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A NOTE ON GENERALIZED RAINBOW CONNECTION OF CONNECTED GRAPHS AND THEIR NUMBER OF EDGES

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Abstract. Let $l \geq 1, k \geq 1$ be two integers. Given an edge-coloured connected graph G. A path P in the graph G is called l-rainbow path if each subpath of length at most $l+1$ is rainbow. The graph G is called (k, l)-rainbow connected if any two vertices in G are connected by at least k pairwise internally vertex-disjoint l-rainbow paths. The smallest number of colours needed in order to make G (k, l)-rainbow connected is called the (k, l)-rainbow connection number of G and denoted by $rc_{k,l}(G)$. In this paper, we first focus to improve the upper bound of the $(1, l)$-rainbow connection number depending on the size of connected graphs. Using this result, we characterize all connected graphs having the large $(1, 2)$-rainbow connection number. Moreover, we also determine the $(1, l)$-rainbow connection number in a connected graph G containing a sequence of cut-edges.
Keywords: edge-colouring, rainbow connection, (k, l)-rainbow connection.

1. Introduction

We use [1] for terminology and notation not defined here and consider simple, finite and undirected graphs only. Let G be a graph. We denote by $V(G)$, $E(G)$, $n(G)$, $m(G)$ the vertex set, the edge set, the number of vertices, the number of edges, respectively. Let uv be an edge of G and $c(uv)$ be its colour. A cut-edge of a graph is an edge whose deletion increases the number of components. Let $p(G)$ denote the number of edges of the longest path in G. We abbreviate the set $\{1, 2, \ldots, k\}$ by $[k]$.

In the last years, the connection concepts of connected graphs appeared in graph theory and received much attention. They have many applications in the transmission of information in networks. Let G be a connected and edge-coloured graph.

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The first connection concept introduced by Chartrand et al. [2] is *rainbow connection*. A *rainbow path* in an edge-coloured graph G is a path P whose edges are assigned distinct colours. An edge-coloured graph G is *rainbow connected* if every two vertices are connected by at least one rainbow path in G. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the smallest number of colours required to make it rainbow connected. After that, many researchers have studied problems with rainbow connection. Moreover, it has been shown in [3] that computing $rc(G)$ for a given connected graph G is an NP-hard problem. Readers who are interested in this topic are referred to [4, 5].

Motivated by proper colouring and rainbow connection, Borozan et al. [6] and Andrews et al. [7], independently introduced the concept of *proper connection*. A path P in an edge-coloured graph G is a *proper path* if any two consecutive edges receive distinct colours. An edge-coloured graph G is *properly connected* if every two vertices are connected by at least one proper path in G. For a connected graph G, the *proper connection number* of G, denoted by $pc(G)$, is defined as the smallest number of colours required to make it properly connected. Very recently, it has been shown in [8] that computing $pc(G)$ for a given graph G is an NP-hard problem. For more details we refer to the survey [9].

Let $k \geq 1$,$l \geq 1$ be two integers. Very recently, the new concept of connection that is (k, l)-*rainbow connection* was defined in [11] as a generalization of rainbow connection and proper connection. A path P in an edge-coloured graph G is called an l-*rainbow path* if each subpath of length at most $l + 1$ of P is rainbow. An edge-coloured graph G is called (k, l)-*rainbow connected* if every two vertices are connected by at least k pairwise internally vertex-disjoint l-rainbow paths in G. For a connected graph G, the (k, l)-*rainbow connection number* of G, denoted by $rc_{k,l}(G)$, is defined as the smallest number of colours required to make it (k, l)-rainbow connected. From this definition, it can be readily seen that the $(1, 1)$-rainbow connection number of a connected graph G is actually its proper connection number, i.e $rc_{1,1}(G) = pc(G)$. Meanwhile, the $(1, l)$-rainbow connection number of a connected graph G can be its rainbow connection number as long as l is large enough, i.e. if G is rainbow connected then G is $(1, l)$-rainbow connected. Moreover, if each edge of G is assigned by exact one different colour from $[m(G)]$ then G is rainbow connected. Let P_n be a path of order n, where $n \geq 4$. Hence, $pc(P_n) = 2$ (by Andrews et al. [7]), $rc(P_n) = n - 1$ (by Chartrand et al. [2]), $rc_{1,l}(P_n) = l + 1$, where $l \geq 2$ (by Li et al. [11]). Recently, there is a few results on this topic. By the above concepts, it can be readily seen that

$$1 \leq pc(G) \leq rc_{1,2}(G) \leq rc_{1,3}(G) \leq \ldots \leq rc(G) \leq m(G).$$

Moreover, $rc_{1,l}(G) = 1$ if and only if G is complete.

In this paper, we improve the upper bound of the $(1, l)$-rainbow connection number depending on the size of connected graphs. We investigate the $(1, l)$-rainbow connection number of a connected graph containing a sequence of cut-edges. Moreover, we also characterize all connected graphs having the large $(1, 2)$-rainbow connection number.

2. Auxiliary results

In this section, we introduce some basic notations, results and definitions that will be essential tools in the proof of our results.

Remark 2.1. *Let $P = v_1v_2 \dots v_n$ be a path of order $n(P)$ and l be a positive integer. We alternately colour all edges of P with colours from $[l]$ that means every subpath of length at most l is rainbow.*

Definition 2.1. *Let G be a graph and P be a path of l edges. P is said to be a l-cut-edge of G if each edge of P is a cut-edge of the graph G.*

Similar to the proper connection number and the rainbow connection number, the following proposition is easily obtained in [11].

Proposition 2.1. *(Li et al. [11]) Let G be a nontrivial connected graph. If H is a connected spanning subgraph of G, then $rc_{1,2}(G) \leq rc_{1,2}(H)$. Particularly, $rc_{1,2}(G) \leq rc_{1,2}(T)$ for every spanning tree T of G.*

By using Proposition 2.1, the authors in [11] gave the $(1, 2)$-rainbow connection number of the traceable graph, i.e. graphs containing a Hamiltonian path.

Proposition 2.2. *(Li et al. [11]) Let G be a traceable graph and l be a positive integer, then $rc_{1,l}(G) \leq l + 1$. Particularly, $rc_{1,2}(G) \leq 3$.*

3. Main results

First of all, we improve the upper bound of the $(1, l)$-rainbow connection number by the following result.

Theorem 3.1. *Let G be a connected graph of size $m(G)$ and $l \geq 2$ be an integer. If $p(G)$ is the number of edges of a longest path in G, then $1 \leq rc_{1,l}(G) \leq \min\{m(G), m(G) + l + 1 - p(G)\}$.*

Proof. Clearly, we only consider that $p(G) > l + 1$. Let $P = v_1v_2 \dots v_{p(G)+1}$ be a longest path of G. We alternately colour all edges of P by $l+1$ colours. Hence, P is the l-rainbow path. There are $m(G) - p(G)$ uncoloured edges of G. Next, each uncoloured edge of G is assigned by a new colour from $[m(G) + l + 1 - p(G)] \setminus [l + 1]$. It can be readily seen that every two distinct vertices of G are connected by at least one l-rainbow path. Hence, G is the $(1, l)$-rainbow connected. Therefore, $rc_{1,l}(G) \leq m(G) + l + 1 - p(G)$.

Our result is obtained. $\square$

Next, we determine the $(1, l)$-rainbow connection number of a connected graph G containing a path as an l-cut-edge, where $l \geq 1$ is a positive integer.

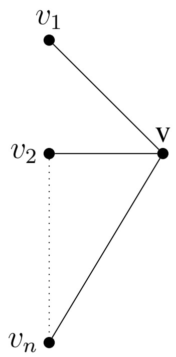

Figure 1. The star graph S_n

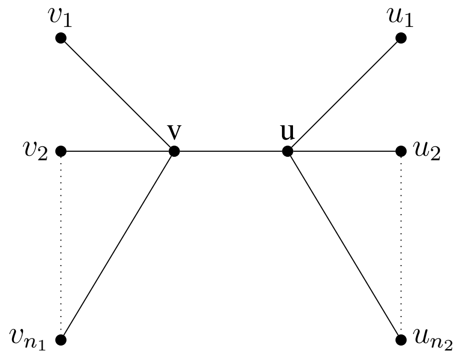

Figure 2. The double star T_{n_1,n_2}

Theorem 3.2. *Given a positive integer $l \geq 1$. Let G be a connected graph with a path as an l-cut-edge, say $P = v_1 v_2 \dots v_{l+1}$ and H_1, H_2 be two components obtained from G by removing all vertices from $V(P) \setminus \{v_1, v_{l+1}\}$. If G_i is a connected graph such that $G_i = G[V(H_i) \cup V(P)]$, where $i \in [2]$, then $rc_{1,l}(G) = \max\{rc_{1,l}(G_1), rc_{1,l}(G_2)\}$.*

Proof. First, it can be readily seen that $rc_{1,l}(G) \geq \max\{rc_{1,l}(G_1), rc_{1,l}(G_2)\}$. Let $rc_{1,l}(G_1) = k_1$ and $rc_{1,l}(G_2) = k_2$. Without loss of generality, we may assume that $k_1 \geq k_2$. Let $i \in [2]$ and c_i be a $(1,l)$-rainbow colouring of G_i with k_i colours, ($c_i(e) \in [k_i]$, for all edges $e \in E(G_i)$) such that $c_1(v_t v_{t+1}) = c_2(v_t v_{t+1})$, where $v_t \in V(P)$ and $t \in [l]$, and $\{c_2(e) : e \in E(G_2)\} \subseteq \{c_1(e) : e \in E(G_1)\}$. Let c be an edge-colouring of G such that $c(e) = c_1(e)$ for any $e \in E(G_1)$ and $c(e) = c_2(e)$ otherwise. Clearly, c is a $(1,l)$- rainbow colouring of G using k_1 colours. We will show that G is the $(1,l)$-rainbow connected. For any two distinct vertices of G, say $u, v \in V(G)$, it can be readily seen that there is a $(1,l)$-rainbow path between them if $u, v \in V(G_1)$ or $u, v \in V(G_2)$. Hence, we only consider that $u \in V(G_1) \setminus V(P)$ and $v \in V(G_2) \setminus V(P)$. Since c_1 is the $(1,l)$-rainbow colouring of G_1, there exists a $(1,l)$-rainbow path connecting u and v_{l+1}. Since c_2 is the $(1,l)$-rainbow colouring of G_2, there exists a $(1,l)$-rainbow path connecting v and v_1. As $c_1(v_t v_{t+1}) = c_2(v_t v_{t+1})$, where $t \in [l]$, so it can be readily deduced that $P_G = uP_1 v_1 P v_{l+1} P_2 v$ is a $(1,l)$-rainbow path connecting u, v in G. Therefore, we have that $rc_{1,l}(G) \leq k_1$.

Our proof is obtained. $\square$

By using Theorem 3.1, we determine all connected graphs having the large $(1,2)$-rainbow connection number. We use S_n to denote the star graph on n vertices and $T(n_1, n_2)$ to denote the double star in which the degrees of its adjacent center vertices are $n_1 + 1$ and $n_2 + 1$, respectively. Clearly, two graphs S_n and T_{n_1,n_2} are already mentioned in [12] but our result is different.

Proposition 3.1. *Let G be a nontrivial connected graph of size $m(G)$. Then $rc_{1,2}(G) = m(G)$ iff $G \cong S_n$, where $n \geq 2$ or $G \cong T(n_1, n_2)$, where $n_1, n_2 \geq 1$.*

Proof. If $G \cong S_n$ or $G \cong T_{n_1,n_2}$ then it can be readily check that $rc_{1,2}(G) = m(G)$. So it remains to verify the converse. Let T be a spanning tree of G. Hence, $m(T) \leq m(G)$. By Proposition 2.1, $rc_{1,2}(G) \leq rc_{1,2}(T)$. Since $rc_{1,2}(G) = m(G)$, we see that $m(G) = rc_{1,2}(G) \leq rc_{1,2}(T)$. Let P be a longest path of T and $p(T)$ be the number of edges of P. By Theorem 3.1, $rc_{1,2}(T) \leq \min\{m(T), m(T) + 3 - p(T)\}$. If $p(T) > 3$, then $rc_{1,2}(T) \leq m(T) + 3 - p(T)$. So $m(G) \leq m(T) + 3 - p(T)$, a contradiction to $m(T) \leq m(G)$.

Now, $p(T) \leq 3$. Since T is a tree, we conclude that $T \cong S_n$ or $T \cong T_{n_1,n_2}$ [1]. On the other hand, $m(G) = rc_{1,2}(G) \leq rc_{1,2}(T) \leq m(T)$. It can be readily deduced that $G \cong T$ i.e. G is a tree. Therefore, $G \cong S_n$ or $\cong T_{n_1,n_2}$.

Our result is obtained. □

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