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ON EXPONENTIAL STABILITY OF A CLASS OF DELAY DIFFERENTIAL EQUATIONS IN NEOCLASSICAL GROWTH MODELS

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Abstract. In this paper, the problem of global exponential stability of delay differential equations in a neoclassical growth model is investigated by comparison techniques via differential inequalities.
Keywords: neoclassical growth model, delay differential equations, exponential stability.

1. Introduction

Neoclassical growth models [1] are widely used to explain long-run economic growth based on capital accumulation and labor growth. The main equation, proposed by Solow in 1956 [2], is based on the assumption that there is only one commodity and its rate of production is defined by a function $P = P(K, L)$, where K and L are the capital stock and the rate of input of labor, respectively. Represented by a variable $x = K/L$ (called the capital-labor rate) and assume the function P is homogeneous of degree one, Solow model can be described by the ordinary differential equation

$$x'(t) = -\alpha x(t) + s(x(t))P(x(t), 1) \tag{1.1}$$

where α is the population growth rate and $s(x)$ is the instantaneous saving rate.

In [3], Day suggested that productivity can be reduced by a "pollution effect" caused by increasing concentrations of capital. This leads to the model

$$x'(t) = -\alpha x(t) + s(x(t))p_1(x(t))p_2(x(t)) \tag{1.2}$$

where $p_1(x) = P(x, 1)$ and $p_2(x)$ is a pollution function, which is a decreasing function of x. Recently, by integrating a production lag in Solow-type models (1.1) and (1.2), Matsumoto and Szidarovsky [4] proposed the following equation

$$x'(t) = -\alpha x(t) + s(x(t))f(x(t-\tau)) \tag{1.3}$$

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where $f(x) = p_1(x)p_2(x)$ and $\tau > 0$ represents the production time-delay. A common choice of production functions is the Cobb-Douglas function $P(K, L) = QK^\gamma L^{1-\gamma}$, where $Q > 0$ refers to the level of labor-augmenting technology and $\gamma \in (0, 1)$ represents the part of the output produced by the capital. With constant saving ratio s and "pollution effect" function $p_2(x) = e^{-\delta x}$, the problem of local stability of the model

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t - \tau))e^{-\delta x(t-\tau)}, \tag{1.4}$$

where $\beta = sQ$, was studied in [5]. By introducing an additional delay, called the depreciation delay, Guerrini et al in [6] studied the effect of two fixed delays of the model

$$x'(t) = -\alpha x(t - \tau_d) + sf(x(t - \tau_p)) \tag{1.5}$$

via the characteristic equation. However, the existing methods, for example, in [4, 5, 6] cannot be utilized for neoclassical growth models with time-varying delays

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t - \tau(t)))e^{-\delta x(t-\tau(t))} \tag{1.6}$$

This motivates the present study.

In this paper, we consider the problem of global exponential stability of neoclassical growth model (1.6), where $\alpha > 0$, $\beta > 0$, $\delta > 0$, $\gamma \in (0, 1)$ and the time-varying delay $0 \le \tau(t) \le \tau$, where τ is a given constant.

2. Global existence of positive solutions

Let $\mathcal{C} = C([-\tau, 0], \mathbb{R})$ be the Banach space consisting of continuous functions endowed with the sup-norm

$$\|\varphi\| = \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|$$

for $\varphi \in \mathcal{C}$. Due to practical reasons in the economy, only positive solutions of (1.6) are admissible. Thus, it is reasonable to specify initial conditions as

$$x(t_0 + \theta) = \varphi(\theta) \ge 0 \text{ for } \theta \in [-\tau, 0] \text{ and } \varphi(0) > 0. \tag{2.1}$$

To explicitly mention the initial condition, we will denote as $x(t; t_0, \varphi)$ the corresponding solution of (1.6) with initial condition (2.1). For convenience, we denote by $\mathcal{C}_+ = \{\varphi \in \mathcal{C} : \varphi(\theta) \ge 0,\ \varphi(0) > 0\}$ the admissible initial functions of equation (1.6). In addition, for a continuous function defined on the interval $[t_0 - \tau, t_f)$, $t_f > t_0$, for $t \in [t_0, t_f)$, the function $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(t + \theta)$.

Let $F : \mathcal{C}_+ \longrightarrow \mathbb{R}$, $\varphi \mapsto F(\varphi)$, be the function defined by

$$F(\varphi) = -\alpha\varphi(0) + \beta\varphi^\gamma(-\tau(0))e^{-\delta\varphi(-\tau(0))}.$$

Then, equation (1.6) can be written in the abstract form

$$x'(t) = F(x_t),\ t > t_0. \tag{2.2}$$

In addition, it can be verified that

$$\left|e^{-\delta u}-e^{-\delta v}\right| \le \delta|u-v|$$

holds for any $u, v \ge 0$. Thus, the function F is continuous and locally Lipschitz with respect to $\varphi \in \mathcal{C}_+$. By the fundamental theory of functional differential equations [7], the problem governed by functional differential equation (1.6) and initial condition (2.1) has a unique solution for each $(t_0, \varphi) \in \mathbb{R}_+ \times \mathcal{C}_+$. Let $x(t; t_0, \varphi)$ denote the corresponding solution of (1.6)-(2.1) and $[t_0, \eta(\varphi))$ is the maximal right interval of the existence of $x_t(t_0, \varphi)$.

Lemma 2.1. *For given $\delta > 0$ and $\gamma > 0$, it holds that*

$$\sup_{x\ge 0} x^{\gamma} e^{-\delta x} = \left(\frac{\gamma}{\delta e}\right)^{\gamma}.$$

Proposition 2.1. *Let $x(t) = x(t; t_0, \varphi)$ be a solution of (1.6) with initial function $\varphi \in \mathcal{C}_+$. Then, $x(t) > 0$ for any $t \in [t_0, \eta(\varphi))$ and $\eta(\varphi) = +\infty$.*

Proof. Since $x(t_0) = \varphi(0) > 0$, by the continuity, $x(t) > 0$ for $t \in [t_0, t_0 + \epsilon)$ with sufficiently small $\epsilon > 0$. If there exists a $t_1 > t_0$ such that

$$x(t_1) = 0 \text{ and } x(t) > 0 \text{ for all } t \in [t_0, t_1). \tag{2.3}$$

Then, it follows from (1.6) that

$$\begin{aligned} x(t) &= e^{-\alpha(t-t_0)}x(t_0) + \beta \int_{t_0}^{t} x^{\gamma}(s-\tau(s)) e^{-\alpha(t-s)-\delta x(s-\tau(s))} ds \\ &\ge e^{-\alpha(t-t_0)}x(t_0) \\ &= e^{-\alpha(t-t_0)}\varphi(0), \quad t \in [t_0, t_1). \end{aligned} \tag{2.4}$$

Let $t \uparrow t_1$, from (2.4), we obtain

$$0 = x(t_1) \ge e^{-\alpha(t_1-t_0)}\varphi(0) > 0.$$

This contradiction shows that $x(t) > 0$ for all $t_0 \le t < \eta(\varphi)$.

On the other hand, by Lemma 2.1

$$\begin{aligned} x'(t) &\le -\alpha x(t) + \sup_{s\ge t_0-\tau} \beta x^{\gamma}(s) e^{-\delta x(s)} \\ &\le -\alpha x(t) + \beta\left(\frac{\gamma}{\delta e}\right)^{\gamma}, \end{aligned}$$

which gives

$$x(t) \le e^{-\alpha(t-t_0)}\varphi(0) + \frac{\beta}{\alpha}\left(\frac{\gamma}{\delta e}\right)^{\gamma}\left(1-e^{-\alpha(t-t_0)}\right) \tag{2.5}$$

If $t_f = \eta(\varphi) < +\infty$ then, from (2.5), we have

$$\limsup_{t\to t_f-0} x(t) \le e^{-\alpha(t_f-t_0)}\varphi(0) + \frac{\beta}{\alpha}\Big(\frac{\gamma}{\delta e}\Big)^{\gamma}\big(1-e^{-\alpha(t_f-t_0)}\big) < +\infty.$$

This contradicts with $\lim_{t\to t_f} x(t) = +\infty$ (see, [7], Theorem 2.3.1). The proof is completed. □

Remark 2.1. *For any solution of* (1.6)*, from* (2.5)*, we have*

$$\limsup_{t\to+\infty} x(t;t_0,\varphi) \le M := \frac{\beta}{\alpha}\Big(\frac{\gamma}{\delta e}\Big)^{\gamma}. \tag{2.6}$$

3. Exponential stability

In this section, we focus on the problem of global exponential stability of a unique positive equilibrium of model (1.6). First, it can be seen that a positive equilibrium of equation (1.6) exists if and only if the equation $F(x) = x$ admits at least a positive solution, where $F(x) = (\beta/\alpha)x^{\gamma}e^{-\delta x}$.

Lemma 3.1. *Given* $\gamma \in (0,1)$ *and consider the function* $\Psi(x) = (\gamma - x)x^{\gamma-1}e^{-x}$, $x \in (0,+\infty)$*. There exists a unique scalar* $\kappa \in (0,\gamma)$ *such that*

$$\Psi(\kappa) = \sup_{x\ge\kappa}|\Psi(x)| = \sqrt{\gamma}\,(\gamma+\sqrt{\gamma})^{\gamma-1}\,e^{-(\gamma+\sqrt{\gamma})}.$$

Proof. It is clear that the function $\Psi(x)$ has a unique critical point $c = (\gamma+\sqrt{\gamma})$, at which $\Psi(x)$ attains it global minimum and $\Psi(x)$ is strictly decreasing on $(0,\gamma)$. The scalar κ is then determined as $\kappa = \Psi^{-1}(-\Psi(c))$. □

Lemma 3.2. *For given positive constants* α*,* β*,* δ *and* $\gamma \in (0,1)$*, model* (1.6) *has a unique positive equilibrium* $x_* \in (\kappa/\delta, M]$.

Proof. Since $F(x) = \frac{\beta}{\alpha}\delta^{1-\gamma}x\psi(\delta x)$ and $\psi(x) = x^{\gamma-1}e^{-x}$ is a strictly decreasing function on $(0,+\infty)$, $\psi(x) \to +\infty$ as $x \to 0^+$ and $\psi(x) \to 0$ as $x \to +\infty$, we can conclude that equation (1.6) has a unique positive equilibrium

$$x_* = \frac{1}{\delta}\psi^{-1}\Big(\frac{\alpha}{\beta}\delta^{\gamma-1}\Big).$$

The proof is completed. □

Definition 3.1. *The positive equilibrium* x_* *of* (1.6) *is said to be globally exponentially stable if there exist positive constants* C *and* σ *such that the inequality*

$$|x(t;t_0,\varphi) - x_*| \le C\|\varphi - x_*\|e^{-\sigma(t-t_0)},\ t \ge t_0,$$

holds for any solution $x(t;t_0,\varphi)$ *of equation* (1.6)-(2.1).

Remark 3.1. *It can be shown by utilizing the fluctuation lemma [8, Lemma A.1] that the estimate*

$$\liminf_{t\to+\infty} x(t;t_0,\varphi) \geq \frac{\kappa}{\delta}$$

holds for any solution $x(t;t_0,\varphi)$ of (1.6)-(2.1). *Thus, for sufficiently large t, it can be assumed that $x(t;t_0,\varphi) \geq \kappa/\delta$.*

Lemma 3.3 (Halanay inequality [9])**.** *Let $v : [t_0-\tau,+\infty) \to \mathbb{R}_+$ be a continuous function satisfying*

$$D^+v(t) \leq -av(t) + b \sup_{s\in[t-\tau,t]} v(s),\ t > t_0, \tag{3.1}$$

where $D^+v(t)$ denotes the upper-right Dini derivation and a, b are given real numbers. If $a > b > 0$ then there exists a positive scalar σ such that

$$v(t) \leq \sup_{s\in[t_0-\tau,t_0]} v(s)e^{-\sigma(t-t_0)},\ t \geq t_0.$$

We are now in a position to present the main result in this section.

Theorem 3.1. *Assume that*

$$\frac{\beta}{\alpha}\delta^{1-\gamma} < \frac{\left(\gamma+\sqrt{\gamma}\right)^{1-\gamma} e^{\gamma+\sqrt{\gamma}}}{\sqrt{\gamma}}. \tag{3.2}$$

Then, the equilibrium point x_* of equation (1.6) *is globally exponentially stable.*

Proof. Let $x(t) = x(t;t_0,\varphi)$ be a solution of equation (1.6), then

$$x'(t) = -\alpha(x(t)-x_*) + \beta\left[x^\gamma(t-\tau(t))e^{-\delta x(t-\tau(t))} - x_*^\gamma e^{-\delta x_*}\right],\ t > t_0. \tag{3.3}$$

We now define the function $v(t) = |x(t) - x_*|$, $t \geq t_0 - \tau$. It follows from (3.3) that

$$\begin{aligned} D^+v(t) &= (x(t)-x_*)'\mathrm{sgn}(x(t)-x_*) \\ &= -\alpha|x(t)-x_*| + \beta\mathrm{sgn}(x(t)-x_*)\left[x^\gamma(t-\tau(t))e^{-\delta x(t-\tau(t))} - x_*^\gamma e^{-\delta x_*}\right] \\ &\leq -\alpha|x(t)-x_*| + \beta\left|x^\gamma(t-\tau(t))e^{-\delta x(t-\tau(t))} - x_*^\gamma e^{-\delta x_*}\right| \\ &= -\alpha v(t) + \alpha\left|F(x(t-\tau(t))) - F(x_*)\right|,\ t > t_0. \end{aligned} \tag{3.4}$$

For any $u_1, u_2 \geq \kappa/\delta$, by mean-value theorem, we have

$$\begin{aligned} \alpha|F(u_1) - F(u_2)| &= \alpha|F'(\xi)||u_1-u_2| \\ &= \beta\delta^{1-\gamma}|\Psi(\delta\xi)||u_1-u_2| \\ &\leq \beta\delta^{1-\gamma}\sup_{x\geq\kappa}|\Psi(x)||u_1-u_2| \\ &= \beta\delta^{1-\gamma}\Psi(\kappa)|u_1-u_2|. \end{aligned}$$

Therefore,

$$\begin{aligned} D^+v(t) &\le -\alpha v(t) + \beta\delta^{1-\gamma}\Psi(\kappa)|x(t-\tau(t)) - x_*| \\ &\le -\alpha v(t) + \beta\delta^{1-\gamma}\Psi(\kappa) \sup_{s\in[t-\tau,t]} v(s). \end{aligned} \tag{3.5}$$

By condition (3.2), we have

$$\alpha > \beta\delta^{1-\gamma}\Psi(\kappa) > 0.$$

Thus, by utilizing the Halanay inequality, there exists a positive constant σ such that

$$|x(t; t_0, \varphi) - x_*| \le \|\varphi - x_*\| e^{-\sigma(t-t_0)},\ t \ge t_0.$$

This shows that the equilibrium point x_* of (1.6) is globally exponentially stable. The proof is completed. □

4. An illustrative exmple

Consider equation (1.6) with $\gamma = 1/2$. According to condition (3.2), the positive equilibrium x_* is globally exponentially stable if

$$\frac{\beta}{\alpha}\sqrt{\delta} < 5.1955.$$

Let $\delta = 0.3$, $\alpha = 2$ and $\beta = 2.65$. Then, the positive equilibrium x_* is obtained as $x_* = 0.9061$. Some state trajectories of equation (1.6) with $\tau(t) = 5|\sin(10\pi t)|$ and different initial constant functions are presented in Figure 1. It can be seen that all state trajectories converge to the positive equilibrium x_*, which illustrates the theoretical result.

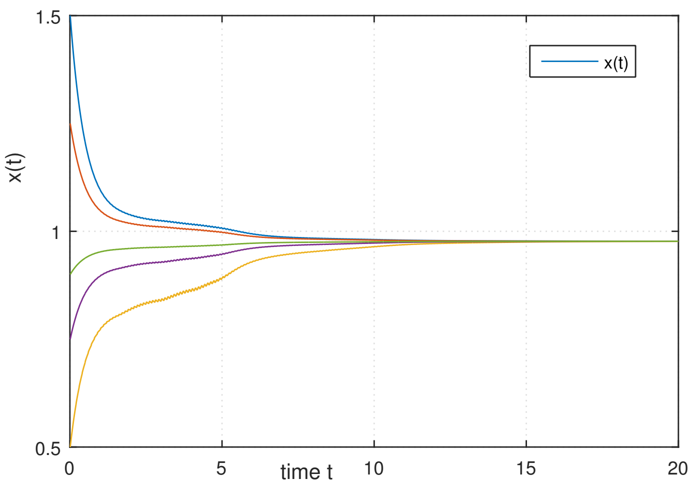

Figure 1. State trajectories of* (1.6) *with $\tau(t) = 5|\sin(10\pi t)|$

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