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# A SECOND MAIN THEOREM FOR ENTIRE CURVES IN A PROJECTIVE VARIETY WHOSE DERIVATIVES VANISH ON INVERSE IMAGE OF HYPERSURFACE TARGETS

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**Abstract.** We establish a second main theorem for algebraically nondegenerate entire curves $f$ in a projective variety $V \subset P^n(\mathbb{C})$ and a hypersurface target $\{D_1, D_2, \ldots, D_q\}$ satisfying $f_{*,z} = 0$ for all $z \in \cup_{j=1}^{q} f^{-1}(D_j)$.
***Keywords:*** second main theorem, Nevanlinna theory.

## 1. Introduction

During the last century, several Second Main Theorems have been established for linearly nondegenerate holomorphic curves in complex projective spaces intersecting (fixed or moving) hyperplanes, and we now have satisfactory knowledge about it. Motivated by a paper of Corvaja-Zannier [1] in Diophantine approximation, in 2004 Ru [2] proved a Second Main Theorem for algebraically nondegenerate holomorphic curves in the complex projective space $\mathbb{CP}^n$ intersecting (fixed) hypersurface targets. One of the most important developments in 15 years pass in Nevanlinna theory is the work on the Second Main Theorem for hypersurface targets. The interested reader is referred to [2-9] for many interesting results on this topic.

In this paper, we establish a second main theorem with a good defect relation for entire curves in a projective variety whose derivatives vanish on inverse image of hypersurface targets. Our method is a combination of the techniques in [7-9].

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## 2. Notations

Let $\nu$ be a nonnegative divisor on $\mathbb{C}$. For each positive integer (or $+\infty$) $p$, we define the counting function of $\nu$ (where multiplicities are truncated by $p$) by

$$N^{[p]}(r,\nu) := \int_1^r \frac{n_\nu^{[p]}}{t}dt \quad (1 < r < \infty)$$

where $n_\nu^{[p]}(t) = \sum_{|z|\leq t} \min\{\nu(z), p\}$. For brevity we will omit the character $[p]$ in the counting function if $p = +\infty$.

For a meromorphic function $\varphi$ on $\mathbb{C}$, we denote by $(\varphi)_0$ the divisor of zeros of $\varphi$. We have the following Jensen's formula for the counting function

$$N(r,(\varphi)_0) - N(r, \left(\frac{1}{\varphi}\right)_0) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|(\varphi(re^{i\theta})\right| d\theta + O(1).$$

Let $f$ be a holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$ with a reduced representation $\widehat{f} = (f_0, \ldots, f_n)$. The characteristic function $T_f(r)$ of $f$ is defined by

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log\|f(e^{i\theta})\| d\theta, \quad r > 1,$$

where $\|f\| = \max\limits_{i=0,\ldots,n} |f_i|$.

Denote by $f_{*,z}$ the tangent mapping at $z \in \mathbb{C}$ of $f$.

Let $D$ be a hypersurface in $P^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[x_0, \ldots, x_n]$, $\deg Q = \deg D$. Asumme that $f(\mathbb{C}) \not\subset D$, then the counting function of $f$ with respect to $D$ is defined by $N_f^{[p]}(r, D) := N^{[p]}(r,(Q(f_0, \ldots, f_n))_0)$.

Let $V \subset P^n(\mathbb{C})$ be a projective variety of dimension $k$. Denote by $I(V)$ the prime ideal in $\mathbb{C}[x_0, ..., x_n]$ defining $V$. Denote by $\mathbb{C}[x_0, ..., x_n]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, ..., x_n]$ of degree $m$ (including 0). Put $I(V)_m := \mathbb{C}[x_0, ..., x_n]_m \cap I(V)$.

Assume that $f(\mathbb{C}) \subset V$, then we say that $f$ is algebraically nondegenerate in $V$ if there is no hypersurface $D \subset P^n(\mathbb{C})$, $V \not\subset D$ such that $f(\mathbb{C}) \subset D$.

The Hilbert function $H_V$ of $V$ is defined by $H_V(m) := \dim \frac{\mathbb{C}[x_0,\ldots,x_n]_m}{I(V)_m}$.

Consider two integer numbers $q, N$ satisfying $q \geq N + 1$, $N \geq k$. Hypersurfaces $D_1, \ldots, D_q$ in $P^n(\mathbb{C})$ are said to be in $N$-subgeneral position with respect to $V$ if $V \cap (\cap_{i=0}^N D_{j_i}) = \varnothing$, for all $1 \leq j_0 < \cdots < j_N \leq q$.

## 3. Main result

**Theorem 3.1.** *Let $V \subset P^n(\mathbb{C})$ be a complex projective variety of dimension $k$ $(1 \leq k \leq n)$. Let $Q_1, \ldots, Q_q$ be hypersurfaces in $P^n(\mathbb{C})$ in $N$-subgeneral position with respect to*

*$V$, $\deg Q_j = d_j$, where $N \geq k$ and $q > (N-k+1)(k+1)$. Denote by $d$ the common multiple of $d_j$'s. Let $f$ be an algebraically entire curve in $V$ satisfying $f_{*,z} = 0$ for all $z \in \cup_{j=1}^{q} f^{-1}(Q_j)$. Then, for each $\epsilon > 0$,*

$$\Big\| \, (q - (N-k+1)(k+1) - \epsilon)\, T_f(r) \leq \frac{M^2+M-1}{M^2+M} \sum_{j=1}^{q} \frac{1}{d_j} N_f(r, Q_j) + o(T_f(r)),$$

*where $M = k + d^k \deg V \left( [(2k+1)(N-k+1)^2(k+1)^2 d^{k-1} \deg V \epsilon^{-1}] + 1 \right)^k$. Here, we denote $[x] := \max\{t \in \mathbb{Z} : t \leq x\}$ for each real number $x$, and as usual, by the notation $\big\| P$ we mean the assertion $P$ holds for all $r \in [1, +\infty)$ excluding a Borel subset $E$ of $(1, +\infty)$ with $\int_E dr < +\infty$.*

We would like to remark that Chen-Ru-Yan [10], Giang [11], Quang [7] established degeneracy second main theorems with truncated counting functions. With notations as in Theorem 3.1, Quang [7] gave the following inequality:

$$\Big\| \, (q - (N-k+1)(k+1) - \epsilon)\, T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[M_0]}(r, Q_j) + o(T_f(r)).$$

*Proof.* Firstly, we prove the theorem for the case where all hypersurfaces $Q_j$'s have the same degree $d$. Denote by $\mathcal{I}$ the set of all permutations of the set $\{1, \ldots, q\}$. We have $n_0 := \#\mathcal{I} = q!$. We write $\mathcal{I} = \{I_1, \ldots, I_{n_0}\}$ and $I_i = (I_i(1), \ldots, I_i(q))$ where $I_1 < I_2 < \cdots < I_{n_0}$ in the lexicographic order. Since $Q_1, \ldots, Q_q$ are in $N$-subgeneral position with respect to $V$, we have $Q_{I_i(1)} \cap \cdots \cap Q_{I_i(N+1)} \cap V = \varnothing$ for all $i \in \{1, \ldots, n_0\}$. Therefore, by Lemma 3.1 in [7], for each $I_i \in \mathcal{I}$, there are linearly combinations $Q_{I_i(1)}, \ldots, Q_{I_i(N+1)}$ in the following forms:

$$P_{i,1} := Q_{I_i(1)}, \quad P_{i,s} := \sum_{j=2}^{N-k+s} b_{sj} Q_{I_i(j)} \quad (2 \leq s \leq k+1, b_{sj} \in \mathbb{C}) \tag{3.1}$$

such that $P_{i,1} \cap \cdots \cap P_{i,k+1} \cap V = \varnothing$.

We define a map $\Phi : V \to P^{\ell-1}(\mathbb{C}) \quad (\ell := n_0(k+1))$ by

$$\Phi(x) = (P_{1,1}(x) : \cdots : P_{1,k+1}(x) : \cdots : P_{n_0,1}(x) : \cdots : P_{n_0,k+1}(x)).$$

Then $\Phi$ is a finite morphism on $V$. We have that $Y := \mathrm{Im}\Phi$ is a complex projective variety of $P^{\ell-1}(\mathbb{C})$ and $\dim Y = k$ and

$$\triangle := \deg Y \geq d^k \deg V.$$

Let $\widehat{f} = (f_0, \ldots, f_n)$ be a reduced presentation of $f$. For each positive integer $u$, we take $v_1, \ldots, v_{H_Y(u)}$ in $\mathbb{C}[y_{1,1}, \ldots, y_{1,k+1}, \ldots, y_{n_0,1}, \ldots, y_{n_0,k+1}]_u$ such that they form a basis of $\frac{\mathbb{C}[y_{1,1},\dots,y_{1,k+1},\dots,y_{n_0,1},\dots,y_{n_0,k+1}]_u}{I_Y(u)}$. We consider an entire curve $F$ in $P^{H_Y(u)-1}(\mathbb{C})$ with a reduced representation

$$\widehat{F}(z) = (v_1(\Phi(\widehat{f}(z))), \dots, v_{H_Y(u)}(\Phi(\widehat{f}(z)))).$$

Since $f$ is algebraically nondegenerate, we have that $F$ is linearly nondegenerate. By (3.12) in [7], for every $\epsilon' > 0$ (which will be chosen later) we have

$$\begin{aligned}
&(q - (N-k+1)(k+1))\, T_f(r) \\
&\quad \le \sum_{j=1}^{q} \frac{1}{d} N_f(r, Q_j) - \frac{(N-k+1)(k+1)}{duH_Y(u)} \Big( N(r, (W(\widehat{F}))_0) - \epsilon' du T_f(r) \Big) \\
&\qquad + \frac{(N-k+1)(2k+1)(k+1)\triangle}{ud} \sum_{1 \le i \le n_0, 1 \le j \le k+1} m_f(r, P_{i,j}).
\end{aligned} \tag{3.2}$$

For each $i \in \{1, \dots, H_Y(u)\}$, we have

$$\Big( v_i(\Phi(\widehat{f}(z))) \Big)' = \sum_{s=0}^{n} \frac{\partial (v_i \Phi)}{\partial x_s} (\widehat{f}(z)) \cdot f_s'(z). \tag{3.3}$$

On the other hand, since $f_{*,z} = 0$ for all $z \in \cup_{j=1}^{q} f^{-1}(Q_j)$, we have

$$(f_0(z) : \cdots : f_n(z)) = (f_0'(z) : \cdots : f_n'(z))$$

for all $z \in \cup_{j=1}^{q} f^{-1}(Q_j)$.
Hence, by (3.3) and by Euler formula (for homogenous polynomials $v_i(\Phi(x)) \in \mathbb{C}[x_0, \dots, x_n]$), for all $z \in \cup_{j=1}^{q} f^{-1}(Q_j)$

$$\begin{aligned}
&\left( \Big( v_1(\Phi(\widehat{f}(z))) \Big)' : \cdots : \Big( v_{H_Y(u)}(\Phi(\widehat{f}(z))) \Big)' \right) \\
&\quad = \left( \sum_{s=0}^{n} \frac{\partial (v_1 \Phi)}{\partial x_s} (\widehat{f}(z)) \cdot f_s'(z) : \cdots : \sum_{s=0}^{n} \frac{\partial (v_{H_Y(u)} \Phi)}{\partial x_s} (\widehat{f}(z)) \cdot f_s'(z) \right) \\
&\quad = \left( \sum_{s=0}^{n} \frac{\partial (v_1 \Phi)}{\partial x_s} (\widehat{f}(z)) \cdot f_s(z) : \cdots : \sum_{s=0}^{n} \frac{\partial (v_{H_Y(u)} \Phi)}{\partial x_s} (\widehat{f}(z)) \cdot f_s(z) \right) \\
&\quad = \Big( v_1(\Phi(\widehat{f}(z))) : \cdots : v_{H_Y(u)}(\Phi(\widehat{f}(z))) \Big).
\end{aligned} \tag{3.4}$$

We consider an arbitrary $a \in \cup_{j=1}^{q} f^{-1}(Q_j)$ (if this set is nonempty). Then there exists $I_p \in \mathcal{I}$ such that

$$(Q_{I_p(1)}(\widehat{f}))_0(a) \ge (Q_{I_p(2)}(\widehat{f}))_0(a) \ge \cdots \ge (Q_{I_p(q)}(\widehat{f}))_0(a). \tag{3.5}$$

Since $Q_1, \dots, Q_q$ are in $N$-subgeneral position with respect to $V$, we have

$$(Q_{I_p(j)}(\widehat{f}))_0(a) = 0 \quad \text{for all} \quad j \in \{N+1, \dots, q\}. \tag{3.6}$$

Set $c_{t,s} := (P_{t,s}(\widehat{f}))_0(a)$ and

$$c := (c_{1,1}, \dots, c_{1,k+1}, \dots, c_{n_0,1}, \dots, c_{n_0,k+1}).$$

Then there are $a_i = (a_{i_{1,1}}, \dots, a_{i_{1,k+1}}, \dots, a_{i_{n_0,1}}, \dots, a_{i_{n_0,k+1}})$, $i = 1, 2, \dots, H_Y(u)$, such that $y^{a_1}, \dots, y^{a_{H_Y(u)}}$ form a basis of $\frac{\mathbb{C}[y_{1,1}, \dots, y_{1,k+1}, \dots, y_{n_0,1}, \dots, y_{n_0,k+1}]_u}{I_Y(u)}$ and

$$S_Y(u, c) = \sum_{i=1}^{H_Y(u)} a_i \cdot c,$$

where $y = (y_{1,1}, \dots, y_{1,k+1}, \dots, y_{n_0,1}, \dots, y_{n_0,k+1})$.
Hence, there are linearly independent (over $\mathbb{C}$) forms $L_1, \dots, L_{H_Y(u)}$ such that $y^{a_i} = L_i(v_1, \dots, v_{H_Y(u)})$ in $\frac{\mathbb{C}[y_{1,1}, \dots, y_{1,k+1}, \dots, y_{n_0,1}, \dots, y_{n_0,k+1}]_u}{I_Y(u)}$. Then we have

$$\begin{aligned} L_i(\widehat{F}) &= L_i(v_1(\Phi(\widehat{f})), \cdots, v_{H_Y(u)}(\Phi(\widehat{f}))) \\ &= P_{1,1}^{a_{i_{1,1}}}(\widehat{f}) \cdots P_{1,k+1}^{a_{i_{1,k+1}}}(\widehat{f}) \cdots P_{n_0,1}^{a_{i_{n_0,1}}}(\widehat{f}) \cdots P_{n_0,k+1}^{a_{i_{n_0,k+1}}}(\widehat{f}), \end{aligned} \tag{3.7}$$

for all $i \in \{1, 2, \dots H_Y(u)\}$.
Hence, for al $i \in \{1, 2, \dots H_Y(u)\}$

$$(L_i(\widehat{F}))_0(a) = \sum_{1 \le u \le n_0, 1 \le v \le k+1} a_{i_{t,s}} (P_{i_{t,s}}(\widehat{f}))_0(a) = a_i \cdot c.$$

Hence,

$$\sum_{i=1}^{H_Y(u)} (L_i(\widehat{F}))_0(a) = \sum_{i=1}^{H_Y(u)} a_i \cdot c = S_Y(u, c). \tag{3.8}$$

By (3.4), we have

$$(L_1(\widehat{F}(a)) : \cdots : L_{H_Y(u)}(\widehat{F}(a))) = ((L_1(\widehat{F}))'(a) : \cdots : (L_{H_Y(u)}(\widehat{F}))'(a)) \tag{3.9}$$

By Laplace expansion Theorem, we have

$$\begin{aligned} & W(L_1(\widehat{F})) : \cdots : L_{H_Y(u)}(\widehat{F})) \\ & = \begin{vmatrix} L_1(\widehat{F}) & L_2(\widehat{F}) & \dots & L_{H_Y(u)}(\widehat{F}) \\ (L_1(\widehat{F}))' & (L_2(\widehat{F}))' & \dots & (L_{H_Y(u)}(\widehat{F}))' \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ (L_1(\widehat{F}))^{(H_Y(u)-1)} & (L_2(\widehat{F}))^{(H_Y(u)-1)} & \dots & (L_{H_Y(u)}(\widehat{F}))^{(H_Y(u)-1)} \end{vmatrix} \\ & = \sum_{1\le s<t\le H_Y(u)} (-1)^{1+s+t} \begin{vmatrix} L_s(\widehat{F}) & L_t(\widehat{F}) \\ L_s(\widehat{F})' & L_t(\widehat{F})' \end{vmatrix} \det A_{st} \qquad (3.10) \end{aligned}$$

where $A_{st}$ is the matrix which is defined from the matrix $\Big(L_i(\widehat{F})^{(v)}\Big)_{1\le i,v+1\le H_Y(u)}$ by omitting two first rows and $s^{th}$, $t^{th}$ columns.
For each $1 \le s < t \le H_Y(u)$, it is clear that

$$(\det A_{st})_0 \ge \sum_{v\in\{1,\dots,H_Y(u)\}\setminus\{s,t\}}^{H_Y(u)} \max\{(L_v(f))_0 - H_Y(u) + 1, 0\}. \qquad (3.11)$$

We now prove that

$$\begin{aligned} \Big(L_s(\widehat{F}) \cdot L_t(\widehat{F})' - L_t(\widehat{F}) \cdot L_s(\widehat{F})'\Big)_0 (a) \ge \max\{(L_s(\widehat{F}))_0(a) - H_Y(u) + 1, 0\} \\ + \max\{(L_t(\widehat{F}))_0(a) - H_Y(u) + 1, 0\} + 1. \qquad (3.12) \end{aligned}$$

We distinguish three cases.
**Case 1.** $(L_s(\widehat{F}))_0(a) \le H_Y(u) - 1$ and $(H_{i_t}(\widehat{F}))_0(a) \le H_Y(u) - 1$.
Then, the right side of (3.12) is equal to 1, but by (3.9), the left side of (3.12) is not less than 1.
**Case 2.** $(L_s(\widehat{F}))_0(a) > H_Y(u) - 1$ and $(L_t(\widehat{F}))_0(a) > H_Y(u) - 1$.
We have

$$\begin{aligned} & \Big(L_s(\widehat{F}) \cdot (L_t(\widehat{F}))' - L_t(\widehat{F}) \cdot (L_s(\widehat{F}))'\Big)_0 (a) \ge \Big(L_s(\widehat{F})\Big)_0 (a) + \Big(L_t(\widehat{F})\Big)_0 (a) - 1 \\ & \quad \ge \Big((L_s(\widehat{F}))_0(a) - H_Y(u) + 1\Big) + \Big((L_t(\widehat{F}))_0(a) - H_Y(u) + 1\Big) + 1 \\ & \quad = \max\{(L_s(\widehat{F}))_0(a) - H_Y(u) + 1, 0\} + \max\{(L_t(\widehat{F}))_0(a) - H_Y(u) + 1, 0\} + 1. \end{aligned}$$

**Case 3.** $(L_s(\widehat{F}))_0(a) > H_Y(u) - 1$ and $(L_t(\widehat{F}))_0(a) < H_Y(u) - 1$ (and similarly for the case where $(L_s(\widehat{F}))_0(a) < H_Y(u) - 1$ and $(L_t(\widehat{F}))_0(a) > H_Y(u) - 1$).

We have

$$\begin{aligned}
\Big(L_s(\widehat{F})\cdot(L_t(\widehat{F}))'-L_t(\widehat{F})\cdot(L_s(\widehat{F}))'\Big)_0(a) &\geq (L_s(\widehat{F}))_0(a)-1\\
&\geq \Big((L_s(\widehat{F}))_0(a)-H_Y(u)+1\Big)+1\\
&= \max\{(L_s(\widehat{F}))_0(a)-H_Y(u)+1,0\}+\max\{(L_t(\widehat{F}))_0(a)-H_Y+1,0\}+1.
\end{aligned}$$

We have completed the proof of (3.12).

By (3.10), (3.11) and (3.12), we have

$$\begin{aligned}
(W(\widehat{F}))_0(a) &= \Big(W(L_1(\widehat{F}),\dots,L_{H_Y(u)}(\widehat{F}))\Big)_0(a)\\
&\geq \sum_{i=1}^{H_Y(u)}\max\{(L_i(\widehat{F}))_0(a)-H_Y(u)+1,0\}+1\\
&= \sum_{i=1}^{H_Y(u)}\Big(\max\{(L_i(\widehat{F}))_0(a)-H_Y(u)+1,0\}+\frac{1}{H_Y(u)}\Big)\\
&\geq \frac{1}{H_Y(u)(H_Y(u)-1)}\sum_{i=1}^{H_Y(u)}(L_i(\widehat{F}))_0(a)
\end{aligned}$$

(note that $\max\{x-y,0\}+\frac{1}{z}\geq\frac{1}{yz}x$ for all $x\geq 0, y,z>1)$.
Combining with (3.8), we get

$$(W(\widehat{F}))_0(a)\geq\frac{1}{H_Y(u)(H_Y(u)-1)}S_Y(u,c). \tag{3.13}$$

By the definition of $P_{i,j}$, $P_{p,1}\cap\dots\cap P_{p,k+1}\cap V=\varnothing$, hence, by Lemma 3.2 in [5] (or Theorem 2.1 and Lemma 3.2 in [3]), we have

$$\begin{aligned}
\frac{1}{uH_Y(u)}S_Y(u,c) &\geq \frac{1}{(k+1)}(c_{p,1}+\dots+c_{p,k+1})-\frac{(2k+1)\triangle}{u}\max_{1\leq t\leq n_0,1\leq s\leq k+1}c_{t,s}\\
&= \frac{1}{(k+1)}\sum_{s=1}^{k+1}(P_{p,s}(\widehat{f}))_0(a)-\frac{(2k+1)\triangle}{u}\sum_{1\leq t\leq n_0,1\leq s\leq k+1}(P_{t,s}(\widehat{f}))_0(a).
\end{aligned}\tag{3.14}$$

By (3.13) and (3.5), we have $(P_{p,1}(\widehat{f}))_0(a)=(Q_{I_p(1)}(\widehat{f}))_0(a)$ and

$$(P_{p,s}(\widehat{f}))_0(a)\geq(Q_{I_p(N-k+s)}(\widehat{f}))_0(a)$$

for all $s\in\{1,\dots,k+1\}$.

Hence, by (3.5), (3.6), we have

$$\begin{aligned}
\sum_{j=1}^{q}(Q_j(\widehat{f}))_0(a) &= \sum_{t=1}^{N+1}(Q_{I_p(t)}(\widehat{f}))_0(a) \\
&\leq (N-k+1)(Q_{I_p(1)}(\widehat{f}))_0(a) + \sum_{t=N-k+2}^{N+1}(Q_{I_p(t)}(\widehat{f}))_0(a) \\
&\leq (N-k+1)(P_{p,1}(\widehat{f}))_0(a) + \sum_{s=2}^{k+1}(P_{p,s}(\widehat{f}))_0(a) \\
&\leq (N-k+1)\sum_{s=1}^{k+1}(P_{p,s}(\widehat{f}))_0(a).
\end{aligned}$$

Hence, by (3.14), we have

$$\begin{aligned}
\frac{1}{uH_Y(u)}S_Y(u,c) \geq{}& \frac{1}{(k+1)(N-k+1)}\sum_{j=1}^{q}(Q_j(\widehat{f}))_0(a) \\
&- \frac{(2k+1)\triangle}{u}\sum_{1\leq t\leq n_0, 1\leq s\leq k+1}(P_{t,s}(\widehat{f}))_0(a).
\end{aligned}$$

Combining with (3.13) we get

$$\begin{aligned}
\frac{(N-k+1)(k+1)}{duH_Y(u)}(W(\widehat{F}))_0(a) &\geq \frac{(N-k+1)(k+1)}{duH_Y^2(u)(H_Y(u)-1)}S_Y(u,c) \\
&\geq \frac{1}{dH_Y(u)(H_Y(u)-1)}\sum_{j=1}^{q}(Q_j(\widehat{f}))_0(a) \\
&\quad - \frac{(2k+1)(N-k+1)(k+1)\triangle}{duH_Y(u)(H_Y(u)-1)}\sum_{1\leq t\leq n_0, 1\leq s\leq k+1}(P_{t,s}(\widehat{f}))_0(a) \\
&\geq \frac{1}{dH_Y(u)(H_Y(u)-1)}\sum_{j=1}^{q}(Q_j(\widehat{f}))_0(a) \\
&\quad - \frac{(2k+1)(N-k+1)(k+1)\triangle}{du}\sum_{1\leq t\leq n_0, 1\leq s\leq k+1}(P_{t,s}(\widehat{f}))_0(a),
\end{aligned}$$

for all $a \in \cup_{j=1}^{q} f^{-1}(Q_j)$.
Hence,

$$\begin{aligned}
\frac{(N-k+1)(k+1)}{duH_Y(u)}N(r, W(\widehat{F}))_0) \geq{}& \frac{1}{dH_Y(u)(H_Y(u)-1)}\sum_{j=1}^{q}N_f(r,Q_j) \\
&- \frac{(2k+1)(N-k+1)(k+1)\triangle}{du}\sum_{1\leq t\leq n_0, 1\leq s\leq k+1}N(r, P_{t,s}(\widehat{f}))_0).
\end{aligned}$$

Combining with (3.2) we have

$$\begin{aligned}
\Big\| \, (q-(N-k+1)(k+1))\, T_f(r) &\le \sum_{j=1}^{q} \frac{1}{d} N_f(r,Q_j) \\
&\quad - \frac{1}{dH_Y(u)(H_Y(u)-1)} \sum_{j=1}^{q} N_f(r,Q_j) + \frac{(N-k+1)(k+1)\epsilon'}{H_Y(u)} T_f(r) \\
&\quad + \frac{(2k+1)(N-k+1)(k+1)\triangle}{du} \sum_{1\le i\le n_0, 1\le j\le k+1} (N(r,P_{t,s}(\widehat{f}))_0) + m_f(r,P_{i,j})) \\
&\le \frac{H_Y(u)(H_Y(u)-1)-1}{H_Y(u)(H_Y(u)-1)} \sum_{j=1}^{q} \frac{1}{d} N_f(r,Q_j) \\
&\quad + \left( \frac{(N-k+1)(k+1)\epsilon'}{H_Y(u)} + \frac{(2k+1)(N-k+1)(k+1)\triangle}{du} \right) T_f(r). \qquad (3.15)
\end{aligned}$$

For each $\epsilon > 0$, we choose $u = u_0 := [\frac{(2k+1)(N-k+1)^2(k+1)^2\triangle}{d\epsilon}] + 1$, and $\epsilon' := \frac{\epsilon}{(N-k+1)(k+1)} - \frac{(2k+1)(N-k+1)(k+1)\triangle}{du}$.
Then, we have

$$\begin{aligned}
H_Y(u_0) &\le k + \deg Y u_0^k \\
&\le k + d^k \deg V \left( [(2k+1)(N-k+1)^2(k+1)^2 d^{k-1} \deg V \epsilon^{-1}] + 1 \right)^k \\
&= M,
\end{aligned}$$

(note that $\deg Y = \triangle \le d^k \deg V$).
Hence, by (3.15) we have

$$\Big\| \, (q-(N-k+1)(k+1)-\epsilon)\, T_f(r) \le \frac{M(M-1)-1}{M(M-1)} \sum_{j=1}^{q} \frac{1}{d} N_f(r,Q_j).$$

$\square$

## REFERENCES

[1] P. Corvaja and U. Zannier, 2004. On a general Thue's equation. *Amer. J. Math*. **126**, pp. 1033-1055.
[2] M. Ru, 2004. A defect relation for holomorphic curves intersecting hypersurfaces. *Amer. J. Math.*, **126**, pp. 215-226.
[3] M. Ru, 2009. Holomorphic curves into algebraic varieties. *Ann. of Math.*, **169**, pp. 255-267.

[4] G. Dethloff and T. V. Tan, 2011. A second main theorem for moving hypersurface targets. *Houston J. Math.*, **37**, pp. 79-111.

[5] G. Dethloff, T. V. Tan and D. D. Thai, 2011. An extension of the Cartan-Nochka second main theorem for hypersurfaces. *Int. J. Math.*, **22**, pp. 863-885.

[6] G. Dethloff and T. V. Tan, 2020. Holomorphic curves into algebraic varieties intersecting moving hypersurface targets. *Acta Math Vietnam*, **45**, pp. 291-308.

[7] S. D. Quang, 2019. Degeneracy second main theorems for meromorphic mappings into projective varieties with hypersurfaces. *Trans. Amer. Math. Soc.*, **371**, pp. 2431-2453.

[8] T. V. Tan, Higher dimensional generalizations of some theorems on normality of meromorphic functions, to appear in the *Michigan Math. J.*

[9] N. T. Son and T. V. Tan, A property of the spherical derivative of an entire curve in complex projective space, to appear in the *J. Math. Anal. Appl.*

[10] Z. Chen, M. Ru and Q. Yan, 2012. The degenerated second main theorem and Schmidts subspace theorem. *Sci. China Math.*, **55**, pp. 1367-1380.

[11] L. Giang, 2016. An explicit estimate on multiplicity truncation in the degenerated second main theorem. *Houston J. Math*. **42**, pp. 447-462.