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A NOTE ON S. TAKAHASHI'S ARTICLE

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Abstract. We prove that the necessary and sufficient conditions given by S. Takahashi for the Nevanlinna-Pick-Carathéodory-Fejér interpolation problem can be obtained from the theory of D. Sarason.
Keywords: Nevanlinna-Pick, interpolation, reproducing kernel, Hardy spaces.

1. Introduction

Nevanlinna-Pick interpolation problem is a classical problem in mathematics. The problem can be generally stated as follows: Let X be a domain in $\mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ respectively where m, n are some positive integers. Let $z_1, z_2, \ldots, z_N \in X$ be N distinct points and $w_1, w_2, \ldots, w_N$ be any points in Y. The problem which asks if there exists a holomorphic function $f\colon X \to Y$ such that $f(z_j) = w_j$ for $1 \leq j \leq N$ is called the *Nevanlinna-Pick interpolation problem for the domains X, Y and the interpolation data* $\{z_1, z_2, \ldots, z_N; w_1, w_2, \ldots, w_N\}$. The function f is called a *solution* to the problem. The points $z_1, z_2, \ldots, z_N$ are called *interpolation nodes* and $w_1, w_2, \ldots, w_N$ are called *interpolation values*.

If $X = \mathbb{D}$ and $Y = \bar{\mathbb{D}}$ where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, then the corresponding interpolation problem is called the *classical Nevanlinna-Pick interpolation problem*. This problem is completely solved by G. Pick [1] and R. Nevanlinna [2]. In particular, G. Pick proves that the problem has a solution if the following Pick matrix

$$\left[\frac{1 - w_i\bar{w}_j}{1 - z_i\bar{z}_j}\right]_{1\leq i,j\leq N}$$

is positive semi-definite.

This problem attracts much attention since it is deeply related to other fields like robust control theory (cf. A. Tannenbaum [3]) or operator theory (cf. D. Sarason [4]). Nowadays the interpolation problem of Nevanlinna-Pick type which is of interest is the

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spectral Nevanlinna-Pick interpolation i.e. the target domain Y is the set of all square matrices of order n with spectral radius less than or equal to 1.

If we put more conditions on the derivatives at the interpolation nodes then we obtain the Carathéodory-Fejér interpolation (cf. [5], however, we follow the denomination in [6]), but in this note, we prefer to call this problem the Nevanlinna-Pick-Carathéodory-Fejér interpolation problem.

In this note, we are interested in the condition given by S. Takahashi [7] where the author proves the necessary and sufficient condition to the Nevanlinna-Pick-Carathéodory-Fejér interpolation problem by the method presented in D. Marshall [8]. We would like to deduce this condition by applying D. Sarason's theory, i.e. to give another proof by operator-theoretical approach.

2. Content

2.1. Statement of the Nevanlinna-Pick-Carathéodory-Fejér interpolation problem

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be N distinct points in $\mathbb{D}$. To each α_i, we associate n_i complex numbers $c_{i0}, c_{i1}, \ldots, c_{i,n_i-1}$.

Problem 2.1. *What are the conditions for which there exists a holomorphic function $\phi\colon \mathbb{D} \to \bar{\mathbb{D}}$ such that*

$$\phi(z) = \sum_{j=0}^{n_i-1} c_{ij}(z-\alpha_i)^j + O((z-\alpha_i)^{n_i}) \qquad (i = 1 \ldots N)? \tag{2.1}$$

2.2. S. Takahashi's condition

In this section we cite the necessary and sufficient condition in the article of S. Takahashi [7].

Firstly, we associate two points α_i and α_j with the matrix Γ_{ij} as follows: We consider the expansion

$$\frac{1}{1-z\bar{\zeta}} = \sum_{k,l\geq 0} a_{kl}(\alpha_i, \alpha_j)(z-\alpha_i)^k (\overline{\zeta - \alpha_j})^l$$

where the coefficients $a_{kl}(\alpha_i, \alpha_j)$ depend on α_i and α_j. Next, we put

$$\Gamma_{ij} = \begin{pmatrix} a_{00}(\alpha_i,\alpha_j) & \cdots & a_{0,n_j-1}(\alpha_i,\alpha_j) \\ \vdots & & \vdots \\ a_{n_i-1,0}(\alpha_i,\alpha_j) & \ldots & a_{n_i-1,n_j-1}(\alpha_i,\alpha_j) \end{pmatrix} \text{ and } \Gamma = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1N} \\ & \cdots\cdots & \\ \Gamma_{N1} & \cdots & \Gamma_{NN} \end{pmatrix}.$$

Then we put

$$C_i = \begin{pmatrix} c_{i0} & & \\ \vdots & \ddots & \\ c_{i,n_i-1} & \cdots & c_{i0} \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_N \end{pmatrix},$$

and finally

$$\boxed{A = \Gamma - C \cdot \Gamma \cdot C^*} \tag{2.2}$$

where C^* is the conjugate transpose of C, i.e. $C^* = \bar{C}^T$. Remark that A is a hermitian matrix of order $n_1 + n_2 + \ldots + n_N$.

Theorem 2.1 (Condition of S. Takahashi)**.** *In order that there exists a holomorphic function ϕ which interpolates the data in* (2.1)*, the necessary and sufficient condition is that the matrix A above is positive semi-definite.*

2.3. Deduction of interpolation conditions from D. Sarason's article

In this section, we apply the theory of D. Sarason [4] to the interpolation Problem 2.1 above to obtain interpolation conditions. We begin with some notations.

Notations: Denote by $H(\mathbb{D})$ the space of all holomorphic functions on the open unit disc $\mathbb{D}$. Denote by H^2 and H^∞ the Hardy spaces on the disc $\mathbb{D}$ where

$$H^2 = \left\{ f \in H(\mathbb{D}) \ : \ \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{\partial\mathbb{D}} |f(rz)|^2 |dz| \right)^{1/2} < +\infty \right\}$$

and H^∞ is the space of all bounded holomorphic functions on the unit disc $\mathbb{D}$.

The space H^2 has an additional structure which is a Hilbert space with the hermitian product presented as follows: For $f, g \in H^2$, $\langle f, g\rangle$ means the hermitian product in H^2. To present explicitly the hermitian product in H^2, we follow W. Rudin [9, Theorem 17.10, pages 332-334] to introduce the following notation: For each unimodular complex number z_0, put

$$f^*(z_0) = \lim_{\substack{r>0 \\ r\to 1^-}} f(rz_0).$$

The limit exists for almost all z_0 with respect to the Lebesgue measure on the unit circle. Then the hermitian product of $f, g \in H^2$ is equal to

$$\langle f, g\rangle = \frac{1}{2\pi} \int_{\partial\mathbb{D}} f^*(z)\overline{g^*(z)}|dz| = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}d\theta.$$

Denote by $S\colon H^2 \to H^2$ the shift to the right, i.e. $(Sf)(z) = zf(z)$ for every $f \in H^2$. Its adjoint S^* is the shift to the left which is given by

$$S^*f(z) = \frac{f(z) - f(0)}{z}$$

for $f \in H^2$.

For each $\phi \in H^\infty$, denote by $M_\phi\colon H^2 \to H^2$ the multiplication operator by ϕ, i.e. $M_\phi f = \phi f$ for every $f \in H^2$, and denote by M_ϕ^* the adjoint of M_ϕ.

And finally, put

$$\psi(z) = \prod_{i=1}^{N} \left(\frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \right)^{n_i}$$

for $z \in \mathbb{D}$. Remark that ψ is an inner function and it plays the same role as ψ in [4] or as the function $m(z)$ in [10, page 754].

Deduction of interpolation conditions: Firstly, we look for functions h which allow us to reproduce the derivatives at a certain point of the disc $\mathbb{D}$, i.e. for $f \in H^2$,

$$\langle f, h \rangle = f^{(m)}(\alpha)$$

for $\alpha \in \mathbb{D}$ and $m \in \mathbb{N}$ fixed.

To do it, we make use of Cauchy's integral. Let $f \in H^2$, $\alpha \in \mathbb{D}$ and $m \in \mathbb{N}$. We recall that f is the Cauchy integral of f^*, i.e.,

$$f(\alpha) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f^*(z)}{z - \alpha} dz$$

for $\alpha \in \mathbb{D}$ (cf. W. Rudin [9, Theorem 17.10, pages 332-334] for a proof). Therefore

$$f^{(m)}(\alpha) = \frac{m!}{2\pi i} \int_{\partial\mathbb{D}} \frac{f^*(z)}{(z - \alpha)^{m+1}} dz.$$

We transform this integral into a Lebesgue integral on the unit circle as follows:

$$\begin{aligned} \frac{f^{(m)}(\alpha)}{m!} &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f^*(e^{i\theta})}{(e^{i\theta} - \alpha)^{m+1}} i e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(e^{i\theta})}{(1 - e^{-i\theta}\alpha)^{m+1}} e^{-(m+1)i\theta} e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(e^{i\theta})}{(1 - e^{-i\theta}\alpha)^{m+1}} e^{-mi\theta} d\theta \\ &= \left\langle f, \frac{z^m}{(1 - \bar{\alpha} z)^{m+1}} \right\rangle. \end{aligned}$$

We put

$$k_{\alpha,m}(z) = \frac{z^m}{(1 - \bar{\alpha} z)^{m+1}}$$

then $k_{\alpha,m}$ allows us to reproduce the coefficient of the term $(z - \alpha)^m$ in the Taylor expansion of $f \in H^2$ at $z = \alpha$.

Then we consider

$$K = \operatorname{span}\{k_{\alpha_i,m} \ : \ 0 \leq m \leq n_i - 1, 1 \leq i \leq N\}$$

and we have $\dim K = n_1 + \ldots + n_N$ and $K = H^2 \ominus \psi H^2$.

Now suppose that ϕ is a function which interpolates the data in Problem 2.1. We have to compute the action of M_ϕ^* on K. To do it, we take a test function $f \in H^2$, and we have

$$\begin{aligned}\langle f, M_\phi^* k_{\alpha,m}\rangle &= \frac{(\phi f)^{(m)}(\alpha)}{m!} = \sum_{i=0}^{m} \frac{C_m^i \phi^{(m-i)}(\alpha) f^{(i)}(\alpha)}{m!} \\ &= \sum_{i=0}^{m} \frac{\phi^{(m-i)}(\alpha)}{(m-i)!} \frac{f^{(i)}(\alpha)}{i!} \\ &= \left\langle f, \sum_{i=0}^{m} \overline{\frac{\phi^{(m-i)}(\alpha)}{(m-i)!}} \cdot k_{\alpha,i} \right\rangle.\end{aligned}$$

Therefore

$$M_\phi^* k_{\alpha,m} = \sum_{i=0}^{m} \overline{\frac{\phi^{(m-i)}(\alpha)}{(m-i)!}} \cdot k_{\alpha,i}.$$

We put

$$D_i = \begin{pmatrix} \bar{c}_{i0} & \bar{c}_{i1} & \cdots & \bar{c}_{i,n_i-2} & \bar{c}_{i,n_i-1} \\ & \bar{c}_{i0} & \bar{c}_{i1} & \cdots & \bar{c}_{i,n_i-2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & & \bar{c}_{i0} \end{pmatrix} \text{ and } D = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_N \end{pmatrix}.$$

We can see that D is the matrix of $M_\phi^* \colon K \to K$ with respect to the basis $\{k_{\alpha_i,m} \ : \ 1 \le i \le N, 0 \le m \le n_i - 1\}$.

Now we look for conditions for which there exists a function ϕ which interpolates the data in Problem 2.1. Put $T \colon K \to K$ to be an endomorphism which satisfies that

$$T k_{\alpha_i,m} = \sum_{k=0}^{m} \bar{c}_{ik} \cdot k_{\alpha_i,k},$$

in other words, D is the matrix of T with respect to the basis $\{k_{\alpha_i,m} \ : \ 1 \le i \le N, 0 \le m \le n_i - 1\}$. Remark that T commutes with the shift to the left $S^* \colon K \to K$.

Interpolation condition: *By the theory of D. Sarason (cf. [4, Section 4, pages 186-187] or [11, Theorem 1.1.2, page 7] for details), the necessary and sufficient condition for which there exists a holomorphic function ϕ which interpolates the data in* (2.1) *is that T must be a contraction, i.e. $1 - T^*T \ge 0$.*

It is equivalent to the positive semi-definiteness of the following matrix

$$[\langle (1 - T^*T)v_i, v_j\rangle]_{1 \le i,j \le n_1 + \ldots + n_N}$$

with $\{v_1, \ldots, v_{n_1+\ldots+n_N}\}$ any basis of K. It means that

$$[\langle v_i, v_j\rangle - \langle Tv_i, Tv_j\rangle]_{1\le i,j\le n_1+\ldots+n_N} \ge 0.$$

Let G be the Gramian matrix of the basis $\{k_{\alpha_i,m} \;:\; 1 \le i \le N, 0 \le m \le n_i - 1\}$.

The necessary and sufficient condition is therefore

$$\boxed{G - D^T G\bar{D} \ge 0}. \tag{2.3}$$

We make precise the entries of

$$G = \begin{pmatrix} G_{11} & \cdots & G_{1N} \\ & \cdots\cdots & \\ G_{N1} & \cdots & G_{NN} \end{pmatrix}$$

where

$$G_{ij} = \begin{pmatrix} \langle k_{\alpha_i,0}, k_{\alpha_j,0}\rangle & \cdots & \langle k_{\alpha_i,0}, k_{\alpha_j,n_j-1}\rangle \\ \vdots & & \vdots \\ \langle k_{\alpha_i,n_i-1}, k_{\alpha_j,0}\rangle & \cdots & \langle k_{\alpha_i,n_i-1}, k_{\alpha_j,n_j-1}\rangle \end{pmatrix}$$

for all $1 \le i, j \le N$.

2.4. Comparison with the condition given by S. Takahashi

We see immediately that $D = C^*$. Now we prove that $G = \bar{\Gamma}$.

To do it, we compute the following scalar product

$$\begin{aligned} \langle k_{\alpha,m}, k_{\beta,n}\rangle &= \left\langle \frac{z^m}{(1-\bar{\alpha}z)^{m+1}}, \frac{z^n}{(1-\bar{\beta}z)^{n+1}} \right\rangle \\ &= \frac{1}{n!}\left.\frac{d^n}{dz^n}\right|_{z=\beta} \frac{z^m}{(1-\bar{\alpha}z)^{m+1}}. \end{aligned}$$

When $\alpha = \alpha_i$ and $\beta = \alpha_j$, the value above is the entry at the position (m, n) of G_{ij}.

Then we consider the entry at the same position of Γ_{ij}. It is the coefficient $a_{mn}(\alpha_i, \alpha_j)$ in the expansion

$$\frac{1}{1 - z\bar{\zeta}} = \sum_{k,l\ge 0} a_{kl}(\alpha, \beta)(z-\alpha)^k(\overline{\zeta - \beta})^l$$

where we reput $\alpha = \alpha_i$ and $\beta = \alpha_j$ for generalization. We remark that

$$\begin{aligned} \sum_{i\ge 0} a_{mi}(\alpha,\beta)(\overline{\zeta-\beta})^i &= \frac{1}{m!}\left.\frac{d^m}{dz^m}\right|_{z=\alpha} \frac{1}{1-z\bar{\zeta}} \\ &= \left.\frac{\bar{\zeta}^m}{(1-z\bar{\zeta})^{m+1}}\right|_{z=\alpha} = \frac{\bar{\zeta}^m}{(1-\alpha\bar{\zeta})^{m+1}}. \end{aligned}$$

Therefore

$$a_{mn}(\alpha,\beta) = \frac{1}{n!}\left.\frac{d^n}{d\bar{\zeta}^n}\right|_{\zeta=\beta} \frac{\bar{\zeta}^m}{(1-\alpha\bar{\zeta})^{m+1}} = \overline{\langle k_{\alpha,m}, k_{\beta,n}\rangle}.$$

We deduce that $G = \bar{\Gamma}$ and that

$$G - D^T G \bar{D} = \overline{\Gamma - C\Gamma C^*},$$

i.e. the two conditions are identical.

3. Conclusions

In this note, we reproduce the interpolation condition given by S. Takahashi [7] by applying D. Sarason's theory. Similarly to the classical Nevanlinna-Pick interpolation problem, Sarason's approach gives a more natural proof, however, to understand the theory, we have to have a background in operator theory.

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