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ON THE GENERATION OF THE CREMONA GROUP

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Abstract. Let $\mathbb{k}$ be an algebraically closed field of characteristic 0. We show that any set of generators of the Cremona group $\mathrm{Cr}_{\mathbb{k}}(n)$ of the projective space $\mathbb{P}^n_{\mathbb{k}}$ with n greater than 2 contains an infinite and uncountable number of non trivial birational isomorphisms.

Keywords: birational isomorphism, birational map, birational transformation, cremona group.

1. Introduction

Let $\mathrm{Cr}_{\mathbb{k}}(n) = \mathrm{Bir}(\mathbb{P}^n_{\mathbb{k}})$ denote the set of all birational maps of the projective space $\mathbb{P}^n_{\mathbb{k}}$ on a field $\mathbb{k}$. It is clear that $\mathrm{Cr}_{\mathbb{k}}(n)$ is a group under the composition of dominant rational maps; called the Cremona group of order n. It contains the group of automorphisms of $\mathbb{P}^n_{\mathbb{k}}$, i.e. the group of projective linear transformations $\mathrm{PGL}_{\mathbb{k}}(n+1)$. This group is naturally identified with the Galois group of $\mathbb{k}$-automorphisms of the field $\mathbb{k}(x_1, \ldots, x_n)$ of rational fractions in n-variables $x_1, \ldots, x_n$. It was studied for the first time by Luigi Cremona (1830 - 1903), an Italian mathematician. Although it has been studied since the 19th century by many famous mathematicians, it is still not well understood. For example, we still don't know if it has the structure of an algebraic group of infinite dimensions (see [1, 2]).

In Dimension 1, it is not difficult to see that $\mathrm{Cr}_{\mathbb{k}}(1) \cong \mathrm{PGL}_{\mathbb{k}}(2)$, because each element $f \in \mathrm{Cr}_{\mathbb{k}}(1)$ is of the form

$$\begin{array}{rcl} f : \mathbb{P}^1_{\mathbb{k}} & \dashrightarrow & \mathbb{P}^1_{\mathbb{k}} \\ [x : y] & \longmapsto & [ax + by : cx + dy] \,. \end{array}$$

where $a, b, c, d \in \mathbb{k}$ and $ad - bc \neq 0$. Hence $\mathrm{Cr}_{\mathbb{k}}(1) \cong \mathrm{PGL}_{\mathbb{k}}(2)$ via the following isomorphism

$$\begin{array}{rcl} \mathrm{Cr}_{\mathbb{k}}(1) & \xrightarrow{\simeq} & \mathrm{PGL}_{\mathbb{k}}(2) \\ f & \longmapsto & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

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where $f([x:y]) = [ax+by:cx+dy]$.

In Dimension 2, we consider the standard quadratic transformation

$$\begin{array}{rcl} \omega \, : \, \mathbb{P}^2_{\Bbbk} & \dashrightarrow & \mathbb{P}^2_{\Bbbk} \\ [x:y:z] & \longmapsto & [yz:zx:xy] \end{array}$$

i.e. in affine coordinates $\omega(x,y) = \left(\frac{1}{x}, \frac{1}{y}\right)$. Note that $\omega^{-1} = \omega$. We have a well-known theorem of Noether proved by Castelnuovo.

Theorem 1.1. ***(Noether, Castelnuovo).*** *If the field $\Bbbk$ is algebraically closed, then the Cremona group $Cr_{\Bbbk}(2)$ of Dimension 2 is generated by the group of projective linear transformations $PGL_{\Bbbk}(3)$ of the projective space $\mathbb{P}^2_{\Bbbk}$ and the standard quadratic transformation ω:*

$$Cr_{\Bbbk}(2) = \langle PGL_{\Bbbk}(3), \omega \rangle$$

i.e. every element $f \in Cr_{\Bbbk}(2)$ is a product of projective linear transformations of $PGL_{\Bbbk}(3)$ and the standard quadratic transformation ω

$$f = \varphi_1 \circ \omega \circ \cdots \circ \varphi_r \circ \omega \circ \varphi_{r+1}$$

where $\varphi_i \in PGL_{\Bbbk}(3)$ for all i.

Noether stated this theorem in 1871 and Castelnuovo proved it in 1901 (see [3]). This statement is only true if the dimension $n = 2$. In the case of the dimension $n > 2$, we have Theorem 2.1.

2. Main results

In classic algebraic geometry (see [4]), we know that a rational map of the projective space $\mathbb{P}^n_{\Bbbk}$ is of the form

$$\mathbb{P}^n_{\Bbbk} \ni [x_0 : \ldots : x_n] = x \dashrightarrow \varphi(x) = \left[P_0(x) : \ldots : P_n(x)\right] \in \mathbb{P}^n_{\Bbbk},$$

where $P_0, \ldots, P_n$ are homogeneous polynomials of same degree in $(n+1)$-variables $x_0, \ldots, x_n$ and are mutually prime. The common degree of P_i is called the degree of φ; denoted $\deg \varphi$. In the language of linear systems; giving a rational map such as φ is equivalent to giving a linear system without fixed components of $\mathbb{P}^n_{\Bbbk}$

$$\varphi^\star |\mathcal{O}_{\mathbb{P}^n}(1)| = \left\{ \sum_{i=0}^{n} \lambda_i P_i | \lambda_i \in \Bbbk \right\}.$$

Clearly, the degree of φ is also the degree of a generic element of $\varphi^\star|\mathcal{O}_{\mathbb{P}^n}(1)|$ and the undefined points of φ are exactly the base points of $\varphi^\star|\mathcal{O}_{\mathbb{P}^n}(1)|$.

Note that a rational map $\varphi : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ is not in general a map of the set $\mathbb{P}^n_{\Bbbk}$ to $\mathbb{P}^n_{\Bbbk}$; it is only the map defined on its domain of definition $\mathrm{Dom}(\varphi) = \mathbb{P}^n_{\Bbbk} \setminus V(P_0, \dots, P_n)$. We say that φ is *dominant* if its image $\varphi(\mathrm{Dom}(\varphi))$ is dense in $\mathbb{P}^n_{\Bbbk}$. By the Chevalley theorem, the image $\varphi(\mathrm{Dom}(\varphi))$ is always a constructible subset of $\mathbb{P}^n_{\Bbbk}$, hence, it is dense in $\mathbb{P}^n_{\Bbbk}$ if and only if it contains a non-empty Zariski open subset of $\mathbb{P}^n_{\Bbbk}$ (see the page 94, in [4]). In general, we can not compose two rational maps. However, the composition $\psi \circ \varphi$ is always defined if φ is dominant so that the set of all the dominant rational maps $\varphi : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ is identified with the set of injective field homomorphisms $\varphi^\star$ of the field of all the rational fractions $\Bbbk(x_1, \dots, x_n)$ in n-variables $x_1, \dots, x_n$. We say that a rational map $\varphi : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ is *birational* (a *birational automorphism, or birational transformation*) if there exists a rational map $\psi : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ such that $\psi \circ \varphi = \mathrm{id}_{\mathbb{P}^n} = \varphi \circ \psi$ as rational maps. Clearly, if such a ψ exists, then it is unique and is called the *inverse* of φ. Moreover, φ and ψ are both dominant. If we denote by $\mathrm{Cr}_{\Bbbk}(n) = \mathrm{Bir}(\mathbb{P}^n_{\Bbbk})$ the set of all birational maps of the projective space $\mathbb{P}^n_{\Bbbk}$, then $\mathrm{Cr}_{\Bbbk}(n)$ is a group under composition of dominant rational maps and is called the *Cremona group of order* n or the *Cremona group of dimension* n. This group is naturally identified with the Galois group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1, \dots, x_n)$ of rational fractions in n-variables $x_1, \dots, x_n$. We immediately have the main following result:

Theorem 2.1. ***(Main theorem).*** *If n is a positive integer, $n > 2$ then every set of generators of the Cremona group $Cr_{\Bbbk}(n)$ of the projective space $\mathbb{P}^n_{\Bbbk}$ must contain an infinite and uncountable number of birational transformations of degree > 1.*

In order to prove this theorem, we need the following results. The first discusses on the existence of birational transformations:

If $f, q \in \Bbbk[x_0, x_1, \dots, x_n]$ and $t_1, \dots, t_n \in \Bbbk[x_1, \dots, x_n]$ are homogeneous polynomials with $\deg(f) = \deg(qt_i)$ for all i, we note $T_{f,q,t} : \mathbb{P}^n_{\Bbbk} \dashrightarrow \mathbb{P}^n_{\Bbbk}$ and $\overline{T} : \mathbb{P}^{n-1}_{\Bbbk} \dashrightarrow \mathbb{P}^{n-1}_{\Bbbk}$ the rational maps defined respectively by

$$T_{f,q,t} := [f, qt_1, \dots, qt_n], \quad \text{and} \quad \overline{T} := [t_1, \dots, t_n].$$

Lemma 2.1. *Suppose that d, l are integers with $d \geq l + 1 \geq 2$. Consider homogeneous polynomials without common factors $f, q \in \Bbbk[x_0, x_1, \dots, x_n]$ of degrees d, l respectively and $t_1, \dots, t_n \in \Bbbk[x_1, \dots, x_n]$ of degree $d - l$. Suppose that $f = x_0 f_{d-1} + f_d$ and $q = x_0 q_{l-1} + q_l$ with $f_{d-1}, f_d, q_{l-1}, q_l \in \Bbbk[x_1, \dots, x_n]$ and $f_{d-1} \neq 0$ or $q_{l-1} \neq 0$. Then $T_{f,q,t}$ is birational if and only if $\overline{T}$ is birational.*

Proof. On the one hand, $\Bbbk(T) = \Bbbk\left(\overline{T}\right)(\alpha)$ with $\alpha := \dfrac{f}{qt_1}$. On the other hand because that $\gcd(f, q) = 1$, the hypothesis on f_{d-1} and q_{l-1} is equivalent to $f_{d-1}q_l - f_d q_{l-1} \neq 0$, therefore $\alpha \in \mathrm{PGL}_{\Bbbk\left(\mathbb{P}^{n-1}_{\Bbbk}\right)}(2)$. Hence we obtain the assertion. ☐

Remark 2.1. *The transformations constructed in Lemma 2.1 above are studied in detail in the article [5].*

We recall that if $S \subset \mathbb{P}^n_{\Bbbk}$ is a hypersurface of equation $q' = 0$ and a point $P \in \mathbb{P}^n_{\Bbbk}$, the multiplicity of S at P is the order of zero of $q' = 0$ at P.

Corollary 2.1. *Suppose that $n \geq 2$ and $S \subset \mathbb{P}^n_{\Bbbk}$ is a hypersurface of degree $l \geq 1$ and suppose that S has a point of multiplicity $\geq l - 1$, we denote this point by O and d is an integer $\geq l + 1$. Then there exists a birational transformation ω of degree d of $\mathbb{P}^n_{\Bbbk}$ so that this hypersurface is contracted to a point by ω.*

Proof. Without loss of generality, we can suppose $O := [1 : 0 : \ldots : 0]$. Note that $q' = 0$ the equation of S and take $h = 0$ the equation of a generic plane passing through O. Finally, we choose $f := x_0 f_{d-1} + f_d$ with $f_{d-1} \neq 0$ and verifying $\gcd(f, hq') = 1$. If $q := h^{d-l-1}q'$ and $t_i = x_i$ for $i = 1, 2, \ldots, n$, then the rational map $\omega = T_{f,q,t}$ satisfies the conclusion of the corollary ☐

Let $\varphi \in \mathrm{Cr}_{\Bbbk}(n)$ and suppose that $X \subset \mathbb{P}^n_{\Bbbk}$ is a subvariety. We will say that φ is generically injective on X if there exists an open subset non-empty $U \subset \mathbb{P}^n_{\Bbbk}$, $U \cap X \neq \emptyset$ on which φ is defined and injective. The proof of the following lemma is trivial.

Lemma 2.2. *Let $\varphi = \varphi_1 \circ \cdots \circ \varphi_r$ with $\varphi_i \in$ Cr$_{\Bbbk}(n)$ and suppose that $X \subset \mathbb{P}^n_{\Bbbk}$ is a subvariety on which φ is not generically injective. Then there exists $1 \leq i \leq n$ so that X is birationally equivalent to a subvariety on which φ_i is not generically injective.*

Proof. Now, we prove Theorem 2.1 We observe that the set of hypersurfaces on which a birational transformation is not generically injective is finite. According to Corollary 2.1 and Lemma 2.2, it suffices to construct an uncountable family of hypersurfaces of $\mathbb{P}^n_{\Bbbk}$ of some degree $l \geq 1$, in pairs non birationally equivalent, that contain $O := [1 : 0 : \ldots : 0]$ as point of multiplicity exactly l.

Consider the family of hypersurfaces of equation $q(x_1, x_2, x_3) = 0$ where $q = 0$ defines a smooth curve C_q of degree l on the plane of equations $x_0 = x_4 = \cdots = x_n = 0$; the surface $q = 0$ is birationally equivalent to $\mathbb{P}^{n-2}_{\Bbbk} \times C_q$, and then two such surfaces are birationally equivalent if and only if C_q and $C_{q'}$ are isomorphic. The proof follows from the fact that for $l = 3$, the set of all the classes of isomorphisms of smooth plane cubics is a family with a parameter (see Chapter IV, Theorem 4.1 and Proposition 4.6. in [4]). ☐

Remark 2.2. *The argument above shows that Lemma 5 can be a useful instrument in order to decide if a rational map belongs to or not a subgroup of Cr$_{\Bbbk}(n)$ whose a subset of generators is known; as a particular case, by the theorem of Noether, a rational map of the plane that constracts a non-rational curve is not birational; this fact is well-known.*

3. Conclusion

In this paper, the author has acquired the main following result: if n is a positive integer, $n > 2$ then every set of generators of the Cremona group $\mathrm{Cr}_{\Bbbk}(n)$ of the projective space $\mathbb{P}^n_{\Bbbk}$ must contain an infinite and uncountable number of birational transformations of degree > 1.

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