HNUE JOURNAL OF SCIENCE DOI: 10.18173/2354-1059.2021-0026
Natural Science, 2021, Volume 66, Issue 2, pp. 37-47
This paper is available online at http://stdb.hnue.edu.vn

GLOBAL ATTRACTORS FOR A CLASS OF SEMILINEAR PARABOLIC EQUATIONS INVOLVING GRUSHIN OPERATOR AND NOLINEARITIES OF ARBITRARY ORDER

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Abstract. We study the existence and uniqueness of weak solutions and the existence of global attractors to a class of semilinear parabolic equations involving the Grushin operator and nonlinearities of arbitrary order. The main novelty of our result is that no restriction on the upper growth of the nonlinearities is imposed.

Keywords: degenerate parabolic equation, Grushin operator, exponential nonlinearity, weak solution, global attractor.

1. Introduction

In recent years, a number of papers has been devoted to the study of existence and asymptotic behavior of solutions to degenerate parabolic equations. In this paper we consider the following semilinear parabolic equation involving an operator of Grushin type

$$\begin{cases} \dfrac{\partial u}{\partial t} - G_s u + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, the nonlinearity f and the external force g satisfy some conditions specified later. The Grushin operator G_s was first introduced in [1], is defined by

$$G_s u = \Delta_{x_1} u + |x_1|^{2s} \Delta_{x_2} u, \ \ (x_1, x_2) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, s \geq 0.$$

Noting that $G_0 = \Delta$ and G_s, when $s > 0$, is not elliptic in domains $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ intersecting with the hyperplane $\{x_1 = 0\}$. The local properties of G_s were investigated in [1, 2].

Received April 2, 2021. Revised June 8, 2021. Accepted June 15, 2021.
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To study problem (1.1) we have usually used the natural energy space $S_0^1(\Omega)$ defined as the completion of $C_0^1(\overline{\Omega})$ in the following norm

$$\|u\|^2_{S_0^1(\Omega)} := \int_\Omega (|\nabla_{x_1} u|^2 + |x_1|^{2s}|\nabla_{x_2} u|^2)dx.$$

We have the continuous embedding $S_0^1(\Omega) \hookrightarrow L^r(\Omega)$, for $1 \le r \le 2_s^* = \dfrac{2N(s)}{N(s)-2}$, where $N(s) = N_1 + (s+1)N_2$. Moreover, this embedding is compact if $1 \le r < 2_s^*$ (see [3]).

In [4], the authors considered problem (1.1) with $f : \mathbb{R} \to \mathbb{R}$ being locally Lipschitz continuous and satisfying a Sobolev growth condition

$$|f(u) - f(v)| \le C_0|u - v|(1 + |u|^\gamma + |v|^\gamma)\ , 0 \le \gamma < \frac{4-2\alpha}{N-2+\alpha},$$

and some dissipativity conditions. Under the above assumptions of f, the authors proved that problem (1.1) defines a semigroup $S(t) : S_0^1(\Omega) \to S_0^1(\Omega)$, which possesses a compact global attractor in the space $S_0^1(\Omega)$.

When the nonlinearity f is supposed to satisfy a growth and dissipativity condition of polynomial type, that is,

$$C_1|u|^p - C_0 \le f(u)u \le C_2|u|^p + C_0, \quad \text{for some } p \ge 2,$$
$$f'(u) \ge -C_3, \quad \text{for all } u \in \mathbb{R},$$

the existence of a global attractor in $L^2(\Omega)$ in the autonomous case [5] and the existence of an uniform attractor in $L^2(\Omega)$, $L^p(\Omega)$ and $S_0^1(\Omega)$ in the non-autonomous case [6].

Note that for both above classes of nonlinearities, some restriction on the growth of the nonlinearity is imposed and an exponential nonlinearity, for example $f(u) = e^u$, does not hold. In this paper we try to remove this restriction and we were able to prove the existence of a weak solution and of a global attractor for a very large class of nonlinearities that particularly covers both the above classes and even exponential nonlinearities. This is the main novelty of our paper.

In this paper we assume that the initial datum $u_0 \in L^2(\Omega)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'(u) \ge -\ell, \tag{1.2}$$

$$f(u)u \ge -\mu u^2 - C_1, \tag{1.3}$$

where C_1 and ℓ are two positive constants, $0 < \mu < \lambda_1$ with $\lambda_1 > 0$ is the first eigenvalue of the operator $Au = -G_s u$ in Ω, and $F(u) = \int_0^u f(s)ds$ is a primitive of f;

(G) $g \in L^2(\Omega)$.

It follows from (1.2) that $0 \leq \int_0^u (f'(s)s+\ell s)ds$, and so by integrating by parts, we obtain

$$F(u) = f(u)u + \frac{\ell u^2}{2} - \int_0^u (sf'(s) + \ell s)ds \leq f(u)u + \frac{\ell u^2}{2} \quad \text{for all } u \in \mathbb{R}.$$

Thus,

$$F(u) \leq f(u)u + \frac{\ell u^2}{2} \quad \text{for all } u \in \mathbb{R}.$$

Using (1.2) we have

$$F(u) = \int_0^u ((f(s) - f(0)) + f(0))ds = \int_0^u (f'(c)s + f(0))ds \geq -\frac{\ell u^2}{2} + f(0)u.$$

Therefore,

$$F(u) \geq -\frac{\ell u^2}{2} + f(0)u, \quad \forall u \in \mathbb{R}. \tag{1.4}$$

It is noticed that the class of nonlinearities satisfying **(F)** is very large in the sense that no upper bound on the growth of nonlinearity is imposed, besides the standard dissipative condition (1.3) and the well-known condition (1.2) ensuring the uniqueness of solutions. In particular, this class contains all nonlinearities of Sobolev type and polynomial type, and even exponential nonlinearities.

The paper is organized as follows: In Section 2, we prove the existence and uniqueness of weak solutions by utilizing the compactness method and weak convergence techniques in Orlicz spaces [7]. In Section 3, we prove the existence of global attractors for the semigroup generated by the problem in various spaces. The main novelty of the paper is that the nonlinearity can grow arbitrarily fast, and in particular, the results obtained here extend previous ones in [8, 9, 10, 11].

2. Existence and uniqueness of a weak solution

Definition 2.1. *A function u is called a weak solution of problem* (1.1) *on* $(0,T)$ *if* $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; S_0^1(\Omega))$, $f(u) \in L^1(Q_T)$, $u(0) = u_0$, *and*

$$\int_\Omega \frac{\partial u}{\partial t} w dx + \int_\Omega (\nabla_{x_1} u \nabla_{x_1} w + |x_1|^{2s} \nabla_{x_2} u \nabla_{x_2} w)dx + \int_\Omega f(u)w dx = \int_\Omega gw dx \tag{2.1}$$

for all test functions $w \in W := S_0^1(\Omega) \cap L^\infty(\Omega)$ *and for a.e.* $t \in (0,T)$.

Theorem 2.1. *Assume* **(F)–(G)** *hold. Then for any* $u_0 \in L^2(\Omega)$ *and* $T > 0$ *given, problem* (1.1) *has a unique weak solution* u *on the interval* $(0,T)$. *Moreover, the weak solution* u *depends continuously on the initial data in* $L^2(\Omega)$.

Proof. i) Existence. We will prove the existence of a weak solution by using the compactness method. To overcome the essential difficulty due to no restriction on the upper bound of the nonlinearity is imposed, so the nonlinear term $f(u)$ only belongs to $L^1(\Omega)$, we will exploit the weak convergence techniques in Orlicz spaces introduced in [7].

Let $\{e_j\}_{j=1}^{\infty}$ be a basis of $S_0^1(\Omega)$ consisting of eigenvectors of the operator $Au = -G_s u$ in Ω with the homogeneous Dirichlet boundary condition, that is orthonormal in $L^2(\Omega)$. We look for an approximate solution $u_n(t)$ of the form

$$u_n(t) = \sum_{j=1}^{n} u_{nj}(t) e_j$$

that solves the following problem

$$\begin{cases} \left\langle \dfrac{\partial u_n}{\partial t}, e_j \right\rangle + \langle A u_n, e_j \rangle + \langle\, f(u_n), e_j \rangle = (g, e_j), \\ (u_n(0), e_j) = (u_0, e_j),\ j = 1, \ldots, n. \end{cases}$$

This is a system of first-order ordinary differential equations for the functions $u_{n1}, u_{n2}, \ldots, u_{nn}$

$$\begin{cases} u'_{nj} + \lambda_j u_{nj} + \langle f(u_n), e_j \rangle = (g, e_j),\ j = 1, \ldots, n \\ u_{nj}(0) = (u_0, e_j). \end{cases}$$

By the theory of ODEs, we obtain the existence of approximate solutions $u_n(t)$.

We now establish some *a priori* estimates for u_n. Multiplying the first equation in (1.1) by $u_{nj}(t)$, then summing from 1 to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^2_{S_0^1(\Omega)} + \int_\Omega f(u_n) u_n dx = \int_\Omega g u_n dx. \tag{2.2}$$

Hence using (1.3) and the Cauchy inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \|u_n(t)\|^2_{S_0^1(\Omega)} - \mu\|u_n(t)\|^2_{L^2(\Omega)} - C_1|\Omega| \\ \le\ &\frac{1}{2(\lambda_1 - \mu)}\|g\|^2_{L^2(\Omega)} + \frac{\lambda_1 - \mu}{2}\|u_n(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + 2\|u_n(t)\|^2_{S_0^1(\Omega)} - 2\mu\|u_n(t)\|^2_{L^2(\Omega)} - (\lambda_1 - \mu)\|u_n(t)\|^2_{L^2(\Omega)} \\ \le\ &\frac{1}{\lambda_1 - \mu}\|g\|^2_{L^2(\Omega)} + 2C_1|\Omega|. \end{aligned} \tag{2.3}$$

Since $\|u\|^2_{S^1_0(\Omega)} \geq \lambda_1 \|u\|^2_{L^2(\Omega)}$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $Au = -G_s u$, we get

$$\frac{d}{dt}\|u_n(t)\|^2 + (\lambda_1 - \mu)\|u_n(t)\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda_1 - \mu}\|g\|^2_{L^2(\Omega)} + 2C_1|\Omega|.$$

By the Gronwall inequality, we obtain

$$\|u_n(t)\|^2_{L^2(\Omega)} \leq C = C(\|u_0\|_{L^2(\Omega)}, \|g\|_{L^2(\Omega)}, \lambda_1, \mu, |\Omega|, C_1, T), \text{ for all } t \in [0,T]. \quad (2.4)$$

Integrating (2.3) from 0 to $t, 0 \leq t \leq T$, and using (2.4), we arrive at

$$\|u_n(t)\|^2 + \int_0^t \|u_n(s)\|^2_{S^1_0(\Omega)} ds \leq C, \text{ for all } t \in [0,T].$$

This inequality yields

$$\begin{aligned}
&\{u_n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\
&\{u_n\} \text{ is bounded in } L^2(0,T;S^1_0(\Omega)).
\end{aligned}$$

Using the boundedness of $\{u_n\}$ in $L^2(0,T;S^1_0(\Omega))$, it is easy to check that $\{Au_n\}$ is bounded in $L^2(0,T;S^{-1}(\Omega)$, where $S^{-1}(\Omega)$ is the dual space of $S^1_0(\Omega)$. From the above results, we can assume that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } L^2(0,T;S^1_0(\Omega)),\\
&u_n \rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega)),\\
&Au_n \rightharpoonup Au \text{ in } L^2(0,T;S^{-1}(\Omega)).
\end{aligned}$$

On the other hand, integrating (2.2) from 0 to T, using the Cauchy inquality and $\|u\|^2_{S^1_0(\Omega)} \geq \lambda_1 \|u\|^2_{L^2(\Omega)}$, we have

$$\int_0^T \|u_n\|^2_{S^1_0(\Omega)} dt + 2\int_{Q_T} f(u_n)u_n dxdt \leq \|u_0\|^2_{L^2(\Omega)} + \frac{1}{\lambda_1}\|g\|^2_{L^2(\Omega)} T.$$

Hence

$$\int_{Q_T} f(u_n)u_n dxdt \leq C.$$

We now prove that $\{f(u_n)\}$ is bounded in $L^1(Q_T)$. Putting $h(s) = f(s) - f(0) + \kappa s$, where $\kappa > \ell$. Note that $h(s)s = (f(s) - f(0))s + \kappa s^2 = f'(c)s^2 + \kappa s^2 \geq (\kappa - \ell)s^2 \geq 0$

for all $s \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{Q_T} |h(u_n)|dxdt &\leq \int_{Q_T\cap\{|u_n|>1\}} |h(u_n)u_n|dxdt + \int_{Q_T\cap\{|u_n|\leq 1\}} |h(u_n)|dxdt \\
&\leq \int_{Q_T} h(u_n)u_n dxdt + \sup_{|s|\leq 1} |h(s)||Q_T| \\
&\leq \int_{Q_T} f(u_n)u_n dxdt + \kappa\|u_n(t)\|^2_{L^2(Q_T)} + |f(0)|\|u_n(t)\|_{L^1(Q_T)} \\
&\quad + \sup_{|s|\leq 1} |h(s)||Q_T| \\
&\leq C.
\end{aligned}$$

Hence it implies that $\{h(u_n)\}$, and therefore $\{f(u_n)\}$ is bounded in $L^1(Q_T)$. Since

$$\frac{du_n}{dt} = -Au_n - f(u_n) + g,$$

we deduce that $\{\frac{du_n}{dt}\}$ is bounded in $L^2(0,T;S^{-1}(\Omega)) + L^1(Q_T)$, and therefore in $L^1(0,T;S^{-1}(\Omega) + L^1(\Omega))$. Because $S_0^1(\Omega) \subset\subset L^2(\Omega) \subset S^{-1}(\Omega) + L^1(\Omega))$, by the Aubin-Lions-Simon compactness lemma (see e.g. [12, Theorem II.5.16, p. 102]), we have that $\{u_n\}$ is compact in $L^2(0,T;L^2(\Omega))$. Hence we may assume, up to a subsequence, that $u_n \to u$ a.e. in Q_T. Applying Lemma 6.1 in [13], we obtain that $h(u) \in L^1(Q_T)$ and for all test function $\xi \in C_0^\infty([0,T];S_0^1(\Omega)\cap L^\infty(\Omega))$,

$$\int_{Q_T} h(u_n)\xi dxdt \to \int_{Q_T} h(u)\xi dxdt.$$

Hence $f(u) \in L^1(Q_T)$ and

$$\int_{Q_T} f(u_n)\xi dxdt \to \int_{Q_T} f(u)\xi dxdt, \text{ for all } \xi \in C_0^\infty([0,T];S_0^1(\Omega)\cap L^\infty(\Omega)).$$

Thus, u satisfies equality (2.1).

It remains to be shown that $u(0) = u_0$. To do this, we choose test functions $\varphi \in C^1([0,T];S_0^1(\Omega)\cap L^\infty(\Omega))$ with $\varphi(T) = 0$. Integrating by parts in the t variable, we have

$$\int_0^T -(u,\varphi')dt + \int_0^T ((u,\varphi))_{S_0^1(\Omega)}dt + \int_{Q_T} (f(u) - g)\,\varphi dxdt = (u(0),\varphi(0)).$$

Doing the same in the Galerkin approximations yields

$$\begin{aligned}
\int_0^T -(u_n,\varphi')dt + \int_0^T ((u_n,\varphi))_{S_0^1(\Omega)}dt + \int_{Q_T} (f(u_n) - g)\,\varphi dxdt \\
= (u_n(0),\varphi(0)).
\end{aligned}$$

Taking limits as $n \to \infty$ we arrive at

$$\int_0^T -(u, \varphi')dt + \int_0^T ((u,\varphi))_{S_0^1(\Omega)}dt + \int_{Q_T} (f(u) - g)\,\varphi dxdt = (u_0, \varphi(0))$$

since $u_n(0) \to u_0$. Thus, $u(0) = u_0$ and this implies that u is a weak solution to problem (1.1).

ii) Uniqueness and continuous dependence on the initial data. Let u and v be two weak solutions of (1.1) with initial data $u_0, v_0 \in L^2(\Omega)$. Putting $w = u - v$, we have

$$\begin{cases} \dfrac{dw}{dt} + Aw + \tilde{f}(u) - \tilde{f}(v) - \ell w = 0 \\ w(0) = u_0 - v_0, \end{cases} \tag{2.5}$$

where $\tilde{f}(s) = f(s) + \ell s$. Here because $w(t)$ does not belong to $W := S_0^1(\Omega) \cap L^\infty(\Omega)$, we cannot choose $w(t)$ as a test function as in [8]. Consequently, the proof will be more involved.

We use some ideas in [7]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function

$$B_k(s) = \begin{cases} k & \text{if } s > k \\ s & \text{if } |s| \le k \\ -k & \text{if } s < -k. \end{cases}$$

Consider the corresponding Nemytskii mapping $\hat{B}_k : W \to W$ defined as follows:

$$\hat{B}_k(w)(x) = B_k(w(x)), \quad \text{for all } x \in \Omega.$$

By Lemma 2.3 in [7], we have that $\|\hat{B}_k(w) - w\|_W \to 0$ as $k \to \infty$. Now multiplying the first equation in (2.5) by $\hat{B}_k(w)$, then integrating over $\Omega \times (\varepsilon, t)$, where $t \in (0,T)$, we get

$$\begin{aligned} &\int_\varepsilon^t \int_\Omega \frac{d}{ds}(w(s)\hat{B}_k(w(s))dxds - \int_\varepsilon^t \int_\Omega w \frac{d}{ds}\hat{B}_k(w(s))dxds \\ &+ \int_\varepsilon^t \int_{\{x : |w(x,s)| \le k\}} (|\nabla_{x_1} w|^2 + |x_1|^{2s}|\nabla_{x_2} w|^2)dxds \\ &+ \int_\varepsilon^t \int_\Omega \left(\tilde{f}(u) - \tilde{f}(v)\right) \hat{B}_k(w)dxds - \ell \int_\varepsilon^t \int_\Omega w\hat{B}_k(w)dx \\ = 0. \end{aligned}$$

Noting that $w\dfrac{d}{dt}\hat{B}_k(w(s)) = \dfrac{1}{2}\dfrac{d}{dt}(\hat{B}_k(w(s))^2)$ we have

$$\begin{aligned}
&\int_\Omega w(t)\hat{B}_k(w)(t)dx - \frac{1}{2}\|\hat{B}_k(w)(t)\|^2_{L^2(\Omega)} \\
&+ \int_\varepsilon^t \int_{\{x:|w(x,s)|\leq k\}} (|\nabla_{x_1} w|^2 + |x_1|^{2s}|\nabla_{x_2} w|^2)dxds \\
&+ \int_\varepsilon^t \int_\Omega (f'(\xi))\, w\hat{B}_k(w)dxds \\
= &\int_\Omega w(\varepsilon)\hat{B}_k(w)(\varepsilon)dx - \frac{1}{2}\|\hat{B}_k(w)(\varepsilon)\|^2_{L^2(\Omega)} + \ell\int_\varepsilon^t \int_\Omega w\hat{B}_k(w)dx.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\int_\Omega w(t)\hat{B}_k(w)(t)dx - \frac{1}{2}\|\hat{B}_k(w)(t)\|^2_{L^2(\Omega)} - \ell\int_\varepsilon^t \int_\Omega w\hat{B}_k(w)dx \\
\leq &\int_\Omega w(\varepsilon)\hat{B}_k(w)(\varepsilon)dx - \frac{1}{2}\|\hat{B}_k(w)(\varepsilon)\|^2_{L^2(\Omega)} + \ell\int_\varepsilon^t \int_\Omega w\hat{B}_k(w)dx.
\end{aligned}$$

Note that $\tilde{f}'(s) \geq 0$ and $sB_k(s) \geq 0$ for all $s \in \mathbb{R}$, by letting $\varepsilon \to 0$ and $k \to \infty$ in the above inequality, we obtain

$$\|w(t)\|^2_{L^2(\Omega)} \leq \|w(0)\|^2_{L^2(\Omega)} + 4\ell\int_0^t \|w(t)\|^2_{L^2(\Omega)}.$$

Hence by the Gronwall inequality of integral form, we get

$$\|w(t)\|^2_{L^2(\Omega)} \leq \|w(0)\|^2_{L^2(\Omega)}(1 + 4\ell te^{4\ell t}), \text{ for all } t \in [0, T].$$

This implies the desired result. □

3. Existence of global attractors in $\mathbf{L^2}(\Omega)$

By Theorem 2.1, we can define a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ associated to problem (1.1) as follows:

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique weak solution of (1.1) with the initial datum u_0. We will prove that the semigroup $S(t)$ has a global attractor $\mathcal{A}$ in the space $S^1_0(\Omega)$.

For the sake of brevity, in the following lemmas we give some formal calculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [14].

We first prove the existence of a bounded absorbing set in $L^2(\Omega)$.

Lemma 3.1. *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $L^2(\Omega)$*, i.e., there exists a positive constant* ρ_1 *such that for any bounded subset* B *in* $L^2(\Omega)$*, the corresponding solution* $u(.)$ *of* (1.1) *with initial datum* $u_0 \in B$ *satisfies*

$$\|u(t)\|^2_{L^2(\Omega)} \leq \rho_1, \text{ for all } t \geq T_1 = T_1(B). \tag{3.1}$$

Proof. Multiplying the first equation in (1.1) by u, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_{S^1_0(\Omega)} + \int_\Omega f(u)udx = \int_\Omega gudx. \tag{3.2}$$

Using (1.3), the inequality $\|u\|^2_{S^1_0(\Omega)} \geq \lambda_1\|u\|^2_{L^2(\Omega)}$ and the Cauchy inequality, we arrive at

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + (\lambda_1 - \mu)\|u(t)\|^2_{L^2(\Omega)} \leq 2C_1|\Omega| + \frac{1}{\lambda_1 - \mu}\|g\|^2_{L^2(\Omega)}.$$

Hence, thanks to the Gronwall inequality, we obtain

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2 e^{-(\lambda_1-\mu)t} + R_1,$$

where $R_1 = R_1(\lambda_1, \mu, |\Omega|, \|g\|_{L^2(\Omega)})$. This completes the proof if we choose, for instance, $\rho_1 = 2R_1$. □

We now prove the existence of a bounded absorbing set in $S^1_0(\Omega)$.

Lemma 3.2. *The semigroup* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in* $S^1_0(\Omega)$*, i.e., there exists a positive constant* ρ_2 *such that for any bounded subset* B *in* $L^2(\Omega)$*, the corresponding solution* $u(.)$ *of* (1.1) *with initial datum* $u_0 \in B$ *satisfies*

$$\|u(t)\|^2_{S^1_0(\Omega)} \leq \rho_2, \text{ for all } t \geq T_2 = T_2(B).$$

Proof. Multiplying the first equation in (1.1) by u_t, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{S^1_0(\Omega)} + \int_\Omega F(u)dx - \int_\Omega gudx\right) = -\|u_t\|^2_{L^2(\Omega)} \leq 0. \tag{3.3}$$

On the other hand, integrating (3.2) from t to $t+1$ we get

$$\int_t^{t+1}\left[\frac{1}{2}\frac{d}{ds}\|u(s)\|^2_{L^2(\Omega)} + \|u(s)\|^2_{S^1_0(\Omega)} + \int_\Omega f(u)udx\right]ds = \int_t^{t+1}\int_\Omega gudxds.$$

Using (3.1) we have

$$
\begin{aligned}
&\int_t^{t+1}\left[\frac{1}{2}\|u(s)\|^2_{S_0^1(\Omega)}+\int_\Omega F(u)dx-\int_\Omega gudx\right]ds\\
&=\int_t^{t+1}\left[-\frac{1}{2}\|u(s)\|^2_{S_0^1(\Omega)}-\frac{1}{2}\frac{d}{ds}\|u(s)\|^2_{L^2(\Omega)}+\int_\Omega (F(u)-f(u)u)dx\right]ds\\
&\le -\frac{1}{2}\|u(t+1)\|^2_{L^2(\Omega)}+\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}+\int_t^{t+1}\frac{\ell}{2}\|u(s)\|^2_{L^2(\Omega)}ds\\
&\le \int_t^{t+1}\frac{\ell}{2}\|u(t)\|^2_{L^2(\Omega)}dt+\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}\\
&\le \frac{\rho_1(\ell+1)}{2}, \forall t\ge T_1.
\end{aligned}
$$

Hence

$$
\int_t^{t+1}\left[\frac{1}{2}\|u(s)\|^2_{S_0^1(\Omega)}+\int_\Omega F(u)dx-\int_\Omega gudx\right]ds\le\frac{\rho_1(\ell+1)}{2}, \forall t\ge T_1. \tag{3.4}
$$

By the uniform Gronwall inequality, from (3.3) and (3.4) we deduce that

$$
\frac{1}{2}\|u(t)\|^2_{S_0^1(\Omega)}+\int_\Omega F(u)dx-\int_\Omega gudx\le R_2,\ \text{for all } t\ge T_2=T_1+1. \tag{3.5}
$$

Using (1.4) and the Cauchy inequality, we have

$$
\begin{aligned}
R_2&\ge\frac{1}{2}\|u(t)\|^2_{S_0^1(\Omega)}+\int_\Omega F(u)dx-\int_\Omega gudx\\
&\ge\frac{1}{2}\|u(t)\|^2_{S_0^1(\Omega)}-\frac{\ell}{2}\|u(t))\|^2_{L^2(\Omega)}+f(0)\int_\Omega udx-\int_\Omega gudx\\
&\ge\frac{1}{2}\|u(t)\|^2_{S_0^1(\Omega)}-\frac{\ell}{2}\|u(t))\|^2_{L^2(\Omega)}-\Big(\frac{|f(0)|}{2}\|u(t)\|^2_{L^2(\Omega)}+\frac{|f(0)|}{2}|\Omega|\Big)\\
&\quad-\frac{\ell}{2}\|u(t))\|^2_{L^2(\Omega)}-\frac{1}{2\ell}\|g\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Using (3.1) we obtain

$$
\|u(t)\|^2_{S_0^1(\Omega)}\le\rho_2=\rho_2(\lambda_1,\ell,|\Omega|,\|g\|_{L^2(\Omega)},|f(0)|),\ \text{for all } t\ge T_2.
$$

This completes the proof. □

Finally, we have the following result about the existence of global attractors in $L^2(\Omega)$.

Theorem 3.1. *The semigroup* $S(t)$ *genenated by problem* (1.1) *has a compact connected global attractor* $\mathcal{A}_{L^2}$ *in* $L^2(\Omega)$.

Proof. From [14], we need to show the existence of an absorbing set in $L^2(\Omega)$ and prove that $S(t)$ is asymptotically compact in $L^2(\Omega)$. Indeed, the former is obtained from Lemma 3.1. Since $S_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and using Lemma 3.2, we obtain that $S(t)$ is asymptotically compact in $L^2(\Omega)$. Note that $L^2(\Omega)$ is connected, we immediately get the following theorem. □

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