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ON L_∞-INDUCED STABILIZATION OF POSITIVE LINEAR SYSTEMS WITH DISTRIBUTED DELAYS

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Abstract. The problem of L_∞-gain control is studied for positive linear systems with distributed time delays. By a novel comparison technique involving the monotonicity of the so-called upper scaled systems with peak values of exogenous disturbances, a characterization of L_∞-induced norm is first reformulated. Then, necessary and sufficient LP-based conditions subject to L_∞-induced performance with prescribed level are derived and utilized to address the design problem of state-feedback controllers that make the closed-loop systems positive, stable and have prescribed L_∞-gain performance level. A numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: Positive systems; L_∞-gain control; distributed delays; feasibility.

1. Introduction

Positive systems are widely used to model various applied phenomena whose relevant states are always nonnegative [1, 2]. Applications of positive systems can be found in a variety of disciplines from biology, ecology, and epidemiology, chemistry, pharmacokinetics to communication systems, and many other models that are subject to conservation laws. In addition, positive systems possess many elegant properties that have yet no counterpart in general dynamical systems [3]. Due to practical and theoretical applications, the systems and control theory of linear positive systems has been one of the most active research topics in the past decade (see, e.g., [4-7]).

On the other hand, exogenous disturbances are often encountered in modeling of practical systems due to the inaccuracy of data processing, linear approximations or measurement errors. In practice, external disturbances such as wind shear on

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aircraft wings or continuous road excitation on vehicle suspension systems are generally persistent and amplitude-bounded rather than specifications on the total energy of disturbances are required [8]. In such models, the worst-case amplification from input disturbance to the regulated output represents a more reasonable performance index, which gives rise to the so-called L_∞-induced control problem. Roughly speaking, L_∞-induced optimal design is to minimize the maximum peak-to-peak gain of a closed-loop system that is driven by bounded amplitude disturbances. Thus, the L_∞-gain minimization is a useful and effective approach to the problem of examining the responses of dynamic systems corrupted by persistently bounded disturbances [9]. In some existing works, L_∞-gain analysis results have been established by using certain types of co-positive Lyapunov functions [10], fundamental solution representation [11] or by utilizing the positivity characteristic [9, 12], which give a characterization of the exact value of L_∞-gain of the systems. However, the obtained characterization results are typically not tractable for the design problem of desired controllers that make the closed-loop systems positive, stable and have prescribed L_∞-gain performance.

In this paper, we consider the stabilization problem under L_∞-gain scheme for positive linear systems with distributed time delays. Novelty and main contribution of this paper are two points. First, characterization of L_∞-gain is reformulated using novel comparison techniques involving steady states of upper scaled systems with peak values of exogenous disturbances. Second, tractable LP-based conditions to the design problem of a state-feedback controller that minimizes the worst-case amplification from disturbances to regulated outputs subject to L_∞-gain is presented.

2. Content

2.1. Preliminaries

Notation. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional vector space and the set of $m \times n$-matrices, respectively. $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector with all entries equal one. $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ and $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$ denote the max-norm of a vector $x = (x_i) \in \mathbb{R}^n$ and a matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, respectively. L_∞-norm of a function $w : \mathbb{R}_+ \to \mathbb{R}^n$ is defined as $\|w\|_{L_\infty} = \mathrm{esssup}_{t\ge 0}\|w(t)\|_\infty$ and $L_\infty(\mathbb{R}^n) = \{w : \mathbb{R}_+ \to \mathbb{R}^n : \|w\|_{L_\infty} < \infty\}$. For two vectors $x = (x_i) \in \mathbb{R}^n$ and $y = (y_i) \in \mathbb{R}^n$, we write $x \preceq y$ if $x_i \le y_i$ and $x \prec y$ if $x_i < y_i$ for $i = 1, 2, \dots, n$; $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$ and $|x| = (|x_i|) \in \mathbb{R}^n_+$. A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is nonnegative, $A \succeq 0$, if $a_{ij} \ge 0$ for all i, j.

Consider the following continuous-time system with distributed delays.

$$\begin{cases} \dot{x}(t) = Ax(t) + \displaystyle\int_{-\tau}^{0} A_d(s)x(t+s)ds + Bu(t) + B_w w(t), \\ z(t) = Cx(t) + \displaystyle\int_{-\kappa}^{0} C_d(s)x(t+s)ds + Du(t) + D_w w(t), \\ x(s) = \phi(s), \quad s \in [-d_*, 0], \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^p$ and $w(t) \in \mathbb{R}^q$ are the state, regulated output and exogenous disturbance input vectors, respectively. τ, κ are positive scalars representing distributed time-delays in the state and output. A, B, B_w, C, D, D_w are known real matrices, $A_d(s)$, $C_d(s)$ are continuous matrix-valued functions defined on $[-\tau, 0]$ and $[-\kappa, 0]$, respectively. $d_* = \max\{\tau, \kappa\}$ and $\phi \in \mathcal{C} \triangleq C([-d_*, 0], \mathbb{R}^n)$ is the initial condition. The ∞-norm of $\phi \in \mathcal{C}$ is defined as $\|\phi\|_{\mathcal{C}} = \sup_{-d_* \le s \le 0} \|\phi(s)\|_\infty$. To explicitly mention the initial condition, we will denote as $x(t, \phi)$ the corresponding solution of (1) with initial function ϕ.

Definition 2.1 (see [1]). *System* (2.1) *is said to be (internally) positive if for any initial state $\phi(s) \succeq 0$ ($s \in [-d_*, 0]$) and inputs $u(t) \succeq 0$, $w(t) \succeq 0$, $t \geq 0$, the state trajectory $x(t) \succeq 0$ and output $z(t) \succeq 0$ for all $t \geq 0$.*

Similar to [9], we have the following positivity characterization.

Proposition 2.1. *System* (2.1) *is positive if and only if the matrix A is Metzler, B, B_w, C, D, D_w are nonnegative, and $A_d(s)$, $C_d(s)$ are nonnegative for $s \in [-\tau, 0]$ and $s \in [-\kappa, 0]$, respectively.*

The following result is similar to that of [13, Theorem III.1]. First, we recall here that, for a matrix $A \in \mathbb{R}^{n\times n}$, let $\sigma(A)$ be the spectrum (the set of eigenvalues) of A, we denote by $\mu(A) = \max\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ the spectral abscissa or the growth constant of A. It is well-known that the inequality

$$\left\|e^{At}\right\| \leq e^{\mu(A)t}$$

for all $t \geq 0$. Thus, for any $x \in \mathbb{R}^n$, $e^{At}x \to 0$ as $t \to \infty$ if and only if $\mu(A) < 0$. Equivalently, $\mathrm{Re}\lambda_j < 0$ for any $\lambda_j \in \sigma(A)$ and the spectrum $\sigma(A)$ of A lies within the left-hand side of complex plane.

Theorem 2.1. *Suppose that system* (2.1) *is positive. The following statements are equivalent.*

(i) *The unforced system of* (2.1) with $w = 0$ is globally exponentially stable (GES).

(ii) The Metzler matrix $\mathcal{A} = A + \int_{-\tau}^{0} A_d(s)ds$ is Hurwitz, that is, the spectral abscissa $\mu(\mathcal{A}) < 0$.

(iii) There exists a vector $\eta \in \mathbb{R}^n$, $\eta \succ 0$, such that

$$\Big(A + \int_{-\tau}^{0} A_d(s)ds\Big)\eta \prec 0. \tag{2.2}$$

(iv) The matrix $A + \int_{-\tau}^{0} A_d(s)ds$ is invertible and

$$\Big(A + \int_{-\tau}^{0} A_d(s)ds\Big)^{-1} \preceq 0.$$

Proof. The proof is similar to that of Theorem III.1 in [13] and thus it is omitted here. □

Since $\mu(\mathcal{A}) = \mu(\mathcal{A}^\top)$, an equivalent condition of (2.2) is that there exists a positive vector ν such that $\nu^\top \mathcal{A} \prec 0$. Thus, it can be shown under equivalent conditions of Theorem 2.1 that for any $w \in L_\infty(\mathbb{R}^q)$, we have $x \in L_\infty(\mathbb{R}^n)$ and hence $z \in L_\infty(\mathbb{R}^p)$. It is natural to assume that system (2.1) is stable (GES) to ensure the L_∞-gain exists. More specifically, we define the input-output operator

$$\Sigma : L_\infty(\mathbb{R}^q) \longrightarrow L_\infty(\mathbb{R}^p), \quad w \mapsto z$$

and L_∞-gain of system (2.1) under zero initial condition is defined as

$$\|\Sigma\|_{(L_\infty, L_\infty)} = \sup_{\|w\|_{L_\infty} \neq 0} \frac{\|z\|_{L_\infty}}{\|w\|_{L_\infty}}. \tag{2.3}$$

Definition 2.2. *For a given $\gamma > 0$, system*(2.1) *is said to have L_∞-induced performance of level γ if $\|\Sigma\|_{(L_\infty, L_\infty)} < \gamma$.*

The main objective here is to address the stabilization problem under L_∞-induced performance index via state-feedback scheme for positive control systems described by (2.1).

2.2. L_∞-gain analysis

In this section, we consider positive system (2.1) with zero initial condition. Let $x(t,w)$, $z(t,w)$ denote the state and output trajectories of system (2.1) with respect to input w. Similar to [9, Lemma 5], it can be verified that for any $w_1, w_2 \in L_\infty(\mathbb{R}^q)$, if $w_1(t) \preceq w_2(t)$, $t \geq 0$, then $x(t,w_1) \preceq x(t,w_2)$ and $z(t,w_1) \preceq z(t,w_2)$ for all $t \geq 0$. Therefore,

$$-x(t,|w|) \preceq x(t,w) \preceq x(t,|w|),\ t \geq 0,$$

which yields $|x(t,w)| \preceq x(t,|w|)$ for all $t \geq 0$. Moreover, for a $w \in L(\mathbb{R}^q)$, we have

$$|w(t)| \preceq \overline{w} \triangleq \|w\|_{L_\infty} \mathbf{1}_q.$$

To further facilitate the analysis of system (2.1), we consider the following auxiliary system

$$\begin{cases} \dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + B_w\overline{w}, \\ \bar{z}(t) = \mathcal{C}\bar{x}(t) + D_w\overline{w}, \\ \bar{x}(0) = 0. \end{cases} \tag{2.4}$$

where $\mathcal{A} = A + \int_{-\tau}^{0} A_d(s)ds$, $\mathcal{C} = C + \int_{-\kappa}^{0} C_d(s)ds$ and $\overline{w} = \|w\|_{L_\infty}\mathbf{1}_q$.

Lemma 2.1 (see, [9, Lemma 7])**.** *The state trajectory $\bar{x}(t)$ of* (2.4) *is monotonically nondecreasing, that is, $\bar{x}(t_1) \preceq \bar{x}(t_2)$ for any $0 \le t_1 < t_2$.*

Based on Lemma 2.1, the following lemma is obtained.

Lemma 2.2. *For any state and output trajectories $x(t)$, $\bar{x}(t)$, $z(t)$, $\bar{z}(t)$ of systems* (2.1) *and* (2.4)*, it holds that $x(t) \preceq \bar{x}(t)$ and $z(t) \preceq \bar{z}(t)$ for all $t \ge 0$.*

Proof. For state trajectories $x(t)$, $\bar{x}(t)$ of (2.1) and (2.4), we define $e(t) = \bar{x}(t) - x(t)$ as the error vector of $x(t)$ and $\bar{x}(t)$. It follows from (2.1) and (2.4) that

$$\begin{aligned} \dot{e}(t) = Ae(t) &+ \int_{-\tau}^{0} A_d(s)e(t+s)ds + B_w(\overline{w} - w(t)) \\ &+ \int_{-\tau}^{0} A_d(s)\left[\bar{x}(t) - \bar{x}(t+s)\right]ds. \end{aligned} \tag{2.5}$$

By Lemma 2.1, $\bar{x}(t) - \bar{x}(t+s) \succeq 0$ for $s \in [-\tau, 0]$. It follows that $e(t) \succeq 0$ regarding $\int_{-\tau}^{0} A_d(s)\left[\bar{x}(t) - \bar{x}(t+s)\right]ds + B_w(\overline{w} - w(t))$ as nonnegative input of positive system (2.5). The comparison $z(t) \preceq \bar{z}(t)$ can be shown by similar lines used in the derivation of $x(t) \preceq \bar{x}(t)$. $\square$

Remark 2.1. According to Lemma 2.2, the state and output trajectories of system (2.1) will be compared with those of the auxiliary system (2.4). Since the state trajectories of the system (2.4) are monotonically nondecreasing, they are expected to monotonically converge to the equilibrium point of the system (2.4). More specifically, by the assumption that system (2.1) is stable, the matrix $A + \int_{-\tau}^{0} A_d(s)ds$ is Metzler and Hurwitz. Therefore, system (2.4) has a unique equilibrium point, which is given as

$$x_* = -\mathcal{A}^{-1}B_w\overline{w}.$$

The following important result shows that the state trajectory $\overline{x}(t)$ of (2.4) monotonically converges to x_*.

Lemma 2.3. *Assume that the Metzler matrix $\mathcal{A} = A + \int_{-\tau}^{0} A_d(s)ds$ is Hurwitz. Then, there exist scalars $\beta > 0$ and $\epsilon > 0$ such that the state trajectory $\bar{x}(t)$ of system* (2.4) *satisfies*

$$\max(0, 1 - \beta e^{-\epsilon t})x_* \preceq \bar{x}(t) \preceq x_*, \ t \ge 0.$$

In particular, it holds that $\lim_{t\to\infty} \bar{x}(t) = -\mathcal{A}^{-1}B_w\overline{w}$.

Proof. Clearly, $e(t) = x_* - \bar{x}(t)$ is a solution of the system

$$\dot{e}(t) = \mathcal{A}e(t), \quad e(0) = x_* \succeq 0. \tag{2.6}$$

Since system (2.6) is positive, it follows that $e(t) \succeq 0$ and hence $\bar{x}(t) \preceq x_*$ for $t \geq 0$.

On the other hand, the Metzler matrix $\mathcal{A}$ is Hurwitz, there exists a positive vector $\nu \in \mathbb{R}^n$ such that $\nu^\top \mathcal{A} \prec 0$. Thus, for a sufficiently small $\epsilon > 0$, we have $\nu^\top \mathcal{A} \preceq -\epsilon\nu^\top$. Consider the co-positive Lyapunov function $v(t) = \nu^\top e(t)$. We have

$$\dot{v}(t) = \nu^\top \mathcal{A}e(t) \leq -\epsilon\nu^\top e(t)$$

which gives $v(t) \leq \nu^\top e_0 e^{-\epsilon t}$. By this, we readily obtain

$$x_* - \bar{x}(t) \preceq C_\nu x_* e^{-\epsilon t},\ t \geq 0,$$

where $C_\nu = (\max_{1\leq i\leq n} \nu_i)/(\min_{1\leq i\leq n}\nu_i)$. This completes the proof. $\square$

Remark 2.2. Since $C_\nu x_* e^{-\epsilon t}$ is a decreasing function, it follows from Lemma 2.3 that the state trajectory $\bar{x}(t)$ of (2.4) is increasingly approaching the equilibrium point x_* as $t \to \infty$.

We now establish the following result.

Theorem 2.2. *Assume that system* (2.1) *is positive and stable. The value of L_∞-induced norm of system* (2.1) *can be represented as*

$$\|\Sigma\|_{(L_\infty,L_\infty)} = \left\|D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right\|_\infty, \tag{2.7}$$

where $\mathcal{C} = C + \int_{-\kappa}^0 C_d(s)ds$ and $\mathcal{A} = A + \int_{-\tau}^0 A_d(s)ds$.

Proof. Let $\overline{w} = \|w\|_{L_\infty}\mathbf{1}_q$ for a $w \in L_\infty(\mathbb{R}^q)$. By Lemmas 2.2 and 2.3, we have

$$|z(t,w)| \preceq z(t,|w|) \preceq \bar{z}(t),\ t \geq 0,$$

where $\bar{z}(t)$ is determined by (2.4). Since $\bar{x}(t) \preceq x_*$, it follows from (2.4) that

$$\begin{aligned}\bar{z}(t) &\preceq \mathcal{C}x_* + D_w\overline{w} \\ &= \left(D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right)\mathbf{1}_q\|w\|_{L_\infty}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|z(t,w)\|_\infty &\leq \left\|(D_w - \mathcal{C}\mathcal{A}^{-1}B_w)\mathbf{1}_q\right\|_\infty \|w\|_{L_\infty} \\ &= \left\|D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right\|_\infty \|w\|_{L_\infty}\end{aligned}$$

by which we readily obtain

$$\|\Sigma\|_{(L_\infty,L_\infty)} \le \left\|D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right\|_\infty .$$

On the other hand, for $\hat{x} = -\mathcal{A}^{-1}B_w\mathbf{1}_q$ and with $w(t) = \overline{w} = \mathbf{1}_q$, let $\psi(t) = \hat{x} - x(t)$. It follows from (2.1), (2.4) and Lemma 2.3 that

$$\hat{x} - \psi(t) = x(t) \preceq \hat{x}, \tag{2.8}$$

where $\psi(t)$ is solution of the system

$$\begin{cases} \dot{x}(t) = Ax(t) + \displaystyle\int_{-\tau}^{0} A_d(s)x(t+s)ds \\ x(s) = \hat{x},\ s \in [-\tau, 0]. \end{cases} \tag{2.9}$$

Since the matrix $A + \int_{-\tau}^{0} A_d(s)ds$ is Hurwitz, by Theorem 2.1, $\lim_{t\to\infty}\psi(t) = 0$ and hence $\lim_{t\to\infty} x(t) = \hat{x}$. It follows from (2.1) and (2.8) that

$$z(t) = \left(D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right)\mathbf{1}_q - \left(C\psi(t) + \int_{-\kappa}^{0} C_d(s)\psi(t+s)ds\right).$$

Thus, $\lim_{t\to\infty} z(t) = (D_w - \mathcal{C}\mathcal{A}^{-1}B_w)\mathbf{1}_q$. By this, we can conclude that

$$\sup_{\|w\|_{L_\infty}=1} \|z\|_{L_\infty} \ge \left\|D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right\|_\infty .$$

The proof is completed. □

The result of Theorem 2.2 is an important and efficient tool that can be utilized to address the stabilization problem involving L_∞-gain performance. However, the formula (2.7) cannot be directly used for the controller design problem. Thus, the following performance result is necessary.

Theorem 2.3. *For a given $\gamma > 0$, positive system* (2.1) *is stable and has L_∞-induced performance of level γ if and only if there exists a vector $\eta \in \mathbb{R}^n$, $\eta \succ 0$, that satisfies the following LP-based conditions:*

$$\left(A + \int_{-\tau}^{0} A_d(s)ds\right)\eta + B_w\mathbf{1}_q \prec 0, \tag{2.10a}$$

$$\left(C + \int_{-\kappa}^{0} C_d(s)ds\right)\eta + D_w\mathbf{1}_q - \gamma\mathbf{1}_p \prec 0. \tag{2.10b}$$

Proof. (*Necessity*) Let $v \succ 0$ be a vector such that $\left(A + \int_{-\tau}^{0} A_d(s)ds\right)v \prec 0$. For a given $\epsilon > 0$, we define $\eta = \epsilon v - \mathcal{A}^{-1}B_w\mathbf{1}_q \succ 0$, then

$$\mathcal{A}\eta + B_w\mathbf{1}_q = \epsilon\mathcal{A}v \prec 0. \tag{2.11}$$

In addition to this, by Theorem 2.2, $\|\Sigma\|_{(L_\infty,L_\infty)} < \gamma$ if and only if

$$\chi_1 = \left(D_w - \mathcal{C}\mathcal{A}^{-1}B_w\right)\mathbf{1}_q \prec \gamma\mathbf{1}_p.$$

Therefore, $\chi_2 = \gamma\mathbf{1}_p - \chi_1 \succ 0$ and we have

$$\mathcal{C}\eta + D_w\mathbf{1}_q - \gamma\mathbf{1}_p = \epsilon\mathcal{C}v - \chi_2 \prec 0$$

for sufficiently small ϵ which, together (2.11), yields (2.10).

(*Sufficiency*) It can be deduced from condition (2.10a) that the matrix $\mathcal{A}$ is Hurwitz and, by Theorem 2.1, system (2.1) is GES. Let $\tilde{\eta} = \mathcal{A}\eta + B_w\mathbf{1}_{n_w} \prec 0$, we have $\eta = \mathcal{A}^{-1}\left(\tilde{\eta} - B_w\mathbf{1}_{n_w}\right)$ and it follows from (2.10b) that

$$\gamma\mathbf{1}_p \succ \mathcal{C}\eta + D_w\mathbf{1}_q = \mathcal{C}\mathcal{A}^{-1}\tilde{\eta} + \chi_1. \tag{2.12}$$

For any $w \in L_\infty(\mathbb{R}^q)$ with $\|w\|_{L_\infty} = 1$, since $\mathcal{A}^{-1}\tilde{\eta} \succ 0$, from (2.12), we obtain $\|z\|_{L_\infty} \leq \|\chi_1\|_\infty < \gamma$. The proof is completed. □

2.3. State-feedback L_∞-induced performance stabilization

In this section, we address the L_∞-induced stabilization problem for system (2.1). A state-feedback controller in the form

$$u(t) = Kx(t) \tag{2.13}$$

will be designed to make the closed-loop system positive, stable and has prescribed L_∞-induced performance. By integrating controller (2.13), the closed-loop system of (2.1) is presented as

$$\begin{cases} \dot{x}(t) = A_c x(t) + \displaystyle\int_{-\tau}^{0} A_d(s)x(t+s)ds + B_w w(t), \\ z(t) = C_c x(t) + \displaystyle\int_{-\kappa}^{0} C_d(s)x(t+s)ds + D_w w(t), \end{cases} \tag{2.14}$$

where $A_c = A + BK$ and $C_c = C + DK$. For a given $\gamma > 0$, by Proposition 2.1 and Theorem 2.3, system (2.14) is positive, stable and has L_∞-induced performance of level γ if and only if

$$A_c = A + BK \text{ is Metzler}, \tag{2.15a}$$

$$C_c = C + DK \succeq 0, \tag{2.15b}$$

$$\exists \eta \succ 0: \quad \begin{bmatrix} \mathcal{A} & B_w \\ \mathcal{C} & D_w \end{bmatrix} \begin{bmatrix} \eta \\ \mathbf{1}_q \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} K\eta \prec \gamma \begin{bmatrix} 0 \\ \mathbf{1}_p \end{bmatrix}. \tag{2.15c}$$

For a vector $\eta \in \mathbb{R}^n$, $\eta \succ 0$, we have $\eta = \mathrm{d}iag(\eta_i)\mathbf{1}_n$, where $\eta = (\eta_i)$ and $\mathrm{d}iag(\eta_i)$ is the diagonal matrix formulated by stacking components η_i. We define the transformation

$$K\mathrm{d}iag(\eta_i) = Z \in \mathbb{R}^{m\times n} \tag{2.16}$$

then condition (2.15c) is reduced to the following one

$$\begin{bmatrix} \mathcal{A} & B_w \\ \mathcal{C} & D_w \end{bmatrix} \begin{bmatrix} \eta \\ \mathbf{1}_q \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} Z\mathbf{1}_n \prec \gamma \begin{bmatrix} 0 \\ \mathbf{1}_p \end{bmatrix}. \tag{2.17}$$

On the other hand, it follows from (2.16) that $K = Z\mathrm{d}iag(\eta_i^{-1})$. Thus, condition (2.15b) holds if and only if

$$C\mathrm{d}iag(\eta_i) + DZ \succeq 0. \tag{2.18}$$

We now tackle with condition (2.15a). For this, we decompose

$$B = \begin{bmatrix} b_1^\top \\ \vdots \\ b_n^\top \end{bmatrix},\ b_i \in \mathbb{R}^m, \quad Z = \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix},\ z_j \in \mathbb{R}^m.$$

Then, $BZ = (b_i^\top z_j)$. In addition, for any matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ and $M_\eta = (m_{ij}^\eta) = M\mathrm{d}iag(\eta_i)$, we have

$$m_{ij}^\eta = \sum_{k=1}^n m_{ik}\delta_{kj}\eta_j,$$

where δ_{kj} is the Kronecker delta notation. Thus, $m_{ij}^\eta \geq 0$ if and only if $m_{ij} \geq 0$. In other words, the matrix M is Metzler if and only if $M\mathrm{d}iag(\eta_i)$ is Metzler for any positive vector η. By this, and assume that $A = (a_{ij})$, condition (2.15a) is satisfied if and only if

$$a_{ij}\eta_j + b_i^\top z_j \geq 0,\ \forall i \neq j. \tag{2.19}$$

In summary, we have the following result.

Theorem 2.4. *For a given $\gamma > 0$, there exists a state-feedback controller in the form of* (2.13) *that makes the closed-loop system* (2.14) *positive, stable and has L_∞-induced performance of level γ if and only if there exist a vector $\eta = (\eta_i) \in \mathbb{R}^n$, $\eta \succ 0$, and a matrix $Z = [z_1 \dots z_n]$, $z_j \in \mathbb{R}^m$, that satisfy the following LP-based conditions*

$$\left(A + \int_{-\tau}^0 A_d(s)ds\right)\eta + BZ\mathbf{1}_n + B_w\mathbf{1}_q \prec 0, \tag{2.20a}$$

$$\left(C + \int_{-\kappa}^0 C_d(s)ds\right)\eta + DZ\mathbf{1}_n + D_w\mathbf{1}_q \prec \gamma\mathbf{1}_p, \tag{2.20b}$$

$$C\mathrm{d}iag(\eta_i) + DZ \succeq 0, \tag{2.20c}$$

$$a_{ij}\eta_j + b_i^\top z_j \geq 0,\ i \neq j, \tag{2.20d}$$

where $A = (a_{ij})$ *and* $B^\top = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$. *The controller gain is obtained as*

$$K = Z\mathrm{d}iag(\eta_i^{-1}). \tag{2.21}$$

2.4. An illustrative example

Consider system (2.1) with $n = 2$, $p = q = 1$ and the system matrices

$$A = \begin{bmatrix} -1.5 & 1 \\ 0.5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 1, \quad D_w = 0.2,$$
$$A_d(s) = \frac{1}{4}\begin{bmatrix} -s & 0 \\ 0 & -s \end{bmatrix}, \; C_d(s) = \frac{1}{4}\begin{bmatrix} 0 & -s \end{bmatrix}, \; s \in [-2, 0].$$

Then, we have

$$\mathcal{A} = A + \int_{-2}^{0} A_d(s)ds = \begin{bmatrix} -1 & 1 \\ 0.5 & -0.5 \end{bmatrix}, \; \mathcal{C} = C + \int_{-2}^{0} C_d(s)ds = \begin{bmatrix} 1 & 0.5 \end{bmatrix}.$$

It is clear that the matrix $\mathcal{A}$ is not Hurwitz. Thus, with the given system parameters, the open-loop system is unstable. We now apply Theorem 2.4. By solving the LP-based conditions (2.20a)-(2.20d) via the `linprog` toolbox in Matlab, it is found that the derived conditions in (2.20) are feasible for $\gamma \geq \gamma_* = 0.41$. With $\gamma = 0.41$, an optimal feasible solution is obtained as

$$\eta = \begin{bmatrix} 0.2078 \\ 0.2046 \end{bmatrix}, \quad Z = \begin{bmatrix} -0.1032 & 0.0008 \end{bmatrix}.$$

According to (2.21), the controller gain K is given by

$$K = \begin{bmatrix} -0.4967 & 0.0038 \end{bmatrix}.$$

By Theorem 2.4, the closed-loop system (2.14) is positive, stable and has L_∞-induced performance of level γ_*.

3. Conclusions

In this paper, the problem of L_∞-gain control has been studied for positive linear systems with distributed delays in the state and output vectors. Based on a novel comparison technique involving the monotonicity state trajectories of scaled systems with peak values of exogenous disturbances, necessary and sufficient LP-based conditions subject to L_∞-induced performance have been derived. The obtained analysis result has been then utilized to address the design problem of a state-feedback controller that makes the closed-loop system positive, stable and has prescribed L_∞-gain performance level. A numerical example has been provided to illustrate the effectiveness of the proposed method.

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