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REGULARITY AND CONVERGENCE TO EQUILIBRIUM FOR A CLASS OF NONLOCAL EVOLUTION EQUATIONS

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Abstract. We study a class of semilinear nonlocal partial differential equations, which model different problems related to processes in materials with memory. Our aim is to derive sufficient conditions ensuring the global solvability, regularity, and convergence to equilibrium of solutions.

Keywords: nonlocal PDE, regularity, convergence to equilibrium.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following problem

$$\partial_t u - \Delta u - \partial_t(m * \Delta u) = f(u) \text{ in } \Omega, t > 0, \tag{1.1}$$
$$\mathcal{B}u = 0 \text{ on } \partial\Omega,\ t \geq 0, \tag{1.2}$$
$$u(\cdot, 0) = \xi \text{ in } \Omega, \tag{1.3}$$

where $\partial_t = \frac{\partial}{\partial t}$, $m \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function, and the notation '*' stands for the Laplace convolution with respect to the time t, i.e.,

$$(m * v)(t) = \int_0^t m(t-s)v(s)ds.$$

In our model, Δ denotes the Laplacian, f is a nonlinear function and $\xi \in L^2(\Omega)$ is given, $\mathcal{B}$ is a boundary operator in one of the following forms

$$\mathcal{B}u = u \text{ or } \mathcal{B}u = \nu \cdot \nabla u + \eta u,\ \eta > 0,$$

where ν is the outward normal vector to $\partial\Omega$.

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We first mention some special cases of (1.1). If m is a nonnegative constant then (1.1) is the classical reaction-diffusion equation with nonlinear sources. In the case $m(t) = m_0 g_{1-\alpha}(t) = \dfrac{m_0 t^{-\alpha}}{\Gamma(1-\alpha)}, m_0 > 0$, our equation reads

$$\partial_t u - (1 + m_0 \partial_t^\alpha)\Delta u = f(u),$$

which is the generalized Rayleigh-Stokes equation (see, e.g. [1]), here ∂_t^α denotes the fractional derivative of order α in the sense of Riemann-Liouville. This equation is employed to describe the behavior of non-Newtonian fluids. In addition, if m is a regular function, e.g. $m \in C^1(\mathbb{R}^+)$, then (1.1) is a diffusion equation with memory, namely

$$\partial_t u - (1 + m(0))\Delta u - \int_0^t m'(t-s)\Delta u(s)ds = f(u),$$

which has been a topic of an extensive study, see e.g., [2-8].

In this paper, we consider the problem (1.1)-(1.3) in a general form, where the kernel function m is possibly singular. To our knowledge, no attempt has been made to investigate the solvability and regularity of this problem, and we aim at closing this gap. In addition, based on the regularity result, we will prove the convergence to equilibrium of solutions. Precisely, under the assumption that f is Lipschitzian, the unique strong solution of the elliptic problem

$$-\Delta w = f(w) \text{ in } \Omega, \ w = 0 \text{ on } \partial\Omega,$$

becomes an attractor for solution of (1.1)-(1.3) with arbitrary initial data.

Our work is organized as follows: In the next section, we recall the theory of completely positive functions and the theory of resolvent operators, which leads to a representation of solutions of (1.1)-(1.3). Section 3 is devoted to the global solvability and regularity results. Under appropriate conditions imposed on m and f, we show that the mild solution of (1.1)-(1.3) is classical. In Section 4, a result on the convergence of solutions of our problem to equilibrium is presented.

2. Preliminaries

Consider the scalar equation

$$\omega'(t) + \lambda\omega(t) + \lambda(m * \omega)'(t) = 0 \text{ for } t > 0, \ \omega(0) = 1, \tag{2.1}$$

where λ is a positive number, $m \in L^1_{loc}(\mathbb{R}^+)$ satisfies the following assumption

(M) *The function $m \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative such that the function $a(t) := 1 + m(t)$ is completely positive.*

Recall that the complete positivity of a means that the solution of the following integral equations

$$s(t) + \theta \int_0^t a(t-\tau)s(\tau)d\tau = 1,\ t \geq 0, \tag{2.2}$$

$$r(t) + \theta \int_0^t a(t-\tau)r(\tau)d\tau = a(t),\ t > 0, \tag{2.3}$$

are nonnegative for each $\theta > 0$. It should be noted that, if the function m is completely monotone, i.e. $(-1)^k m^{(k)}(t) \geq 0$ for every $k \in \mathbb{N}$, then $1 + m$ is completely monotone as well. As mentioned in [9, 10], this property ensures the complete positivity of the function a.

We mention another case when m is smooth and positive on $(0, \infty)$ such that $\log m$ is a convex function. This implies that m is also convex, then $\dfrac{m'}{m}$ and m' are increasing. It follows that the function $\dfrac{m'}{1+\gamma m}$ with $\gamma > 0$, is also increasing. That means, $\log(1 + \gamma m)$ is convex and the function $1 + \gamma m$ is completely positive for any $\gamma > 0$, according to [10]. It should be noted that if m is completely monotone, then $\log m$ is convex.

We recall some properties of s and r in the following proposition.

Proposition 2.1. *Assume that the assumption **(M)** is satisfied. Let* $s = s(\cdot, \theta)$ *and* $r = r(\cdot, \theta)$ *be the solutions of* (2.2) *and* (2.3)*, respectively. Then*

- *The function* $s(\cdot, \theta)$ *is nonnegative and nonincreasing. Moreover,*

$$s(t, \theta)\left[1 + \theta \int_0^t a(\tau)d\tau\right] \leq 1,\ \forall t \geq 0. \tag{2.4}$$

- *The function* $r(\cdot, \theta)$ *is nonnegative and the following relation holds*

$$s(t, \theta) = 1 - \theta \int_0^t r(\tau, \theta)d\tau,\ t \geq 0. \tag{2.5}$$

- *For each* $t > 0$*, the function* $\theta \mapsto s(t, \theta)$ *is nonincreasing.*

Proof. The justification for (2.4) and (2.5) can be found in [9]. The last statement was proved in [11, Lemma 5.1]. □

We make use of Proposition 2.1 to get some useful properties of solution to (2.1).

Proposition 2.2. *Let* $\omega = \omega(\cdot, \lambda)$ *be the solution of* (2.1)*. Then*

1. ω *is nonincreasing on* $\mathbb{R}^+$ *and*

$$0 < \omega(t, \lambda) \leq \frac{1}{1 + \lambda \int_0^t (1 + m(\tau))d\tau},\ \forall t \geq 0,\ \lambda > 0.$$

Consequently, $\lim_{t\to\infty} \omega(t, \lambda) = 0.$

2. *The following estimate holds*

$$\int_0^t \omega(\tau,\lambda)d\tau \le \lambda^{-1}(1-\omega(t,\lambda)),\ \forall t\ge 0, \lambda>0.$$

3. *For each $t>0$, the function $\lambda\mapsto\omega(t,\lambda)$ is nonincreasing.*

Proof. Taking integration of (2.1), we get

$$\omega(t)+\lambda\int_0^t(1+m(t-\tau))\omega(\tau)d\tau = 1. \tag{2.6}$$

This implies that ω is the solution of (2.2) with $\theta=\lambda$. So the statement (1) and (3) follows from Proposition 2.1. Since $\omega(\cdot,\lambda)$ is nonincreasing, it is deduced from (2.6) that

$$\omega(t)+\lambda\omega(t,\lambda)\int_0^t(1+m(t-\tau))d\tau\le 1,$$

which implies the statement (2). □

Consider the inhomogeneous equation

$$z'(t)+\lambda z(t)+\lambda(m*z)'(t)=g(t),\ t>0, z(0)=z_0, \tag{2.7}$$

where $\lambda>0$ and $g\in C(\mathbb{R}^+)$. The following proposition gives a representation for the solution of (2.7).

Proposition 2.3. *The function*

$$z(t)=\omega(t,\lambda)z_0+\int_0^t\omega(t-\tau,\lambda)g(\tau)d\tau, \tag{2.8}$$

is the unique solution of (2.7).

Proof. Denote $L[y]=y'+\lambda y+\lambda(m*y)'$, $y\in C^1(\mathbb{R}^+)$. Then by formulation, $L[\omega]=0$. In addition, we have

$$L[z]=L[\omega]z_0+L[\omega*g]=L[\omega*g].$$

We will show that $L[\omega*g]=g$. Indeed, one sees that

$$\begin{aligned}(\omega*g)'+\lambda\omega*g+\lambda(m*\omega*g)' &= g+\omega'*g+\lambda\omega*g+\lambda(m*\omega)'*g\\ &= g+[\omega'+\lambda\omega+\lambda(m*\omega)']*g\\ &= g+L[\omega]*g=g.\end{aligned}$$

Conversely, if z is a solution of (2.7), we get

$$q\hat{z}(q)+\lambda\hat{z}(q)+\lambda q\hat{m}(q)\hat{z}(q)=z_0+\hat{g}(q),$$

where $\hat{z}$ is the Laplace transform of z. Then

$$\begin{aligned}\hat{z}(q) &= (q+\lambda+\lambda q\hat{m}(q))^{-1}z_0 + (q+\lambda+\lambda q\hat{m}(q))^{-1}\hat{g}(q)\\ &= \hat{\omega}(q)z_0 + \hat{\omega}(q)\hat{g}(q).\end{aligned}$$

Taking the inverse Laplace transform yields $z = \omega z_0 + \omega * g$, which is (2.8). The proof is complete. $\square$

Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ subject to the homogeneous boundary condition, i.e.,

$$-\Delta\varphi_n = \lambda_n\varphi_n \text{ in } \Omega,\ \mathcal{B}\varphi_n = 0 \text{ on } \partial\Omega,$$

where one can assume that $0 < \lambda_1 \le \lambda_2 \le \ldots,\ \lambda_n \to \infty$ as $n \to \infty$. We find a representation for solution of the linear problem

$$\partial_t u - \Delta u - \partial_t(m * \Delta u) = F \text{ in } \Omega, t \in (0,T], \tag{2.9}$$

$$\mathcal{B}u = 0 \text{ on } \partial\Omega,\ t \in [0,T], \tag{2.10}$$

$$u(\cdot,0) = \xi \text{ in } \Omega, \tag{2.11}$$

where $F \in C([0,T];L^2(\Omega))$.

Assume that

$$u(\cdot,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n,\ F(\cdot,t) = \sum_{n=1}^{\infty} F_n(t)\varphi_n.$$

Substituting into (2.9), we get

$$\begin{aligned}&u_n'(t) + \lambda_n u_n(t) + \lambda_n(m * u_n)'(t) = F_n(t),\\ &u_n(0) = \xi_n := (\xi,\varphi_n).\end{aligned}$$

Employing Proposition 2.3, we obtain

$$u_n(t) = \omega(t,\lambda_n)\xi_n + \int_0^t \omega(t-\tau,\lambda_n)F_n(\tau)d\tau.$$

This implies

$$u(\cdot,t) = S(t)\xi + \int_0^t S(t-\tau)F(\cdot,\tau)d\tau, \tag{2.12}$$

where $S(t)$ is *the resolvent operator* defined by

$$S(t)\xi = \sum_{n=1}^{\infty} \omega(t,\lambda_n)\xi_n\varphi_n,\ \xi \in L^2(\Omega). \tag{2.13}$$

Obviously, $S(t)$ is a bounded linear operator on $L^2(\Omega)$ for all $t \ge 0$. Moreover, we have the following result.

Lemma 2.1. *Let $\{S(t)\}_{t\geq 0}$ be the resolvent operator defined by* (2.13)*, $v \in L^2(\Omega)$ and $T > 0$. Then*

1. *$S(\cdot)v \in C([0,T]; L^2(\Omega))$ and $\|S(t)\| \leq \omega(t, \lambda_1)$ for all $t \geq 0$.*
2. *$\Delta S(\cdot)v \in C((0,T]; L^2(\Omega))$ and $\|\Delta S(t)\| \leq (t + 1 * m(t))^{-1}$ for all $t > 0$.*
3. *If m is nonincreasing, then $S(\cdot)v \in C^1((0,T]; L^2(\Omega))$ and it holds that*

$$\|S'(t)\| \leq t^{-1} \text{ for all } t > 0.$$

Proof. (1) It follows from (2.13) that

$$\|S(t)\xi\|^2 = \sum_{n=1}^{\infty} \omega(t, \lambda_n)^2 \xi_n^2$$
$$\leq \omega(t, \lambda_1)^2 \sum_{n=1}^{\infty} \xi_n^2 = \omega(t, \lambda_1)^2 \|\xi\|^2,$$

thanks to Proposition 2.2(3), which implies the uniform convergence of series (2.13) on $[0, T]$ and the estimate $\|S(t)\| \leq \omega(t, \lambda_1)$ for all $t \geq 0$.
(2) We observe that

$$(-\Delta)S(t)\xi = \sum_{n=1}^{\infty} \lambda_n \omega(t, \lambda_n) \xi_n \varphi_n, \tag{2.14}$$
$$\lambda_n \omega(t, \lambda_n) \leq \frac{\lambda_n}{1 + \lambda_n (t + 1 * m(t))} \leq (t + 1 * m(t))^{-1}, \ \forall t > 0,$$

where we utilized Proposition 2.2(1). Thus series (2.14) is uniformly convergent on $[\epsilon, T]$ for any $\epsilon \in (0, T)$. Moreover, we have

$$\|\Delta S(t)\xi\| \leq (t + 1 * m(t))^{-1} \|\xi\|, \ \forall t > 0, \ \xi \in L^2(\Omega).$$

(3) Let $r(\cdot, \lambda)$ be the solution of (2.3) with $\theta = \lambda$ and $a(t) = 1 + m(t)$. Then, due to the assumption that m is nonincreasing, we have

$$r(t, \lambda) + \lambda(1 + m(t)) \int_0^t r(\tau, \lambda) d\tau \leq 1 + m(t).$$

In addition,

$$\int_0^t r(\tau, \lambda) d\tau = \lambda^{-1}(1 - \omega(t, \lambda)) \geq \frac{t + 1 * m(t)}{1 + \lambda(t + 1 * m(t))},$$

thanks to Proposition 2.2(1). Hence

$$r(t,\lambda) \leq [1+m(t)]\left[1 - \frac{\lambda(t+1*m(t))}{1+\lambda(t+1*m(t))}\right] = \frac{1+m(t)}{1+\lambda(t+1*m(t))}. \tag{2.15}$$

Considering the series

$$\sum_{n=1}^{\infty} \omega'(t,\lambda_n)\xi_n\varphi_n,\ t>0, \xi_n = (\xi,\varphi_n),\ \xi \in L^2(\Omega), \tag{2.16}$$

we see that

$$\begin{aligned} |\omega'(t,\lambda_n)| &= \lambda_n r(t,\lambda_n) \\ &\leq \frac{\lambda_n(1+m(t))}{1+\lambda_n(t+1*m(t))} \leq \frac{1+m(t)}{t+1*m(t)} \leq \frac{1+m(t)}{t+tm(t)} = t^{-1}. \end{aligned}$$

According to (2.15) and the fact that $1*m(t) \geq tm(t)$ for $t>0$. This ensures the uniform convergence of series (2.16) on $[\epsilon, T]$ and it holds that

$$S'(t)\xi = \sum_{n=1}^{\infty} \omega'(t,\lambda_n)\xi_n\varphi_n,\ \|S'(t)\xi\| \leq t^{-1}\|\xi\|,\ \forall t>0.$$

The proof is complete. □

In order to get more regularity of $S(\cdot)$, we will impose some additional assumptions on the function m. In regard to these assumptions, we first recall several notions and facts given in [12].

It should be noted that, $S(t)$ defined by (2.13), is the resolvent operator of the problem

$$\partial_t u - \Delta u - \partial_t(m * \Delta u) = 0 \ \text{ in } \Omega, t>0, \tag{2.17}$$

$$\mathcal{B}u = 0 \ \text{ on } \partial\Omega, t \geq 0, \tag{2.18}$$

$$u(\cdot,0) = \xi, \tag{2.19}$$

that is, $u(\cdot,t) = S(t)\xi$. This problem is equivalent to the Volterra equation

$$u(\cdot,t) - \int_0^t a(t-s)Au(\cdot,s)ds = \xi, \tag{2.20}$$

with $a(t) = 1+m(t)$ and $A = \Delta$. This can be seen by taking integration of (2.17) with respect to t.

Definition 2.1. *Let $l \in L^1_{loc}(\mathbb{R}^+)$ be a function of subexponential growth, i.e. $\int_0^\infty |l(t)|e^{-\epsilon t}dt < \infty$ for every $\epsilon > 0$.*

- *Suppose that* $\hat{l}(\lambda) \neq 0$ *for all* $\mathrm{Re}\lambda > 0$. *For* $\theta > 0$, l *is said to be* θ*-sectorial if* $|\arg \hat{l}(\lambda)| \leq \theta$ *for all* $\mathrm{Re}\lambda > 0$.
- *For given* $k \in \mathbb{N}$, l *is called* k*-regular if there exists a constant* $c > 0$ *such that*

$$|\lambda^n \hat{l}^{(n)}(\lambda)| \leq c|\hat{l}(\lambda)| \quad \textit{for all } \mathrm{Re}\lambda > 0, 1 \leq n \leq k.$$

Definition 2.2. *Equation* (2.20) *is called parabolic if the following conditions hold:*

1. $\hat{a}(\lambda) \neq 0$, $1/\hat{a}(\lambda) \in \rho(A)$ *for all* $\mathrm{Re}\lambda \geq 0$.
2. *There is a constant* $M \geq 1$ *such that* $U(\lambda) = \lambda^{-1}(I - \hat{a}(\lambda)A)^{-1}$ *satisfies*

$$\|U(\lambda)\| \leq \frac{M}{|\lambda|} \quad \textit{for all } \mathrm{Re}\lambda > 0.$$

Denote by $\Sigma(\varpi, \theta)$ the open sector with vertex $\varpi \in \mathbb{R}$ and angle 2θ in the complex plane, i.e.

$$\Sigma(\varpi, \theta) = \{\lambda \in \mathbb{C} : |\arg(\lambda - \varpi)| < \theta\}.$$

We have the following sufficient condition for equation (2.20) to be parabolic.

Proposition 2.4. *[12, Proposition 3.1] Assume that* $a \in L^1_{loc}(\mathbb{R}^+)$ *is of subexponential growth and* θ*-sectorial for some* $\theta < \pi$. *If* A *is closed linear densely defined, such that* $\rho(A) \supset \Sigma(0, \theta)$, *and*

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|} \textit{ for all } \lambda \in \Sigma(0, \theta), \tag{2.21}$$

then equation (2.20) *is parabolic.*

Remark 2.1. *Let us mention that,* $A = \Delta$ *generates an analytic semigroup in* $L^2(\Omega)$, *which is given by*

$$e^{tA}v = \sum_{n=1}^{\infty} e^{-t\lambda_n}(v, \varphi_n)\varphi_n, \ t \geq 0, v \in L^2(\Omega).$$

Then (2.21) *holds for* $M = 1$ *and for any* $\theta < \pi$ *(see, e.g. [13]).*

The following result on the regularity of resolvent operator for equation (2.20) will be used in the sequel.

Proposition 2.5. *[12, Theorem 3.1] Assume that* (2.20) *is parabolic and the kernel function* a *is* k*-regular for some* $k \geq 1$. *Then there is a resolvent family* $S(\cdot) \in C^{(k-1)}((0, \infty); \mathcal{L}(L^2(\Omega)))$ *for* (2.20)*, and a constant* $M \geq 1$ *such that*

$$\|t^n S^{(n)}(t)\| \leq M, \quad \textit{for all } t > 0, n \leq k - 1,$$

here $\mathcal{L}(L^2(\Omega))$ *denotes the space of bounded linear operators on* $L^2(\Omega)$.

In the next section, we need the following result.

Corollary 2.1. *Let **(M)** hold. Assume that $a(t) = 1 + m(t)$ is 3-regular and θ-sectorial for some $\theta < \pi$. Then the resolvent family $S(\cdot)$ is continuously differentiable up to second order and there exists $M \geq 1$ such that*

$$\|S'(t)\| \leq Mt^{-1},\ \|S''(t)\| \leq Mt^{-2},\ \forall t > 0.$$

Proof. The conclusion follows from Proposition 2.4, Remark 2.1 and Proposition 2.5. □

3. Solvability and regularity

Dealing with the problem (1.1)-(1.3), we assume that

(F) *$f : L^2(\Omega) \to L^2(\Omega)$ is a locally Lipschitz function, i.e.*

$$\|f(v_1) - f(v_2)\| \leq \kappa(r)\|v_1 - v_2\|,\ \forall v_1, v_2 \in B_r,$$

where B_r is the closed ball in $L^2(\Omega)$ with radius r and center at origin, $\kappa(\cdot)$ is a nonnegative function.

Based on representation (2.12), we give the following definition of mild solution for (1.1)-(1.3).

Definition 3.1. *A function $u \in C([0, T]; L^2(\Omega))$ is said to be a mild solution to the problem* (1.1)-(1.3) *on $[0, T]$ iff*

$$u(\cdot, t) = S(t)\xi + \int_0^t S(t - \tau)f(u(\cdot, \tau))d\tau \text{ for any } t \in [0, T].$$

We now prove a global solvability result for (1.1)-(1.3).

Theorem 3.1. *Let **(M)** hold. Assume that **(F)** is satisfied with $f(0) = 0$ and $\limsup\limits_{r\to 0} \kappa(r) = \ell \in [0, \lambda_1)$. Then there exists $\delta > 0$ such that the problem* (1.1)-(1.3) *has a unique mild solution on $[0, T]$, provided $\|\xi\| \leq \delta$.*

Proof. Let $\Phi : C([0, T]; L^2(\Omega)) \to C([0, T]; L^2(\Omega))$ be the mapping defined by

$$\Phi(u)(t) = S(t)\xi + \int_0^t S(t - \tau)f(u(\cdot, \tau))d\tau \text{ for } t \in [0, T].$$

We first look for $\rho > 0$ such that $\Phi(\mathsf{B}_\rho) \subset \mathsf{B}_\rho$, where B_ρ is the closed ball in $C([0, T]; L^2(\Omega))$ centered at origin with radius ρ. Taking $\epsilon \in (0, \lambda_1 - \ell)$, we can find

$\rho > 0$ such that $\kappa(r) \le \ell + \epsilon$ for any $r \le \rho$. Considering $\Phi : \mathsf{B}_\rho \to C([0,T]; L^2(\Omega))$, we have

$$\begin{aligned}
\|\Phi(u)(\cdot,t)\| &\le \|S(t)\xi\| + \int_0^t \|S(t-\tau)\|\|f(u(\cdot,\tau))\|d\tau \\
&\le \omega(t,\lambda_1)\|\xi\| + \int_0^t \omega(t-\tau,\lambda_1)\kappa(\rho)\|u(\cdot,\tau)\|d\tau \\
&\le \omega(t,\lambda_1)\|\xi\| + (\ell+\epsilon)\rho\int_0^t \omega(t-\tau,\lambda_1)d\tau \\
&\le \omega(t,\lambda_1)\|\xi\| + (\ell+\epsilon)\rho\lambda_1^{-1}(1-\omega(t,\lambda_1)) \\
&= \omega(t,\lambda_1)[\|\xi\| - (\ell+\epsilon)\rho\lambda_1^{-1}] + (\ell+\epsilon)\rho\lambda_1^{-1},\ \forall u \in \mathsf{B}_\rho, t \in [0,T],
\end{aligned}$$

here we used Lemma 2.1(1) and Proposition 2.2(2). Choosing $\|\xi\| \le \delta := \ell\rho\lambda_1^{-1}$, we see that

$$\|\Phi(u)(\cdot,t)\| \le (\ell+\epsilon)\rho\lambda_1^{-1} \le \rho,\ \forall u \in \mathsf{B}_\rho, t \in [0,T],$$

which implies $\Phi(\mathsf{B}_\rho) \subset \mathsf{B}_\rho$. We now prove that Φ is a contraction mapping on B_ρ. For $u_1, u_2 \in \mathsf{B}_\rho$, one gets

$$\begin{aligned}
\|\Phi(u_1)(\cdot,t) - \Phi(u_2)(\cdot,t)\| &\le \int_0^t \omega(t-\tau,\lambda_1)\|f(u_1(\cdot,\tau)) - f(u_2(\cdot,\tau))\|d\tau \\
&\le \kappa(\rho)\int_0^t \omega(t-\tau,\lambda_1)\|u_1(\cdot,\tau) - u_2(\cdot,\tau)\|d\tau \\
&\le (\ell+\epsilon)\|u_1-u_2\|_\infty \int_0^t \omega(t-\tau,\lambda_1)d\tau \\
&\le (\ell+\epsilon)\lambda_1^{-1}(1-\omega(t,\lambda_1))\|u_1-u_2\|_\infty,\ \forall t \in [0,T],
\end{aligned}$$

which ensures that

$$\|\Phi(u_1) - \Phi(u_2)\|_\infty \le (\ell+\epsilon)\lambda_1^{-1}\|u_1-u_2\|_\infty.$$

Hence Φ is a contraction mapping and it admits a fixed point in B_ρ, which is a mild solution to (1.1)-(1.3). In order to testify the uniqueness, we observe that, if $u, v \in C([0,T]; L^2(\Omega))$ are solution of (1.1)-(1.3), then one can assume that $u, v \in \mathsf{B}_R$ for some $R > 0$. So

$$\begin{aligned}
\|u(\cdot,t) - v(\cdot,t)\| &\le \int_0^t \omega(t-\tau,\lambda_1)\kappa(R)\|u(\cdot,\tau) - v(\cdot,\tau)\|d\tau \\
&\le \kappa(R)\int_0^t \|u(\cdot,\tau) - v(\cdot,\tau)\|d\tau,\ \forall t \in [0,T],
\end{aligned}$$

according to the fact that $\omega(t,\lambda_1) \le 1$ for all $t \ge 0$. By using the Gronwall inequality, we get $\|u(\cdot,t) - v(\cdot,t)\| = 0$ for all $t \in [0,T]$, which implies $u = v$. The proof is complete. $\square$

Remark 3.1. *1. If we assume that f satisfies **(F)** without any additional condition, then the problem* (1.1)-(1.3) *has a unique local mild solution, i.e., there exists $t^* \in (0, T]$ such that the problem is uniquely solvable on $[0, t^*]$. This can be proved by standard arguments using the Banach fixed point theorem.*

2. If the nonlinearity function f is global Lipschitzian, i.e. $\kappa(r) = \ell$ is a constant, the assumption $f(0) = 0$ and $\ell \in [0, \lambda_1)$ can be relaxed. In this case, one can prove that Φ is a contraction mapping on $C([0, T]; L^2(\Omega))$ endowed with the norm $\|u\|_\beta = \sup\limits_{t\in[0,T]} e^{-\beta t}\|u(\cdot, t)\|$, where $\beta > 0$ is chosen to be large enough.

The following theorem shows the Hölder regularity of the solution to (1.1)-(1.3).

Theorem 3.2. *Let **(M)** and **(F)** hold. Assume, in addition, that the function m obeys one of the following conditions:*

1. *m is nonincreasing;*

2. *$a = 1 + m$ is 2-regular and θ-sectorial for some $\theta < \pi$.*

Then the mild solution of (1.1)-(1.3) *is Hölder continuous on $(0, T]$.*

Proof. In both cases, the resolvent family $S(\cdot)$ is differentiable on $(0, \infty)$ and there exists $M \geq 1$ such that

$$\|S'(t)\| \leq Mt^{-1}, \text{ for all } t > 0,$$

thanks to Lemma 2.1 and Proposition 2.5.

Let u be the mild solution of (1.1)-(1.3) on $[0, T]$. Then for $t \in (0, T]$ and $h \in (0, T - t)$, we have

$$\begin{aligned}
\|u(\cdot, t + h) - u(\cdot, t)\| &\leq \|[S(t + h) - S(t)]\xi\| \\
&\quad + \int_t^{t+h} \|S(t + h - \tau)f(u(\cdot, \tau))\|d\tau \\
&\quad + \int_0^t \|[S(t + h - \tau) - S(t - \tau)]f(u(\cdot, \tau))\|d\tau \\
&= E_1(t) + E_2(t) + E_3(t).
\end{aligned}$$

Using the mean value formula, we have

$$[S(t + h) - S(t)]\xi = h \int_0^1 S'(t + \zeta h)\xi d\zeta.$$

Then

$$\begin{aligned}
E_1(t) = \|[S(t + h) - S(t)]\xi\| &\leq Mh\|\xi\| \int_0^1 \frac{d\zeta}{t + \zeta h} \\
&= M\|\xi\| \ln\left(1 + \frac{h}{t}\right) \leq M\|\xi\|\beta^{-1}\left(\frac{h}{t}\right)^\beta, \text{ for any } \beta \in (0, 1),
\end{aligned}$$

where we used the inequality $\ln(1+r) \leq \dfrac{r^\beta}{\beta}$ for all $r > 0$, $\beta \in (0,1)$.

Dealing with $E_2(t)$, we put $R = \|u\|_\infty$ and make use of the inequality

$$\|f(u(\cdot,t))\| \leq \kappa(R)\|u(\cdot,t)\| + \|f(0)\| \leq \kappa(R)R + \|f(0)\|.$$

So

$$\begin{aligned} E_2(t) &\leq \int_t^{t+h} \|f(u(\cdot,\tau))\| d\tau \leq [\kappa(R)R + \|f(0)\|]h \\ &\leq [\kappa(R)R + \|f(0)\|]T^{1-\beta}h^\beta. \end{aligned}$$

Regarding $E_3(t)$, we note that

$$[S(t+h-\tau) - S(t-\tau)]f(u(\cdot,\tau)) = h\int_0^1 S'(t-\tau+\zeta h)f(u(\cdot,\tau))d\zeta.$$

Then by the same argument used to estimate $E_1(t)$, we obtain

$$\|[S(t+h-\tau) - S(t-\tau)]f(u(\cdot,\tau))\| \leq M[\kappa(R)R + \|f(0)\|]\beta^{-1}\left(\frac{h}{t-\tau}\right)^\beta.$$

Therefore,

$$\begin{aligned} E_3(t) &\leq M[\kappa(R)R + \|f(0)\|]\beta^{-1}h^\beta \int_0^t \frac{d\tau}{(t-\tau)^\beta} \\ &= M[\kappa(R)R + \|f(0)\|]\beta^{-1}(1-\beta)^{-1}T^{1-\beta}h^\beta. \end{aligned}$$

Summing up, we get

$$E_1(t) + E_2(t) + E_3(t) \leq (C_1 t^{-\beta} + C_2)h^\beta,$$

where

$$\begin{aligned} C_1 &= M\|\xi\|\beta^{-1}, \\ C_2 &= [\kappa(R)R + \|f(0)\|]T^{1-\beta}(1 + M\beta^{-1}(1-\beta)^{-1}). \end{aligned}$$

The proof is complete. □

In the next theorem, we prove the differentiability of the mild solution to (1.1)-(1.3).

Theorem 3.3. *Let (M) and (F) hold. Assume, in addition, that $a = 1 + m$ is 3-regular and θ-sectorial for some $\theta < \pi$. Then the mild solution of* (1.1)-(1.3) *belongs to* $C^1((0,T];L^2(\Omega))$.

Proof. By assumption, the resolvent family $S(\cdot)$ is twice continuously differentiable on $(0, \infty)$ and we get the estimate

$$\|S'(t)\| \leq Mt^{-1},\ \|S''(t)\| \leq Mt^{-2},\ \forall t > 0,$$

with $M \geq 1$, thanks to Corollary 2.1.

Let u be the mild solution of (1.1)-(1.3) and $R = \|u\|_\infty$. Then

$$\begin{aligned} u(\cdot, t) &= S(t)\xi + \int_0^t S(t-\tau) f(u(\cdot, \tau)) d\tau \\ &= u_1(\cdot, t) + u_2(\cdot, t). \end{aligned}$$

Clearly, $u_1 = S(\cdot)\xi \in C^1((0, T]; L^2(\Omega))$. Regarding u_2, we get

$$\partial_t u_2(\cdot, t) = f(u(\cdot, t)) + \int_0^t S'(t-\tau) f(u(\cdot, \tau)) d\tau,$$

where the last term is well-defined by the reasoning as follows. We first observe that

$$\begin{aligned} \int_0^t S'(t-\tau) f(u(\cdot, \tau)) d\tau &= \int_0^t S'(t-\tau)[f(u(\cdot, \tau)) - f(u(\cdot, t))] d\tau \\ &\quad + \int_0^t S'(t-\tau) f(u(\cdot, t)) d\tau \\ &= \int_0^t S'(t-\tau)[f(u(\cdot, \tau)) - f(u(\cdot, t))] d\tau \\ &\quad + [I - S(t)] f(u(\cdot, t)). \end{aligned}$$

In addition, we have

$$\begin{aligned} \int_0^t \|S'(t-\tau)[f(u(\cdot, \tau)) - f(u(\cdot, t))]\| d\tau &\leq M \int_0^t (t-\tau)^{-1} \kappa(R) \|u(\cdot, t) - u(\cdot, \tau)\| d\tau \\ &\leq M\kappa(R) \int_0^t (t-\tau)^{-1} (C_1 \tau^{-\beta} + C_2)(t-\tau)^\beta d\tau \\ &= M\kappa(R) \int_0^t (t-\tau)^{\beta-1} (C_1 \tau^{-\beta} + C_2) d\tau < \infty, \end{aligned}$$

here we utilized the Hölder continuity of u proved in Theorem 3.2. This implies that

$$\int_0^t \|S'(t-\tau) f(u(\cdot, \tau))\| d\tau < \infty,$$

and $\partial_t u_2$ makes sense. It remains to show that $\partial_t u_2$ is continuous with respect to $t \in (0, T]$. Equivalently, we show that the map

$$F(t) = \int_0^t S'(t-\tau) f(u(\cdot, \tau)) d\tau$$

is continuous on $(0,T]$. For $t \in (0,T)$ and $h \in (0, T-t)$, we have

$$
\begin{aligned}
F(t+h) - F(t) &= \int_t^{t+h} S'(t+h-\tau)[f(u(\cdot,\tau)) - f(u(\cdot,t+h))]d\tau \\
&\quad + \int_t^{t+h} S'(t+h-\tau) f(u(\cdot,t+h))d\tau \\
&\quad + \int_0^t [S'(t+h-\tau) - S'(t-\tau)][f(u(\cdot,\tau)) - f(u(\cdot,t))]d\tau \\
&\quad + \int_0^t [S'(t+h-\tau) - S'(t-\tau)] f(u(\cdot,t))d\tau \\
&= F_1(t) + F_2(t) + F_3(t) + F_4(t).
\end{aligned}
$$

It is easily seen that

$$
\begin{aligned}
F_2(t) &= [S(h) - I] f(u(\cdot,t+h)) \to 0 \text{ as } h \to 0, \\
F_4(t) &= [S(t+h) - S(h) - S(t) + I] f(u(\cdot,t)) \to 0 \text{ as } h \to 0.
\end{aligned}
$$

Estimating $F_1(t)$, we see that

$$
\begin{aligned}
\|F_1(t)\| &\le \int_t^{t+h} \|S'(t+h-\tau)[f(u(\cdot,\tau)) - f(u(\cdot,t+h))]\| d\tau \\
&\le M \int_t^{t+h} (t+h-\tau)^{-1} \kappa(R) \|u(\cdot,t+h) - u(\cdot,\tau)\| d\tau \\
&\le M\kappa(R) \int_t^{t+h} (t+h-\tau)^{-1} (C_1 \tau^{-\beta} + C_2)(t+h-\tau)^{\beta} d\tau \\
&\le M\kappa(R)(C_1 t^{-\beta} + C_2) \int_t^{t+h} (t+h-\tau)^{\beta-1} d\tau \\
&\to 0 \text{ as } h \to 0.
\end{aligned}
$$

Regarding $F_3(t)$, using the mean value formula again, we get

$$
\begin{aligned}
\|F_3(t)\| &\le \int_0^t d\tau \int_0^1 h\|S''(t+\zeta h-\tau)[f(u(\cdot,\tau)) - f(u(\cdot,t))]\| d\zeta \\
&\le M \int_0^t d\tau \int_0^1 h(t+\zeta h-\tau)^{-2} \kappa(R) \|u(\cdot,t) - u(\cdot,\tau)\| d\zeta \\
&\le M\kappa(R) \int_0^t [(t-\tau)^{-1} - (t+h-\tau)^{-1}](C_1 \tau^{-\beta} + C_2)(t-\tau)^{\beta} d\tau \\
&= M\kappa(R) \int_0^t Q_h(\tau) d\tau.
\end{aligned}
$$

One observes that $Q_h(\tau) \to 0$ as $h \to 0$ for each $\tau \in (0,t)$. Moreover,

$$\begin{aligned} Q_h(\tau) &\le [(t-\tau)^{-1} + (t+h-\tau)^{-1}](C_1\tau^{-\beta} + C_2)(t-\tau)^{\beta} \\ &\le 2(t-\tau)^{-1}(C_1\tau^{-\beta} + C_2)(t-\tau)^{\beta} \\ &= 2(t-\tau)^{\beta-1}(C_1\tau^{-\beta} + C_2). \end{aligned}$$

Obviously, $Q(\tau) := 2(t-\tau)^{\beta-1}(C_1\tau^{-\beta} + C_2)$ belongs to $L^1(0,t)$. Hence

$$\int_0^t Q_h(\tau)d\tau \to 0 \text{ as } h \to 0,$$

thanks to the Lebesgue dominated convergence theorem. So $F_3(t) \to 0$ as $h \to 0$. The proof is complete. □

The next theorem represents the regularity of the solution with respect to spatial variables.

Theorem 3.4. *Let **(M)** and **(F)** hold. Assume, in addition, that* $1/(1*m) \in L^1(0,T)$. *Then the mild solution of* (1.1)-(1.3) *satisfies* $\Delta u \in C((0,T];L^2(\Omega))$.

Proof. Writing $u = u_1 + u_2$ as in the proof of Theorem 3.3, we see that $\Delta u_1 = \Delta S(\cdot)\xi \in C((0,T];L^2(\Omega))$ due to Lemme 2.1. Considering Δu_2, we have

$$\Delta u_2(\cdot,t) = -\sum_{n=1}^{\infty} \lambda_n \int_0^t \omega(t-\tau,\lambda_n)F_n(\tau)d\tau\varphi_n, \tag{3.1}$$

where $F_n(t) = (f(u(\cdot,t)),\varphi_n)$. It suffices to show that series (3.1) is uniformly convergent on $[0,T]$. Let

$$\delta_n(t) = \lambda_n \int_0^t \omega(t-\tau,\lambda_n)F_n(\tau)d\tau,$$

then using Proposition 2.2(1), we get $\lambda_n\omega(t,\lambda_n) \le (1*m(t))^{-1}$ and

$$|\delta_n(t)|^2 \le \left(\int_0^t \frac{|F_n(\tau)|d\tau}{1*m(t-\tau)}\right)^2 \le \int_0^t \frac{d\tau}{1*m(t-\tau)} \int_0^t \frac{|F_n(\tau)|^2 d\tau}{1*m(t-\tau)},$$

thanks to the Hölder inequality. Since $f(u) \in C([0,T];L^2(\Omega))$, the series $\sum_{n=1}^{\infty} F_n(t)\varphi_n$ is uniformly convergent on $[0,T]$. Then for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sum_{n=N}^{N+p} |F_n(t)|^2 < \epsilon, \ \forall t \in [0,T], N \ge N_\epsilon, p \in \mathbb{N}.$$

So we deduce that

$$\begin{aligned}\sum_{n=N}^{N+p}|\delta_n(t)|^2 &\le \int_0^t \frac{d\tau}{1*m(t-\tau)}\int_0^t \frac{\sum_{n=N}^{N+p}|F_n(\tau)|^2 d\tau}{1*m(t-\tau)}\\ &< \left(\int_0^T \frac{d\tau}{1*m(\tau)}\right)^2 \epsilon,\ \forall t\in[0,T].\end{aligned}$$

Thus series (3.1) converges uniformly to $\int_0^t \Delta S(t-\tau)f(u(\cdot,\tau))d\tau$, which completes the proof. ☐

Theorem 3.5. *Under the assumptions of Theorem 3.3 and 3.4, the mild solution of* (1.1)-(1.3) *satisfies* $m*\Delta u \in C^1((0,T];L^2(\Omega))$*, and consequently, it is a classical solution.*

Proof. It is easily seen that

$$-m*\Delta u(\cdot,t) = \sum_{n=1}^{\infty}\lambda_n m*u_n(t)\varphi_n,$$

where

$$\begin{aligned}u_n(t) &= \omega(t,\lambda_n)\xi_n + \int_0^t \omega(t-\tau,\lambda_n)F_n(\tau)d\tau, t\in[0,T],\\ \xi_n &= (\xi,\varphi_n),\ F_n(t) = (f(u(\cdot,t)),\varphi_n).\end{aligned}$$

Our goal is to prove that the series $\sum\limits_{n=1}^{\infty}\lambda_n(m*u_n)'(t)\varphi_n$ is uniformly convergent on $[\epsilon,T]$ for any $\epsilon\in(0,T)$. One observes that

$$\begin{aligned}\lambda_n(m*u_n)'(t) &= \lambda_n(m*\omega)'(t)\xi_n + \lambda_n(m*\omega*F_n)'(t)\\ &= \lambda_n(m*\omega)'(t)\xi_n + \lambda_n(m*\omega)'*F_n(t)\\ &= -(\omega'(t,\lambda_n)+\lambda_n\omega(t,\lambda_n))\xi_n - (\omega'+\lambda_n\omega)*F_n(t),\end{aligned}$$

due to (2.1). Now we have

- Under the assumption of Theorem 3.2, $S(\cdot)\xi\in C^1((0,T];L^2(\Omega))$. Then the series $\sum\limits_{n=1}^{\infty}\omega'(t,\lambda_n)\xi_n\varphi_n$ converges uniformly to $S'(t)\xi$ on $[\epsilon,T]$.
- By Lemma 2.1(2), $\Delta S(\cdot)\xi\in C((0,T];L^2(\Omega))$. So the series $\sum\limits_{n=1}^{\infty}\lambda_n\omega(t,\lambda_n)\xi_n\varphi_n$ converges uniformly to $-\Delta S(t)\xi$ on $[\epsilon,T]$.

- In the proof of Theorem 3.4, we have proved that the series $\sum_{n=1}^{\infty} \lambda_n(\omega * F_n)(t)\varphi_n$ converges uniformly to $-\int_0^t \Delta S(t-\tau)f(u(\cdot,\tau))d\tau$ on $[0,T]$.
- According to the proof of Theorem 3.3, the function

$$F(t) = \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau$$

is well-defined and continuous on $(0,T]$. It follows that

$$F(t) = \sum_{n=1}^{\infty}(\omega' * F_n)(t)\varphi_n,$$

and then the last series is uniform convergent on $[\epsilon,T]$.

In summary, we have

$$\begin{aligned}-\partial_t(m * \Delta u)(\cdot,t) = -S'(t)\xi - \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau \\ + \Delta S(t)\xi + \int_0^t \Delta S(t-\tau)f(u(\cdot,\tau))d\tau.\end{aligned}$$

Noting that

$$\partial_t u(\cdot,t) = S'(t)\xi + f(u(\cdot,t)) + \int_0^t S'(t-\tau)f(u(\cdot,\tau))d\tau,$$

as pointed out in the proof of Theorem 3.4, we get

$$-\partial_t(m * \Delta u)(\cdot,t) = -\partial_t u(\cdot,t) + f(u(\cdot,t)) + \Delta u(\cdot,t),$$

which means that u obeys (1.1) in the classical sense. The proof is complete. $\square$

We end this section by testing our assumptions in a specific circumstance. Let

$$\begin{aligned}m(t) &= g_{1-\alpha}(t) + \mu g_{1-\beta}(t),\ 0<\alpha<\beta<1,\ \mu \geq 0\\ f(v)(x) &= F\left(\int_\Omega |v(x)|^2 dx\right) g(x,v(x)),\end{aligned}$$

where F and g satisfy the following conditions:

(N1) The function $F \in C^1(\mathbb{R}^+)$ is such that $|F(r)| \leq a + br^p$ for some $a,b>0, p\geq 0$.

(N2) The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(\cdot,0)=0$ and the following Lipschitz condition holds

$$|g(x,q_1) - g(x,q_2)| \leq h(x)|q_1 - q_2|,\ \forall q_1, q_2 \in \mathbb{R},$$

with $h \in L^\infty(\Omega)$ being a nonnegative function.

We will testify that f is locally Lipschitzian. Let $v_1, v_2 \in L^2(\Omega)$, $\|v_1\|, \|v_2\| \leq r$, then

$$\begin{aligned}
&\|f(v_1) - f(v_2)\|^2 = \int_\Omega [F(\|v_1\|^2)g(x, v_1(x)) - F(\|v_2\|^2)g(x, v_2(x))]^2 dx \\
&\leq 2|F(\|v_1\|^2)|^2 \int_\Omega |h(x)|^2 |v_1(x) - v_2(x)|^2 dx \\
&\quad + 2|F(\|v_1\|^2) - F(\|v_2\|^2)|^2 \int_\Omega |g(x, v_2(x))|^2 dx.
\end{aligned}$$

Let $\|h\|_\infty = \operatorname{esssup}_{x\in\Omega}|h(x)|$. Then

$$\begin{aligned}
&\|f(v_1) - f(v_2)\|^2 \leq 2(a + br^{2p})^2 \|h\|_\infty^2 \|v_1 - v_2\|^2 \\
&\quad + 2\|h\|_\infty^2 \|v_2\|^2 (\|v_1\|^2 - \|v_2\|^2)^2 \left(\int_0^1 F'(\|v_1\|^2 + t(\|v_2\|^2 - \|v_1\|^2)) dt \right)^2 \\
&\leq 2(a + br^{2p})^2 \|h\|_\infty^2 \|v_1 - v_2\|^2 + 8\|h\|_\infty^2 r^4 \|v_1 - v_2\|^2 (\sup_{0\leq\rho\leq 2r^2} |F'(\rho)|)^2 \\
&\leq \kappa(r)^2 \|v_1 - v_2\|^2,
\end{aligned}$$

where

$$\kappa(r) = \sqrt{2}(a + br^{2p})\|h\|_\infty + 2\sqrt{2}\|h\|_\infty r^2 \sup_{0\leq\rho\leq 2r^2} |F'(\rho)|.$$

Observing that

$$1 * m(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\mu t^{1-\beta}}{\Gamma(2-\beta)},$$

we get $1/(1 * m) \in L^1(0, T)$. Let $a(t) = 1 + m(t), t > 0$. Then a is completely positive, since m is completely monotone. In addition, we see that

$$\hat{a}(z) = z^{-1} + \hat{m}(z) = z^{-1} + z^{\alpha-1} + \mu z^{\beta-1}.$$

So for $\operatorname{Re} z > 0$, $|\arg \hat{a}(z)| < \frac{\pi}{2}$, i.e. a is $\frac{\pi}{2}$-sectorial. One can check that a is k-regular for any $k \in \mathbb{N}$ by using the fact that, for any $p_1, p_2, p_3 > 0$, there exists $c > 0$ such that

$$|p_1 z^{-1} + p_2 z^{\alpha-1} + p_3 \mu z^{\beta-1}| \leq c|z^{-1} + z^{\alpha-1} + \mu z^{\beta-1}|, \ \forall \operatorname{Re} z > 0,$$

which implies $|z^k \hat{a}^{(k)}(z)| \leq c|\hat{a}(z)|$ for all $\operatorname{Re} z > 0$.

4. Convergence to equilibrium

This section is devoted to considering the long-time behavior of the solution to (1.1)-(1.3). We first have the following theorem.

Theorem 4.1. *Let **(M)** and **(F)** hold and u be a global mild solution of* (1.1)-(1.3). *Assume that $\partial\Omega \in C^2$ and $\mathcal{B} = I$. If there exists the limit $\lim_{t\to\infty} u(\cdot, t) = u^*$ in $L^2(\Omega)$, and $\lim_{z\to 0} z\hat{m}(z) = 0$ in $\mathbb{C}$, then u^* is a strong solution of the elliptic problem*

$$\begin{aligned}
-\Delta w &= f(w) \ \textit{ in } \Omega, \\
w &= 0 \ \textit{ on } \partial\Omega.
\end{aligned}$$

Proof. By formulation, we have

$$\begin{aligned} u(\cdot,t) &= S(t)\xi + \int_0^t S(t-\tau)f(u(\cdot,\tau))d\tau \\ &= S(t)\xi + \int_0^t S(t-\tau)[f(u(\cdot,\tau)) - f(u^*)]d\tau + \int_0^t S(t-\tau)f(u^*)d\tau \\ &= S(t)\xi + G_1(t) + G_2(t). \end{aligned}$$

Obviously, $\|S(t)\xi\| \le \omega(t,\lambda)\|\xi\| \to 0$ as $t \to \infty$. We will show that $\lim_{t\to\infty} G_1(t) = 0$. By assumption, for $\epsilon > 0$, there exists $T_1 > 0$ such that $\|f(u(\cdot,t)) - f(u^*)\| < \epsilon$ for every $t \ge T_1$. So for $t > T_1$, we have

$$\begin{aligned} \|G_1(t)\| &\le \int_0^{T_1} \|S(t-\tau)[f(u(\cdot,\tau)) - f(u^*)]\|d\tau \\ &\quad + \int_{T_1}^t \|S(t-\tau)[f(u(\cdot,\tau)) - f(u^*)]\|d\tau \\ &\le \int_0^{T_1} \omega(t-\tau,\lambda_1)\kappa(R)\|u(\cdot,\tau) - u^*\|d\tau + \epsilon\int_{T_1}^t \omega(t-\tau,\lambda_1)d\tau \\ &\le 2R\kappa(R)\int_0^{T_1} \omega(t-\tau,\lambda_1)d\tau + \epsilon\int_0^t \omega(\tau,\lambda_1)d\tau \\ &\le 2R\kappa(R)\int_{t-T_1}^t \omega(\tau,\lambda_1)d\tau + \epsilon\lambda_1^{-1}, \end{aligned}$$

thanks to Proposition 2.2(2), where $R = \|u^*\| + \|u\|_\infty$. Since $\omega \in L^1(\mathbb{R}^+)$, there exists $T_2 > 0$ such that

$$\int_{T_2}^t \omega(\tau,\lambda_1)d\tau \le \epsilon, \quad \text{for all } t \ge T_2.$$

Owing to the last estimate, we obtain

$$\|G_1(t)\| \le [2R\kappa(R) + \lambda_1^{-1}]\epsilon, \quad \text{for all } t \ge T_1 + T_2,$$

which ensures $G_1(t) \to 0$ as $t \to \infty$. Therefore,

$$u^* = \lim_{t\to\infty} u(\cdot,t) = \lim_{t\to\infty} G_2(t) = \int_0^\infty S(\tau)f(u^*)d\tau = \hat{S}(0)f(u^*).$$

On the other hand, concerning the Laplace transform of $S(\cdot)$, it follows from (2.20) that

$$\begin{aligned} \hat{S}(z) &= z^{-1}(I - \hat{a}(z)\Delta)^{-1} = z^{-1}\hat{a}(z)^{-1}\Big(\frac{1}{\hat{a}(z)} - \Delta\Big)^{-1} \\ &= (1 + z\hat{m}(z))\Big(\frac{z}{1+z\hat{m}(z)} - \Delta\Big)^{-1}. \end{aligned}$$

Hence, $u^* = \int_0^\infty S(\tau)f(u^*)d\tau = \hat{S}(0)f(u^*) = (-\Delta)^{-1}f(u^*)$. Since $\partial\Omega \in C^2$, the regularity result in [14, Sect. 6.3.2] guarantees that $u^* \in H^2(\Omega)$ and therefore, $-\Delta u^* = f(u^*)$ a.e. in Ω. The proof is complete. $\square$

In the rest of this section, we use a stronger assumption imposed on m.

(M*) *The function $m \in L^1_{loc}(\mathbb{R}^+)$ is smooth and positive on $(0,\infty)$ such that*

- *1)* $\log m$ *is a convex function on* $(0,\infty)$ *and* $\lim\limits_{t\to\infty} m(t) = 0$*;*
- *2)* $a = 1 + m$ *is 3-regular and θ-sectorial for some $\theta < \pi$;*
- *3)* $1/(1 * m) \in L^1_{loc}(\mathbb{R}^+)$.

A typical example is

$$m(t) = \sum_{i=1}^{p} \mu_i g_{1-\alpha_i}(t),$$

where $\alpha_i \in (0,1), \mu_i > 0$, for which equation (1.1) reads

$$\partial_t u - \Delta u - \sum_{i=1}^{p} \mu_i \partial_t^{\alpha_i} \Delta u = f(u).$$

The last equation is in the form of Rayleigh-Stokes [1] with multi-term fractional derivative.

As mentioned in the second section, owing to **(M*)**, the function $a = 1 + \gamma m$ is completely positive for any $\gamma > 0$. Let $\omega(\cdot, \lambda, \gamma)$ be the solution of the equation

$$\omega(t) + \lambda \int_0^t (1 + \gamma m(t-\tau))\omega(\tau)d\tau = 1, t \geq 0;\ \lambda, \gamma > 0.$$

Then $\omega(\cdot, \lambda, \gamma)$ possesses all properties stated in Proposition 2.2 with γm in place of m. In addition, the solution of the equation

$$z'(t) + \lambda z(t) + \lambda\gamma(m * z)'(t) = g(t),\ t > 0, \tag{4.1}$$

is given by

$$z(t) = \omega(t, \lambda, \gamma)z(0) + \int_0^t \omega(t-\tau, \lambda, \gamma)g(\tau)d\tau. \tag{4.2}$$

The following Gronwall type inequality will be used in the sequel.

Proposition 4.1. *Let z be a nonnegative function obeying the inequality*

$$z(t) \leq \omega(t, \lambda, \gamma)z_0 + \int_0^t \omega(t-\tau, \lambda, \gamma)[az(\tau) + b(\tau)]d\tau,\ t \geq 0, \tag{4.3}$$

where $a \in [0, \lambda)$, $\gamma > 0$, $b \in L^1_{loc}(\mathbb{R}^+)$. Then

$$z(t) \leq \omega\Big(t, \lambda - a, \frac{\lambda\gamma}{\lambda - a}\Big)z_0 + \int_0^t \omega\Big(t-\tau, \lambda - a, \frac{\lambda\gamma}{\lambda - a}\Big)b(\tau)d\tau.$$

Proof. Let $y(t)$ be the right hand side of (4.3). Then $z(t) \le y(t)$ and y solves the equation

$$y'(t) + \lambda y + \lambda\gamma(m * y)' = az(t) + b(t),\ t > 0, y(0) = z_0.$$

It follows that

$$y'(t) + (\lambda - a)y + (\lambda - a)\frac{\lambda\gamma}{\lambda - a}(m * y)' = a[z(t) - y(t)] + b(t),\ t > 0, y(0) = z_0,$$

and then y admits the representation

$$\begin{aligned} y(t) &= \omega\Big(t, \lambda - a, \frac{\lambda\gamma}{\lambda - a}\Big)z_0 \\ &\quad + \int_0^t \omega\Big(t - \tau, \lambda - a, \frac{\lambda\gamma}{\lambda - a}\Big)\Big(a[z(\tau) - y(\tau)] + b(\tau)\Big)d\tau \\ &\le \omega\Big(t, \lambda - a, \frac{\lambda\gamma}{\lambda - a}\Big)z_0 + \int_0^t \omega\Big(t - \tau, \lambda - a, \frac{\lambda\gamma}{\lambda - a}\Big)b(\tau)d\tau, \end{aligned}$$

thanks to the positivity of ω and the fact that $z(\tau) - y(\tau) \le 0$ for $\tau \ge 0$. So we get the conclusion as desired. ☐

The following theorem represents the main result of this section.

Theorem 4.2. *Let (M*) hold and f satisfy the global Lipschitz condition with constant $\kappa_0 \in [0, \lambda_1)$. Assume that $\partial\Omega \in C^2$ and $\mathcal{B} = I$. Then the solution of* (1.1)-(1.3) *converges to the unique strong solution of the elliptic problem*

$$-\Delta w = f(w) \ \textit{ in } \Omega, \tag{4.4}$$

$$w = 0 \ \textit{ on } \partial\Omega. \tag{4.5}$$

Proof. We first show that if f is Lipschitzian with constant $\kappa_0 < \lambda_1$, the problem (4.4)-(4.5) has a unique strong solution. Indeed, due to [15, Theorem 7.4.1], the problem (4.4)-(4.5) has a unique weak solution $u^* \in H_0^1(\Omega)$ if the Lipschitz constant κ_0 satisfies $\kappa_0 < C_0^{-2}$, where C_0 is the constant of embedding $H_0^1(\Omega) \subset L^2(\Omega)$. By the smoothness of $\partial\Omega$, we have

$$C_0^{-2} = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2}{\|u\|^2} = \lambda_1.$$

Observe that $f(u^*) \in L^2(\Omega)$. Then using the regularity result in [14, Sect. 6.3.2] again, we obtain $u^* \in H^2(\Omega)$ and hence u^* is a unique strong solution of (4.4)-(4.5).

By Theorem 3.5, the solution of (1.1)-(1.3) is classical. Then combining (1.1) with (4.4), one gets

$$\partial_t u - \Delta(u - u^*) - \partial_t(m * \Delta u) = f(u(\cdot, t)) - f(u^*),$$

or equivalently,

$$\partial_t(u-u^*)-\Delta(u-u^*)-\partial_t(m*\Delta(u-u^*))=f(u(\cdot,t))-f(u^*)+m(t)\Delta u^*,$$

which leads to the representation

$$u(\cdot,t)-u^*=S(t)(\xi-u^*)+\int_0^t S(t-\tau)[f(u(\cdot,\tau))-f(u^*)+m(\tau)\Delta u^*]d\tau.$$

So we have the following estimate

$$\begin{aligned}\|u(\cdot,t)-u^*\| &\le \omega(t,\lambda_1,1)\|\xi-u^*\|\\ &\quad+\int_0^t \omega(t-\tau,\lambda_1,1)[\kappa_0\|u(\cdot,\tau)-u^*\|+m(\tau)\|\Delta u^*\|]d\tau.\end{aligned}$$

Applying Proposition 4.1 yields

$$\begin{aligned}\|u(\cdot,t)-u^*\| &\le \omega\Big(t,\lambda_1-\kappa_0,\frac{\lambda_1}{\lambda_1-\kappa_0}\Big)\|\xi-u^*\|\\ &\quad+\int_0^t \omega\Big(t-\tau,\lambda_1-\kappa_0,\frac{\lambda_1}{\lambda_1-\kappa_0}\Big)m(\tau)\|\Delta u^*\|d\tau\\ &= U_1(t)+U_2(t).\end{aligned}$$

It is evident that $U_1(t)=\omega\Big(t,\lambda_1-\kappa_0,\frac{\lambda_1}{\lambda_1-\kappa_0}\Big)\|\xi-u^*\|\to 0$ as $t\to\infty$. Due to the assumption that $m(t)\to 0$ as $t\to\infty$, for any $\epsilon>0$, there exists $T>0$ such that $m(t)<\epsilon$ for all $t\ge T$. So for any $t>2T$, we have

$$\begin{aligned}U_2(t) &= \Big(\int_T^t+\int_0^T\Big)\,\omega\Big(t-\tau,\lambda_1-\kappa_0,\frac{\lambda_1}{\lambda_1-\kappa_0}\Big)m(\tau)\|\Delta u^*\|d\tau\\ &\le \epsilon\|\Delta u^*\|\int_T^t \omega\Big(t-\tau,\lambda_1-\kappa_0,\frac{\lambda_1}{\lambda_1-\kappa_0}\Big)d\tau\\ &\quad+\int_0^T \frac{m(\tau)\|\Delta u^*\|d\tau}{1+(\lambda_1-\kappa_0)\int_0^{t-\tau}[1+\frac{\lambda_1}{\lambda_1-\kappa_0}m(s)]ds},\end{aligned}$$

thanks to Proposition 2.2(1). Then utilizing Proposition 2.2(2), we get

$$\begin{aligned}U_2(t) &\le \epsilon\|\Delta u^*\|\int_0^t \omega\Big(t-\tau,\lambda_1-\kappa_0,\frac{\lambda_1}{\lambda_1-\kappa_0}\Big)d\tau\\ &\quad+\frac{\|\Delta u^*\|(\lambda_1-\kappa_0)^{-1}}{\int_0^{t-T}[1+\frac{\lambda_1}{\lambda_1-\kappa_0}m(s)]ds}\int_0^T m(\tau)d\tau\\ &\le \epsilon\|\Delta u^*\|(\lambda_1-\kappa_0)^{-1}+\frac{\|\Delta u^*\|(\lambda_1-\kappa_0)^{-1}}{t-T+\frac{\lambda_1}{\lambda_1-\kappa_0}\int_0^{t-T}m(s)ds}\int_0^T m(\tau)d\tau\\ &\le \epsilon\|\Delta u^*\|(\lambda_1-\kappa_0)^{-1}+\frac{\|\Delta u^*\|(\lambda_1-\kappa_0)^{-1}}{t-T+\frac{\lambda_1}{\lambda_1-\kappa_0}\int_0^{T}m(\tau)d\tau}\int_0^T m(\tau)d\tau\\ &\le 2\epsilon\|\Delta u^*\|(\lambda_1-\kappa_0)^{-1},\end{aligned}$$

for t large enough such that

$$\frac{\int_0^T m(\tau)d\tau}{t - T + \frac{\lambda_1}{\lambda_1 - \kappa_0}\int_0^T m(\tau)d\tau} < \epsilon.$$

We have proved that $U_2(t) \to 0$ as $t \to \infty$, which implies $\lim_{t\to\infty} \|u(\cdot, t) - u^*\| = 0$. The proof is complete. $\square$

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