit{if } 5 \leq N \leq 12 \\ +\infty & \textit{if } N \geq 13 \end{cases},$$

where $H_N^* = \left(\frac{N(N-4)}{4}\right)^2 + \frac{(N-2)^2}{2} - 1$ *and*

$$\bar{p} = \frac{2 + \bar{N}}{6 - \bar{N}},$$

where $\bar{N} \in (4, 5)$ is the unique root of the algebraic equation $8(N-2)(N-4) = H_N^*$. It is worth to noticing that $p_*(N) > p_0(N)$. Then, Theorem A is only a partial result and Conjecture A is still open.

In this decade, much attention has been paid to study the elliptic equations and elliptic systems involving degenerate operators such as the Grushin operator [13-18], the Δ_λ- Laplacian [3-7] and references given there. Remark that the Grushin operator is a typical example of Δ_λ-Laplacian, see [1] for further properties of the operator Δ_λ.

As far as we know, there has no work dealing with the system (1.1) involving sub-elliptic operators. The main difficulty arises from the fact that there is no spherical mean formula and one cannot use the ODE technique. Inspired by the work [10] and recent progress in studying degenerate elliptic systems [15], we propose, in this paper, to give a classification of stable positive solutions of (1.1). Motivated by [19, 20], we give the following definition.

Definition. *Let $p > 1$. A positive solution $(u, v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ of* (1.1) *is called stable if there are two positive smooth functions ξ and η such that*

$$\begin{cases} \Delta_\lambda \xi = \eta \\ \Delta_\lambda \eta = p u^{-p-1} \xi \end{cases} . \tag{1.5}$$

Theorem 1.1. *Let $p > 1$. The system* (1.1) *has no positive stable solution provided $Q < 4$.*

Theorem 1.2. *Let $p > 1$ and $Q \geq 4$. Assume that*

$$p > \max(\bar{p}, p_*(Q)). \tag{1.6}$$

Here

$$p_*(Q) = \begin{cases} \frac{Q+2-\sqrt{4+Q^2-4\sqrt{Q^2+H_Q^*}}}{6-Q+\sqrt{4+Q^2-4\sqrt{Q^2+H_Q^*}}} & \text{if } 5 \leq Q \leq 12 \\ +\infty & \text{if } Q > 12 \end{cases},$$

where $H_Q^* = \left(\frac{Q(Q-4)}{4}\right)^2 + \frac{(Q-2)^2}{2} - 1$ and

$$\bar{p} = \frac{2 + \bar{Q}}{6 - \bar{Q}},$$

where $\bar{Q} \in (4, 5)$ is the unique root of the algebraic equation $8(Q-2)(Q-4) = H_Q^*$. Then the problem (1.1) *has no stable solution $u(x)$ satisfying*

$$u(x) = O(|x|_\lambda^{\frac{4}{p+1}}), \text{ as } |x| \to \infty.$$

Here, Q is defined in (1.2)*.*

Remark that [21, Theorem 1.1] is a direct consequence of Theorem 1.2 when $\lambda_i = 1$ for $i = 1, 2, ..., N$. In order to prove Theorem 1.1, we borrow some ideas from [20-22] in which the comparison principle and the bootstrap argument play a crucial role. Recall that one can not use spherical mean formula to prove the comparison principle as in [21-23] and then this requires another approach. In this paper, we prove the comparison principle by using the maximum principle argument [15, 24]. In particular, we do not need the stability assumption as in [21, 22].

The rest of the paper is devoted to the proof of the main result.

2. Proof of Theorem 1.2

We begin by establishing an a priori estimate.

Lemma 2.1. *Suppose that* (u, v) *is a stable positive solution of* (1.1) *satisfying* $u(x) = |x|_\lambda^{\frac{4}{p+1}}$ *as* $|x|_\lambda \to \infty$. *Then for* R *large, there holds*

$$\int_{B_R} u^{-p} dx \leq R^{Q - \frac{4p}{p+1}} \tag{2.1}$$

and

$$\int_{B_R} u^2 dx \leq R^{Q + \frac{8}{p+1}}. \tag{2.2}$$

Here and in what follows

$$B_R = \{x \in \mathbb{R}^N; |x_i| \leq R^{\epsilon_i}, i = 1, 2, ..., N\}.$$

Proof. It follows from the growth condition of u that

$$\int_{B_R} u^2 dx \leq C R^{\frac{8}{p+1}} \int_{B_R} dx = C R^{Q + \frac{8}{p+1}}.$$

It remains to prove (2.1). The Hölder inequality gives

$$\int_{B_R} u^{-p} dx \leq C \left(\int_{B_R} u^{-p-1} dx \right)^{\frac{p}{p+1}} R^{\frac{Q}{p+1}}.$$

Put $\chi(x) = \phi(\frac{x_1}{R^{\epsilon_1}}, ..., \frac{x_N}{R^{\epsilon_N}})$ where $\phi \in C_c^\infty(\mathbb{R}^N; [0, 1])$ is a test function satisfying $\phi = 1$ on B_1 and $\phi = 0$ outside B_2. The stability inequality implies that

$$\int_{B_R} u^{-p-1} dx \leq \int_{B_{2R}} u^{-p-1} \chi^2 dx \leq C \int_{B_{2R}} |\Delta_\lambda \chi|^2 dx \leq C R^{Q-4}.$$

Combining these two estimates, we deduce (2.1). □

Remark that Theorem 1.1 is a direct consequence of the last estimate in the proof of Lemma 2.1.

Lemma 2.2. *For any* $\varphi, \psi \in C^4(\mathbb{R}^N)$, *there holds*

$$\begin{aligned} \Delta_\lambda \varphi \Delta_\lambda (\varphi \psi^2) = (\Delta_\lambda(\varphi\psi))^2 - 4(\nabla_\lambda \varphi \cdot \nabla_\lambda \psi)^2 + 2\varphi \Delta_\lambda \varphi |\nabla_\lambda \psi|^2 \\ - 4\varphi \Delta_\lambda \psi \nabla_\lambda \varphi \cdot \nabla_\lambda \psi - \varphi^2 (\Delta_\lambda \psi)^2. \end{aligned}$$

The proof of Lemma 2.2 is elementary, see e.g., [25]. We then omit the details. Consequently, we obtain

Lemma 2.3. *For any $\varphi \in C^4(\mathbb{R}^N)$ and $\psi \in C_c^4(\mathbb{R}^N)$, we have*

$$\int\limits_{\mathbb{R}^N} \Delta_\lambda\varphi\Delta_\lambda(\varphi\psi^2)dx = \int\limits_{\mathbb{R}^N} (\Delta_\lambda(\varphi\psi))^2\,dx + \int\limits_{\mathbb{R}^N} \left(-4(\nabla_\lambda\varphi\cdot\nabla_\lambda\psi)^2 + 2\varphi\Delta_\lambda\varphi|\nabla_\lambda\psi|^2\right)dx + \int\limits_{\mathbb{R}^N} \varphi^2\left(2\nabla_\lambda(\Delta_\lambda\psi)\cdot\nabla_\lambda\psi + (\Delta_\lambda\psi)^2\right)dx \tag{2.3}$$

and

$$2\int\limits_{\mathbb{R}^N} |\nabla_\lambda\varphi|^2|\nabla_\lambda\psi|^2dx = 2\int\limits_{\mathbb{R}^N} \varphi(-\Delta_\lambda\varphi)|\nabla_\lambda\psi|^2dx + \int\limits_{\mathbb{R}^N} \varphi^2\Delta_\lambda(|\nabla_\lambda\psi|^2)dx. \tag{2.4}$$

We next give a preparation to the bootstrap argument.

Lemma 2.4. *Let $p > 1$ and assume that (u, v) is a stable positive solution of* (1.1)*. Then, for $R > 0$,*

$$\int_{B_R} \left(v^2 + u^{-p+1}\right)dx \le CR^{Q-4+\frac{8}{p+1}}.$$

Proof. From (1.1) and an integration by parts, we have for $\varphi \in C_c^4(\mathbb{R}^N)$,

$$\int\limits_{\mathbb{R}^N} u^{-p}\varphi dx = -\int\limits_{\mathbb{R}^N} \Delta_\lambda u\Delta_\lambda\varphi dx. \tag{2.5}$$

On the other hand, the stability assumption (see e.g., [20, Lemma 7]) implies the following stability inequality

$$p\int\limits_{\mathbb{R}^N} u^{-p-1}\varphi^2dx \le \int\limits_{\mathbb{R}^N} |\Delta_\lambda\varphi|^2dx. \tag{2.6}$$

Put $\chi(x) = \phi(\frac{x_1}{R^{\epsilon_1}}, ..., \frac{x_N}{R^{\epsilon_N}})$ where $\phi \in C_c^\infty(\mathbb{R}^N; [0,1])$ is a test function satisfying $\phi = 1$ on B_1 and $\phi = 0$ outside B_2. An elementary calculation combined with the assumptions (H1), (H2) and (H3) gives

$$|\nabla_\lambda\chi| \le \frac{C}{R} \text{ and } |\Delta_\lambda\chi| \le \frac{C}{R^2}.$$

Similarly, we also have

$$|\nabla_\lambda(\Delta_\lambda)\chi| \le \frac{C}{R^3}.$$

Choosing $\varphi = u\chi^2$ in (2.5) and (2.5), there holds

$$\int\limits_{\mathbb{R}^N} u^{-p+1}\chi^2dx = -\int\limits_{\mathbb{R}^N} \Delta_\lambda u\Delta_\lambda(u\chi^2)dx \tag{2.7}$$

and

$$p\int\limits_{\mathbb{R}^N} u^{-p+1}\chi^2 dx \le \int\limits_{\mathbb{R}^N} |\Delta_\lambda(u\chi)|^2 dx. \tag{2.8}$$

It follows from (2.7) and (2.8) and Lemma 2.3 that

$$\begin{aligned}(p+1)\int\limits_{\mathbb{R}^N} u^{p+1}\chi^2 dx &= \int\limits_{\mathbb{R}^N} |\Delta_\lambda(u\chi)|^2 dx - \int\limits_{\mathbb{R}^N} \Delta_\lambda u \Delta_\lambda(u\chi^2) dx \\ &\le \int\limits_{\mathbb{R}^N} \left(4(\nabla_\lambda u\cdot\nabla_\lambda\chi)^2 - 2u\Delta_\lambda u|\nabla_\lambda\chi|^2\right) dx - \int\limits_{\mathbb{R}^N} u^2\left(2\nabla_\lambda(\Delta_\lambda\chi)\cdot\nabla_\lambda\chi + |\Delta_\lambda\chi|^2\right) dx.\end{aligned}$$

By using simple inequality combined with (2.4), we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \left(4(\nabla_\lambda u\cdot\nabla_\lambda\chi)^2 - 2u\Delta_\lambda u|\nabla_\lambda\chi|^2\right) dx &\le \int_{\mathbb{R}^N} 4|\nabla_\lambda u|^2|\nabla_\lambda\chi|^2 dx + \int\limits_{\mathbb{R}^N} 2uv|\nabla_\lambda\chi|^2 dx \\ &\le C\int\limits_{\mathbb{R}^N} uv|\nabla_\lambda\chi|^2 dx + C\int\limits_{\mathbb{R}^N} u^2\Delta_\lambda(|\nabla_\lambda\chi|^2) dx.\end{aligned}$$

Consequently,

$$\begin{aligned}\int\limits_{\mathbb{R}^N} u^{-p+1}\chi^2 dx &\le C\int\limits_{\mathbb{R}^N} uv|\nabla_\lambda\chi|^2 dx \\ &\quad + C\int\limits_{\mathbb{R}^N} u^2\left(\Delta_\lambda(|\nabla_\lambda\chi|^2) + |\nabla_\lambda(\Delta_\lambda\chi)\cdot\nabla_\lambda\chi| + |\Delta_\lambda\chi|^2\right) dx.\end{aligned} \tag{2.9}$$

It is easy to see that $\Delta_\lambda(u\chi) = v\chi + 2\nabla_\lambda u\cdot\nabla_\lambda\chi + u\Delta_\lambda\chi$ or equivalently

$$\Delta_\lambda(u\chi) - v\chi = 2\nabla_\lambda u\cdot\nabla_\lambda\chi + u\Delta_\lambda\chi.$$

Therefore,

$$\int\limits_{\mathbb{R}^N} v^2\chi^2 dx \le C\int\limits_{\mathbb{R}^N} \left(|\nabla_\lambda u\cdot\nabla_\lambda\chi|^2 + u^2|\Delta_\lambda\chi|^2 + |(\Delta_\lambda(u\chi)|^2\right) dx.$$

This together with (2.9), (2.7) and Lemma 2.2 yield

$$\begin{aligned}\int\limits_{\mathbb{R}^N} \left(v^2 + u^{-p+1}\right)\chi^2 dx &\le C\int\limits_{\mathbb{R}^N} uv|\nabla_\lambda\chi|^2 dx \\ &\quad + C\int\limits_{\mathbb{R}^N} u^2\left(|\Delta_\lambda(|\nabla_\lambda\chi|^2)| + |\nabla_\lambda(\Delta_\lambda\chi)\cdot\nabla_\lambda\chi| + |\Delta_\lambda\chi|^2\right) dx.\end{aligned}$$

Next, the function χ in the inequality above is replaced by χ^m, where m is chosen later on, one gets

$$\int_{\mathbb{R}^N} \left(u^{-p+1} + v^2\right) \chi^{2m} dx \le \int_{\mathbb{R}^N} uv\chi^{2(m-1)} |\nabla_\lambda \chi|^2 dx$$
$$+ C \int_{\mathbb{R}^N} u^2 \left(|\Delta_\lambda (|\nabla_\lambda \chi^m|^2)| + |\nabla_\lambda (\Delta_\lambda \chi^m) \cdot \nabla_\lambda \chi^m| + |\Delta_\lambda \chi^m|^2 \right) dx. \tag{2.10}$$

Moreover, it follows from the Young inequality, for $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} uv\chi^{2(m-1)} |\nabla_\lambda \chi|^2 dx \le \varepsilon \int_{\mathbb{R}^N} v^2 \chi^{2m} dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}^N} u^2 \chi^{2(m-2)} |\nabla_\lambda \chi|^4 dx.$$

Combining this and (2.10), one has

$$\int_{\mathbb{R}^N} \left(v^2 + u^{-p+1}\right) \chi^{2m} dx \le C \int_{\mathbb{R}^N} u^2 \chi^{2(m-2)} |\nabla_\lambda \chi|^4 dx$$
$$+ C \int_{\mathbb{R}^N} u^2 \left(|\Delta_\lambda (|\nabla_\lambda \chi^m|^2)| + |\nabla_\lambda (\Delta_\lambda \chi^m) \cdot \nabla_\lambda \chi^m| + |\Delta_\lambda \chi^m|^2 \right) dx.$$

Consequently, for $R > 0$,

$$\int_{B_R} \left(v^2 + u^{-p+1}\right) dx \le \int_{\mathbb{R}^N} \left(v^2 + u^{-p+1}\right) \chi^{2m} dx \le C R^{Q-4-\frac{8}{p-1}}.$$

$\square$

Lemma 2.5. *Let* $p > 1$*. Assume that* (u, v) *is a positive solution of* (1.1)*. Then, pointwise in* $\mathbb{R}^N$*, the following inequality holds*

$$\frac{v^2}{2} \ge \frac{u^{1-p}}{p-1}.$$

Proof. To simplify the notations, let us put

$$l := \sqrt{\frac{2}{p-1}} \text{ and } \sigma := \frac{1-p}{2}.$$

Since $p > 1$, we get

$$0 < l \text{ and } \sigma < 0.$$

It is enough to prove that

$$v \ge l u^\sigma.$$

Set $w = lu^\sigma - v$. We shall show that $w \leq 0$ by contradiction argument. Suppose in contrary that

$$\sup_{\mathbb{R}^N} w > 0.$$

A straightforward computation combined with the relation $-\Delta_\lambda v = u^p$ implies that

$$\begin{aligned}\Delta_\lambda w &= l\sigma u^{\sigma-1}\Delta_\lambda u + l\sigma(\sigma-1)u^{\sigma-2}|\nabla_\lambda u|^2 - \Delta_\lambda v \\ &\geq l\sigma u^{\sigma-1}\Delta_\lambda u - \Delta_\lambda v \\ &= l\sigma u^{\sigma-1}v + u^{-p} \\ &= \frac{1}{l}u^{\sigma-1}w.\end{aligned}$$

Consequently, we arrive at

$$\Delta_\lambda w \geq \frac{1}{l}u^{\sigma-1}w. \tag{2.11}$$

We now consider two possible cases of the supremum of w. First, if there exists x^0 such that

$$\sup_{\mathbb{R}^N} w = w(x^0) = lu^\sigma(x^0) - v(x^0) > 0,$$

then we must have $\frac{\partial w}{\partial x_i} = 0$ and $\frac{\partial^2 w}{\partial x_i^2} \leq 0$ for $i = 1, 2, ..., N$. This together with the assumption (H2) gives

$$\nabla_\lambda w(x^0) = 0 \text{ and } \Delta_\lambda w(x^0) \leq 0.$$

However, the right hand side of (2.11) at x^0 is positive thanks to (2.11). Thus, we obtain a contradiction.

It remains to consider the case where the supremum of w is attained at infinity. Let $\phi \in C_c^\infty(\mathbb{R}^N; [0,1])$ be a cut-off function satisfying $\phi = 1$ on B_1 and $\phi = 0$ outside B_2. Put $\phi_R(x) = \phi^m(\frac{x_1}{R^{\varepsilon_1}}, \frac{x_2}{R^{\varepsilon_2}}, ..., \frac{x_N}{R^{\varepsilon_N}})$ where $m > 0$ chosen later. A simple calculation combined with the assumptions (H1), (H2) show that

$$|\Delta_\lambda \phi_R| \leq \frac{C}{R^2}\phi_R^{\frac{m-2}{m}} \text{ and } \frac{|\nabla_\lambda \phi_R|^2}{\phi_R} \leq \frac{C}{R^2}\phi_R^{\frac{m-2}{m}}. \tag{2.12}$$

Put $w_R(x) = w(x)\phi_R(x)$ and then there exists $x_R \in B_{2R}$ such that $w_R(x_R) = \max_{\mathbb{R}^N} w_R(x)$. Therefore, as above

$$\nabla_\lambda w_R(x_R) = 0 \text{ and } \Delta_\lambda w_R(x_R) \leq 0.$$

This implies that at x_R

$$\nabla_\lambda w = -\phi_R^{-1} w \nabla_\lambda \phi_R \tag{2.13}$$

and

$$\phi_R \Delta_\lambda w \leq (2\phi_R^{-1}|\nabla_\lambda \phi_R|^2 - \Delta_\lambda \phi_R)w. \tag{2.14}$$

From (2.12), (2.13) and (2.14), one has

$$\phi_R \Delta_\lambda w \le \frac{C}{R^2} \phi_R^{\frac{m-2}{m}} w. \tag{2.15}$$

Multiplying (2.11) by ϕ_R and using (2.15), we obtain at x_R

$$\phi_R l \sigma u^{\sigma-1} w \le \frac{C}{R^2} \phi^{\frac{m-2}{m}} \phi_R w$$

or equivalently

$$\phi_R^{\frac{2}{m}}(x_R) u^{\sigma-1}(x_R) \le \frac{C}{R^2}.$$

By choosing $m = \frac{2}{\sigma-1} > 0$, there holds

$$u_R^{\sigma-1} \le \frac{C}{R^2}.$$

Remark that $\sigma < 0$. Thus, $\lim\limits_{R\to+\infty} u_R(x_R) = \infty$ and we obtain a contradiction since

$$\sup_{\mathbb{R}^N} w \le \lim_{R\to+\infty} u_R^\sigma(x_R) = 0.$$

$\square$

With Lemma 2.4 and Lemma 2.5 at hand, it is enough to follow the bootstrap argument in [10] to obtain the proof of Theorem 1.2.

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