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EXPONENTIAL STABILITY OF A CLASS OF POSITIVE NONLINEAR SYSTEMS WITH MULTIPLE TIME-VARYING DELAYS

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Abstract. This paper is concerned with the problem of exponential stability of a class of positive nonlinear systems with heterogeneous time-varying delays which describe a model of Hopfield neural networks with nonlinear self-inhibition rates. Based on a novel comparison technique via a differential and integral inequalities, testable conditions are derived to ensure system state trajectories converge exponentially to a unique positive equilibrium. The effectiveness of the obtained results is illustrated by a numerical example.
Keywords: neural networks, positive equilibrium, exponential stability, time-varying delay, M-matrix.

1. Introduction

In modeling of many applied models in economics, ecology and biology or communication systems, the relevant state variables are subject to positivity constraints according to the nature of the phenomenon itself [1]. These models are typically described by positive systems. Roughly speaking, positive systems are dynamical systems whose states are always nonnegative whenever the inputs and initial conditions are nonnegative [2]. As an essential issue in applications of positive systems, the problem of stability analysis and control of positive systems and, in particular, positive systems with delays, has received considerable attention from researchers in the past few decades [3-7].

During the past two decades, the problem of stability analysis of neural networks including artificial neural networks and biological neural networks has received considerable attention due to its widespread applications in signal processing, pattern recognition, ecosystem evaluation and parallel computation [8-10]. When a neural network model is designed for practical positive systems, for example, in identification [11], control [12] or competitive-cooperation dynamical systems for decision rules, pattern formation, and parallel memory storage, it is inherent that the

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states of the designed networks are nonnegative. In addition, the nonlinearity of activation functions and the negativeness of self-feedback terms make the study of positive neural networks more complicated. Thus, it is of interest to study the problem of stability analysis of positive nonlinear systems involving neural networks models. However, this problem has just received growing research attention in recent years and only a few results have been reported in the literature. For example, Hien (2017) [13] studied the exponential stability of a unique positive equilibrium of positive Hopfield neural networks with linear self-inhibition rates and a bounded time-varying delays based on the theory of M-matrix and linear programming (LP) approach. The results of [13] were later extended to inertial neural networks with multiple delays [14].

In this paper, we further investigate the problem of exponential stability of a unique positive equilibrium point of positive nonlinear systems which describe Hopfield neural networks with heterogeneous time-varying delays. Based on novel comparison techniques, we derive unified conditions in terms of linear programming to ensure simultaneously that the system is positive and, for each nonnegative input vector, there exists a unique positive equilibrium point which is globally exponentially stable.

2. Preliminaries

Notation: We denote $\mathbb{R}^n$ the n-dimensional space with the vector norm $\|x\|_\infty = \max_{1\le i\le n}|x_i|$ and $\mathbb{R}^{m\times n}$ the set of $m\times n$-matrices. For any two vectors $x=(x_i)\in\mathbb{R}^n$ and $y=(y_i)\in\mathbb{R}^n$, $x\preceq y$ if $x_i\le y_i$ for all $i\in[n]\triangleq\{1,2,\dots,n\}$ and $x\prec y$ if $x_i<y_i$ for all $i\in[n]$. $\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x\succeq 0\}$ and $|x|=(|x_i|)\in\mathbb{R}^n_+$ for any $x\in\mathbb{R}^n$. A matrix $A=(a_{ij})\in\mathbb{R}^{m\times n}$ is nonnegative, $A\succeq 0$, if $a_{ij}\ge 0$ for all i,j and A is a Metzler matrix if its off-diagonal entries are nonnegative.

Consider the following nonlinear system with heterogeneous delays

$$\begin{aligned} x_i'(t) = &-d_i\varphi_i(x_i(t)) + \sum_{j=1}^n a_{ij}f_j(x_j(t)) \\ &+\sum_{j=1}^n b_{ij}g_j(x_j(t-\tau_{ij}(t))) + I_i,\ i\in[n],\ t\ge 0. \end{aligned} \tag{2.1}$$

System (2.1) describes a model of Hopfield neural networks, where n is the number of neurons in the network, $x(t)=(x_i(t))\in\mathbb{R}^n$ and $I=(I_i)\in\mathbb{R}^n$ are the state vector and the external input vector, respectively; $f_j(x_j(t))$ and $g_j(x_j(t))$ are neuron activation functions; $\varphi_i(x_i(t))$, $i\in[n]$, are nonlinear self-excitation rates and $d_i>0$, $i\in[n]$, are self-inhibition coefficients; $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and $B=(b_{ij})\in\mathbb{R}^{n\times n}$ are neuron connection weight matrices and $\tau_{ij}(t)$, $i,j\in[n]$, represent heterogeneous time-varying delays satisfying $0\le\tau_{ij}(t)\le\tau_{ij}^+$ for all $t\ge 0$, where τ_{ij}^+ is a known scalar. The initial condition of (2.1) is specified as

$$x(\theta)=\phi(\theta),\ \theta\in[-\tau^+,0]$$

where $\tau^+ = \max_{i,j} \tau_{ij}^+$ and $\phi \in C([-\tau^+, 0], \mathbb{R}^n)$ is a given function.

Let $\mathcal{F}$ be the set of continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi(0) = 0$ and there exist positive scalars c_φ^-, c_φ^+ such that

$$c_\varphi^- \leq \frac{\varphi(u) - \varphi(v)}{u - v} \leq c_\varphi^+ \tag{2.2}$$

for all $u, v \in \mathbb{R}$, $u \neq v$. It is clear that the function class $\mathcal{F}$ includes all linear functions $\varphi(u) = \gamma_\varphi u$ where γ_φ is some positive scalar.

Assumptions

(A1) The decay rate functions φ_i, $i \in [n]$, are assumed to belong the function class $\mathcal{F}$.

(A2) The activation functions $f_j(.)$ and $g_j(.)$ are continuous and satisfy the following conditions

$$0 \leq \frac{f_j(u) - f_j(v)}{u - v} \leq l_j^f, \ 0 \leq \frac{g_j(u) - g_j(v)}{u - v} \leq l_j^g, \ \forall u \neq v, \tag{2.3}$$

where l_j^f and l_j^g, $j \in [n]$, are positive constants.

Remark 2.1. *It follows from Assumption* (A2) *that the functions* $f(x) = (f_i(x_i))$ *and* $g(x) = (g_i(x_i))$, $x = (x_i) \in \mathbb{R}^n$, *are globally Lipschitz continuous on* $\mathbb{R}^n$. *Thus, by utilizing fundamental results in the theory of functional differential equations [15], it can be verified that for any initial function* $\phi \in C([-\tau^+, 0], \mathbb{R}^n)$, *there exists a unique solution* $x(t) = x(t, \phi)$ *of* (2.1) *on the interval* $[0, \infty)$, *which is absolutely continuous in* t. *In the sequel, each solution of* (2.1) *will be denoted simply as* $x(t)$ *if it does not make any confusion.*

Definition 2.1. *System (2.1) is said to be positive if for any nonnegative initial function* $\phi \in C([-\tau^+, 0], \mathbb{R}_+^n)$ *and nonnegative input vector* $I \in \mathbb{R}_+^n$, *the corresponding state trajectory is nonnegative, that is* $x(t) \in \mathbb{R}_+^n$ *for all* $t \geq 0$.

Definition 2.2. *Given an input vector* $I \in \mathbb{R}_+^n$. *A vector* $x_* \in \mathbb{R}_+^n$ *is said to be a positive equilibrium of system (2.1) if it satisfies the following algebraic system*

$$-D\Phi(x_*) + Af(x_*) + Bg(x_*) + I = 0, \tag{2.4}$$

where the function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *is defined as* $\Phi(x) = (\varphi_i(x_i))$

Definition 2.3. *A positive equilibrium* x_* *of (2.1) is said to be globally exponentially stable if there exist positive scalars* β, η *such that any solution* $x(t)$ *of (2.1) satisfies the following inequality*

$$\|x(t) - x_*\|_\infty \leq \beta \|\phi - x_*\|_C e^{-\eta t}, \ t \geq 0. \tag{2.5}$$

We recall here some concepts in nonlinear analysis and the theory of monotone dynamical systems which will be used in the derivation of our results. A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *order-preserving* on $\mathbb{R}^n_+$ if $F(x) \preceq F(y)$ for any $x,\ y \in \mathbb{R}^n_+$ satisfying $x \preceq y$ [1]. Let $A \in \mathbb{R}^{n\times n}_+$, then by Assumption (A2), the vector field $F(x) = Af(x)$ is an order-preserving. A mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is *proper* if $\Psi^{-1}(K)$ is compact for any compact subset $K \subset \mathbb{R}^n$. It is well-known that a continuous mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is proper if and only if Ψ has the property that for any sequence $\{p_k\} \subset \mathbb{R}^n$, $\|p_k\| \to \infty$ then $\|\Psi(p_k)\| \to \infty$ as $k \to \infty$.

Lemma 2.1 (see [16]). *A locally invertible continuous mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism of $\mathbb{R}^n$ onto itself if and only if it is proper.*

3. Main results

In this section, we will derive conditions to ensure that the nonlinear system (2.1) is positive and has a unique positive equilibrium which is globally exponentially stable. First, the positivity of the system (2.1) is presented in the following proposition.

Proposition 3.1. *Let Assumptions* (A1)-(A2) *hold and assume that the neuron connection weight matrices A, B are nonnegative. Then, system* (2.1) *is positive for all bounded delays.*

Proof. Let $x(t)$ be a solution of system (2.1) with initial function $\phi \in C([-\tau^+, 0], \mathbb{R}^n_+)$ and input vector $I \in \mathbb{R}^n_+$. For a given $\epsilon > 0$, let $x_\epsilon(t)$ denote the solution (2.1) with initial condition $\phi_\epsilon(.) = \phi(.) + \epsilon \mathbf{1}_n$, where $\mathbf{1}_n$ denotes the vector in $\mathbb{R}^n$ with all entries equal one. Note that $x_\epsilon(t) \to x(t)$ as $\epsilon \to 0$. Thus, it suffices to show that $x_\epsilon(t) > 0$ for all $t \geq 0$. Suppose in contrary that there exists an index $i \in [n]$ and a $t_* > 0$ such that

$$x_{i\epsilon}(t_*) = 0,\ x_{i\epsilon}(t) > 0 \text{ for all } t \in [0, t_*)$$

and $x_{j\epsilon}(t) \geq 0$ for all $j \in [n]$. Then,

$$q_i(t) = \sum_{j=1}^{n} a_{ij} f_j(x_{j\epsilon}(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_{j\epsilon} j(t - \tau_{ij}(t))) + I_i \geq 0 \tag{3.1}$$

for all $t \in [0, t_*]$.

On the other hand, by condition (2.2), we have

$$c^-_{\varphi_i} \leq \frac{\varphi_i(x_{i\epsilon}(t))}{x_{i\epsilon}(t)} \leq c^+_{\varphi_i}, \quad t \in [0, t_*).$$

Thus, from (2.1), we have

$$x'_{i\epsilon}(t) \geq -c^+_{\varphi_i} x_{i\epsilon}(t) + q_i(t),\ t \in [0, t_*). \tag{3.2}$$

By integrating both sides of inequality (3.2) we then obtain

$$x_{i\epsilon}(t) \geq e^{-c^+_{\varphi_i}t}\left(x_0 + \epsilon + \int_0^t e^{c^+_{\varphi_i}s}q_i(s)ds\right)$$
$$\geq e^{-c^+_{\varphi_i}t}(x_0 + \epsilon), \quad t \in [0, t_*). \tag{3.3}$$

Let $t \uparrow t_*$, inequality (3.3) gives

$$0 < (x_0 + \epsilon)e^{-c^+_{\varphi_i}t_*} \leq x_{i\epsilon}(t_*) = 0$$

which clearly raises a contradiction. This shows that $x_\epsilon(t) \succ 0$ for $t \in [0, \infty)$. The proof is completed. $\square$

Revealed by (2.4), for a given input vector $I \in \mathbb{R}^n$, an equilibrium of system (2.1) exists if and only if the equation $\Psi(x) = 0$ has a solution $x_* \in \mathbb{R}^n$, where the mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\Psi(x) = -D\Phi(x) + Af(x) + Bg(x) + I$. Clearly, Ψ is continuous on $\mathbb{R}^n$. Based on Lemma 2.1, we have the following result.

Proposition 3.2. *Let Assumptions* (A1)-(A2) *hold and A, B are nonnegative matrices. Assume that there exists a vector $\nu \in \mathbb{R}^n$, $\nu \succ 0$, such that*

$$\sum_{i=1}^{n}(a_{ij}l^f_j + b_{ij}l^g_j)\nu_i < d_j c^-_{\varphi_j}\nu_j,\ j \in [n]. \tag{3.4}$$

Then, for a given input vector $I \in \mathbb{R}^n$, system (2.1) *has a unique equilibrium $x_* \in \mathbb{R}^n$.*

Proof. Let $\Psi(x) = -D\Phi(x) + Af(x) + Bg(x) + I$. Then, for any two vectors $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}\Psi(x) - \Psi(y) = &-D(\Phi(x) - \Phi(y)) + A[f(x) - f(y)]\\ &+ B[g(x) - g(y)].\end{aligned} \tag{3.5}$$

We denote a sign matrix $\mathcal{S}(x - y) = \mathrm{diag}\{\mathrm{sgn}(x_i - y_i)\}$. It follows from (A2) that

$$\mathrm{sgn}(x_j - y_j)(f_j(x_j) - f_j(y_j)) \leq l^f_j|x_j - y_j|.$$

By multiplying both sides of (3.5) with $\mathcal{S}(x - y)$, we obtain

$$\mathcal{S}(x - y)\left(\Psi(x) - \Psi(y)\right) \preceq \left(-DC^-_\varphi + AL_f + BL_g\right)|x - y|, \tag{3.6}$$

where $L_f = \mathrm{diag}\{l^f_1, l^f_2, \ldots, l^f_n\}$, $L_g\mathrm{diag}\{l^g_1, l^g_2, \ldots, l^g_n\}$ and $C^-_\varphi = \mathrm{diag}\{c^-_{\varphi_1}, c^-_{\varphi_2}, \ldots, c^-_{\varphi_n}\}$. Due to (3.6), we have

$$|\Psi(x) - \Psi(y)| \succeq \left(DC^-_\varphi - AL_f - BL_g\right)|x - y|$$

and therefore,

$$\nu^\top|\Psi(x) - \Psi(y)| \succeq \nu^\top\left(DC^-_\varphi - AL_f - BL_g\right)|x - y| \tag{3.7}$$

for any $\nu \in \mathbb{R}^n$, $\nu \succ 0$. If $\Psi(x) = \Psi(y)$ then, by condition (3.4),

$$\nu^\top \left(DC_\varphi^- - AL_f - BL_g \right) |x - y| = 0$$

which clearly gives $x = y$. This shows that Ψ is an injective mapping in $\mathbb{R}^n$. On the other hand, inequality (3.7) also gives

$$\|\Psi(x)\|_\infty \geq \frac{1}{\|\nu\|_\infty} \nu^\top \left(DC_\varphi^- - AL_f - BL_g \right) |x| - \|\Psi(0)\|_\infty.$$

The above estimate implies that $\|\Psi(x_k)\|_\infty \to \infty$ for any sequence $\{x_k\} \subset \mathbb{R}^n$ satisfying $\|x_k\|_\infty \to \infty$. By Lemma 2.1, $\Psi(.)$ is a homeomorphism onto $\mathbb{R}^n$, and thus, the equation $\Psi(x) = 0$ has a unique solution $x_* \in \mathbb{R}^n$ which is an equilibrium of system (2.1). The proof is completed. □

Remark 3.1. *Clearly,* $\mathcal{M} = -DC_\varphi^- + AL_f + BL_g$ *is a Metzler matrix and so is* $\mathcal{M}^\top$*. In addition, condition* (3.4) *holds if and only if* $\mathcal{M}^\top \nu \prec 0$*. This condition is feasible if and only if* $\mathcal{M}^\top$*, and thus* $\mathcal{M}$*, is a Metzler-Hurwitz matrix [17]. In the following, we will show that the derived conditions in Propositions 3.1 and 3.2 ensure that system* (2.1) *is positive and the unique equilibrium point* x_* *is positive for each positive input vector* $I \in \mathbb{R}_+^n$ *which is globally exponentially stable.*

Theorem 3.1. *Let Assumptions* (A1)-(A2) *hold and* $A \succeq 0$*,* $B \succeq 0$*. Assume that there exists a vector* $\chi \in \mathbb{R}^n$*,* $\chi \succ 0$*, such that*

$$\mathcal{M}\chi = \left(-DC_\varphi^- + AL_f + BL_g \right) \chi \prec 0. \tag{3.8}$$

Then, for any positive input vector $I \in \mathbb{R}_+^n$*, system* (2.1) *has a unique positive equilibrium* $x_* \in \mathbb{R}_+^n$ *which is globally exponentially stable for any delays* $\tau_{ij}(t) \in [0, \tau_{ij}^+]$*.*

Proof. By Proposition 3.2, there exists a unique equilibrium $x_* \in \mathbb{R}^n$ of system (2.1). We first prove that x_* is globally exponentially stable. Indeed, let $x(t)$ be a solution of (2.1). It follows from systems (2.1) and (2.4) that

$$\begin{aligned} (x_i(t) - x_{*i})' = & - d_i \left(\varphi_i(x_i(t)) - \varphi_i(x_{*i}) \right) + \sum_{j=1}^n a_{ij} [f_j(x_j(t)) - f_j(x_{*j})] \\ & + \sum_{j=1}^n b_{ij} [g_j(x_i(t - \tau_{ij}(t))) - g_j(x_{*j})]. \end{aligned} \tag{3.9}$$

We define $z(t) = |x(t) - x_*|$ then, from (3.9), we have

$$\begin{aligned} D^- z_i(t) &= \mathrm{sign}(x_i(t) - x_{*i})(x_i(t) - x_{*i})' \\ &\leq -d_i c^-_{\varphi_i}|x_i(t) - x_{*i}| + \sum_{j=1}^{n} a_{ij} l^f_j |x_j(t) - x_{*j}| \\ &\quad + \sum_{j=1}^{n} b_{ij} l^g_j |x_j(t - \tau_{ij}(t)) - x_{*j}| \\ &\leq -d_i c^-_{\varphi_i} z_i(t) + \sum_{j=1}^{n} a_{ij} l^f_j z_j(t) + \sum_{j=1}^{n} b_{ij} l^g_j z_j(t - \tau_{ij}(t)). \end{aligned} \tag{3.10}$$

where $D^- z_i(t)$ denotes the upper left Dini derivative of $z_i(t)$.

Now, we utilize the derived condition (3.8) to establish an exponential estimate for $z(t)$. From (3.8), we have

$$-d_i c^-_{\varphi_i} \chi_i + \sum_{j=1}^{n} (a_{ij} l^f_j + b_{ij} l^g_j)\chi_j < 0, \forall i \in [n]. \tag{3.11}$$

Consider the following function

$$H_i(\eta) = (\eta - d_i c^-_{\varphi_i})\chi_i + \sum_{j=1}^{n} a_{ij} l^f_j \chi_j + (\sum_{j=1}^{n} b_{ij} l^g_j \chi_j) e^{\eta\tau^+}, \eta \geq 0.$$

Clearly, $H_i(\eta)$ is continuous on $[0, \infty)$, $H_i(0) < 0$ and $H_i(\eta) \to \infty$ as $\eta \to \infty$. Thus, there exists a unique positive scalar η_i such that $H_i(\eta_i) = 0$. Let $\eta_0 = \min_{1\leq i\leq n} \eta_i$ and define the following functions

$$\rho_i(t) = \frac{\chi_i}{\chi_+} \|\phi - x_*\|_C e^{-\eta_0 t}, \; t \geq 0$$

and $\rho_i(t) = \rho_i(0)$, $t \in [-\tau^+, 0]$, where $\chi_+ = \min_{1\leq i\leq n} \chi_i$. Note that, for any $t \geq 0$, we have

$$\rho_i(t - \tau_{ij}(t)) = e^{\eta_0 \tau_{ij}(t)} \rho_i(t) \leq e^{\eta\tau^+} \rho_i(t).$$

Therefore,

$$\begin{aligned} &-d_i c^-_{\varphi_i} \rho_i(t) + \sum_{j=1}^{n} a_{ij} l^f_j \rho_j(t) + \sum_{j=1}^{n} b_{ij} l^g_j \rho_j(t - \tau_{ij}(t)) \\ &\quad \leq \Big[- d_i c^- \varphi_i \chi_i + \sum_{j=1}^{n} a_{ij} l^f_j \chi_j + (\sum_{j=1}^{n} b_{ij} l^g_j \chi_j) e^{\eta_0 \tau^+} \Big] \frac{1}{\chi_+} \|\phi - x_*\|_C e^{-\eta_0 t} \\ &\quad \leq \frac{H_i(\eta_0) - \eta_0 \chi_i}{\chi_+} \|\phi - x_*\|_C e^{-\eta_0 t}. \end{aligned} \tag{3.12}$$

Since $H_i(\eta)$ is increasing in η, $H_i(\eta_0) \leq 0$ for all $i \in [n]$. Thus, (3.12) gives

$$\rho_i'(t) \geq -d_i c_{\varphi_i}^- \rho_i(t) + \sum_{j=1}^n a_{ij} l_j^f \rho_j(t) + \sum_{j=1}^n b_{ij} l_j^g \rho_j(t - \tau_{ij}(t)) \tag{3.13}$$

for all $t \geq 0$ and $i \in [n]$. Combining (3.10) and (3.13) we obtain

$$D^- \zeta_i(t) \leq -d_i c_{\varphi_i}^- \zeta_i(t) + \sum_{j=1}^n a_{ij} l_j^f \zeta_j(t) + \sum_{j=1}^n b_{ij} l_j^g \zeta_j(t - \tau_{ij}(t)) \tag{3.14}$$

where $\zeta_i(t) = z_i(t) - \rho_i(t)$. It follows from (3.14) that

$$\begin{aligned} \zeta_i(t) \leq e^{-d_i c_{\varphi_i}^- t} \zeta_i(0) &+ \sum_{j=1}^n a_{ij} l_j^f \int_0^t e^{d_i c_{\varphi_i}^- (s-t)} \zeta_j(s) ds \\ &+ \sum_{j=1}^n b_{ij} l_j^g \int_0^t e^{d_i c_{\varphi_i}^- (s-t)} \zeta_j(s - \tau_{ij}(s)) ds, \ t \geq 0. \end{aligned} \tag{3.15}$$

It is obvious that $\zeta(0) \preceq 0$. For any $t_f > 0$, if $\zeta(t) \preceq 0$ for all $t \in [0, t_f)$ then from (3.15), $\zeta(t_f) \preceq 0$. This shows that $\zeta(t) \preceq 0$ for all $t \geq 0$. Consequently,

$$\|x(t) - x_*\|_\infty \leq (\max_{1 \leq i \leq n} \chi_i / \chi_+) \|\phi - x_*\|_C e^{-\eta_0 t}$$

by which we can conclude the exponential stability of the equilibrium x_*.

Finally, for a nonnegative initial function ϕ, by Proposition 3.1, the corresponding trajectory $x(t) \succeq 0$ for all $t \geq 0$. Thus, $x_* = \lim_{t \to \infty} x(t) \succeq 0$. This shows that x_* is a unique positive equilibrium of system (2.1). The proof is completed. □

4. An illustrative example

Consider a class of cooperative neural networks in the form (2.1) with Bolzmann sigmoid activation functions

$$f_j(x_j) = g_j(x_j) = \frac{1 - e^{-\frac{x_j}{\theta_j}}}{1 + e^{-\frac{x_j}{\theta_j}}}, \quad \theta_j > 0 \ (j = 1, 2, 3) \tag{4.1}$$

and a common nonlinear decay rate

$$\varphi(x_i) = 2x_i + \sin^2(0.25 x_i).$$

It is easy to verify that Assumptions (A1) and (A2) are satisfied, where $c_\varphi^- = 1.75$, $c_\varphi^+ = 2.25$ and $l_j^f = l_j^g = \frac{1}{2\theta_j}$. Let

$$A = \begin{bmatrix} 0.35 & 0.64 & 0.25 \\ 0.81 & 0.15 & 0.25 \\ 0.42 & 0.46 & 0.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0.12 & 0.53 & 0.29 \\ 0.23 & 0.18 & 0.36 \\ 0.56 & 0.27 & 0.39 \end{bmatrix},$$
$$D = \mathrm{diag}\{0.8, 0.75, 1.1\}$$

and $\mathrm{diag}\{\theta_j\} = \{2.0, 1.8, 2.5\}$ then

$$\mathcal{M} \triangleq -c_\varphi^- D + AL_f + BL_g = \begin{bmatrix} -1.2825 & 0.325 & 0.108 \\ 0.26 & -1.2208 & 0.122 \\ 0.245 & 0.2028 & -1.737 \end{bmatrix}.$$

Therefore, $\mathcal{M}\mathbf{1}_3 \prec 0$. By Theorem 3.1, for any input vector $I \in \mathbb{R}^3_+$, system (2.1) has a unique positive equilibrium $x_* \in \mathbb{R}^3_+$ which is globally exponentially stable. A simulation result of 20 sample state trajectories with random initial states, input $I = (1.5, 1.8, 2.0)^\top$ and a common delay $\tau(t) = 5|\sin(0.1t)|$ is presented in Figure 1. It can be seen that all the conducted state trajectories converge to the positive equilibrium x_*. This validates the obtained theoretical results.

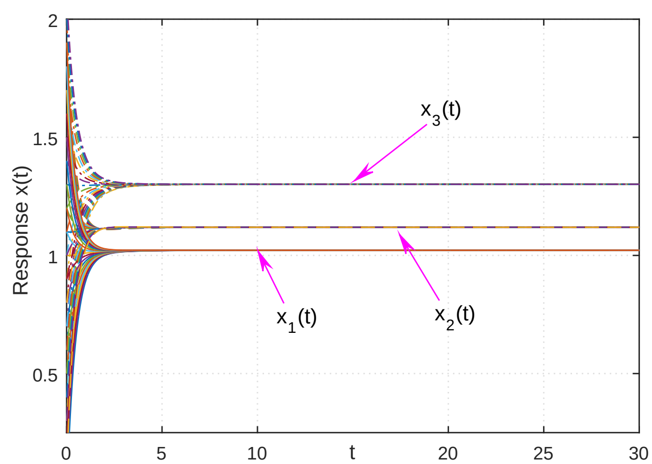

Figure 1. Convergence of state trajectories to positive equilibrium x_*

5. Conclusions

The problem of existence, uniqueness and global exponential stability of a positive equilibrium has been investigated for a class of positive nonlinear systems which describe Hopfield neural networks with heterogeneous time-varying delays. Testable stability conditions in terms of linear programming have been derived using novel comparison techniques via differential and integral inequalities.

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