

COHEN-MACAULAYNESS OF SOME EDGE-WEIGHTED GRAPHS

Ly Thi Kieu Diem and Nguyen Phung Nguyen

Students of Faculty of Mathematics, Hanoi National University of Education

Abstract. In this paper, we will study the characterization of Cohen-Macaulayness of some edge-weighted graphs. For cycle and tree edge-weighted graph, we will reprove the characterization of Cohen-Macaulayness of an edge-weighted cycle and an edge-weighted tree due to C.Paulsen and Wagstaff (2013) [1]. Our proof used a criterion of Hochster for Cohen-Macaulayness of a monomial ideal [2].

Keywords: Cohen-Macaulay, Edge-Weighted graphs, degree complexes.

1. Introduction

Let K be a field, and let R denote a polynomial ring $R = K[x_1, \dots, x_n]$. Let $G = (V(G), E(G))$ be a (finite simple) graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. An edge between vertices x_i and x_j is denoted $x_i x_j$.

An edge-weighted graph G_w is a graph in which each edge has been assigned a weight, i.e, a map $w : E(G) \rightarrow \mathbb{Z}_{>0}$ and the weight of an edge $x_i x_j$ is $w(x_i x_j)$, denote w_{ij} . Consider its “edge ideal” $I(G_w)$ which is an ideals in R generated by all monomials $(x_i x_j)^{w_{ij}}$ where $x_i x_j$ are edges of graph G_w . Then, a graph G_w is called Cohen-Macaulay if $R/I(G_w)$ is a Cohen-Macaulay ring. The purpose of this paper is to study the following question: When an edge-weighted graph G_w is Cohen-Macaulay?

This question fits into an ongoing research program to characterize of Cohen-Macaulayness of edge-weighted graphs. The answer to the general graph is still unknown. However, the complete answer to the above question for edge-weighted cycle and edge-weighted tree graphs had been given thanks to the work of C.Paulsen and Wagstaff in 2013. In their article [1], they had stated two following theorems:

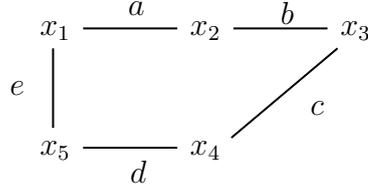
Theorem 1.1. *Consider a weighted d -cycle C_w^d .*

- (i) *If C_w^d is Cohen-Macaulay then $d \in \{3, 5\}$.*
- (ii) *C_w^3 is always Cohen-Macaulay.*

Received October 8, 2022. Revised: October 20, 2022. Accepted October 27, 2022.

Contact Ly Thi Kieu Diem, e-mail address: lykieu diem2001@gmail.com

(iii) C_w^5 is Cohen-Macaulay if and only if it can be written in the form



such that $a \leq b \geq c \leq d \geq e = a$.

Theorem 1.2. Let T_w be a weighted tree. Then T_w is Cohen-Macaulay, if and only if one of the following holds:

- (i) $|V(T_w)| \leq 2$, or
- (ii) T_w is a weighted suspension of a weighted tree H_w such that $w(x_i x_j) \leq w(x_i y_i)$ and $w(x_i x_j) \leq w(y_j x_j)$ for each $x_i x_j \in E(H)$, where y_k is the leaf vertex adjacent to x_k for $x_k \in V(H)$.

Our aim in this article is to approach a different way to those problems and reprove them. Now we explain the organization of the paper. In Section 2, we recall some notation and basic facts about monomial ideals and Hochster’s Theorem [2]. Section 3 is devoted to the proof of Theorem 1.1 Theorem 1.2 is proved in the last section.

2. Preliminaries

In this section, we recall some definitions and properties concerning monomial ideals and graph theory that will be used later. The interested reader is referred to [3] for more details.

Let K be a field and \mathbb{N} be the set of non-negative integers, consider the polynomial ring $K[x_1, \dots, x_n]$. An ideal $I \subseteq K[x_1, \dots, x_n]$ is a monomial ideal if there is a (possibly infinite) subset $A \subseteq \mathbb{N}^n$ such that I consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha \mathbf{x}^\alpha$, where $h_\alpha \in K[x_1, \dots, x_n]$. In this case, we write $I = \langle \mathbf{x}^\alpha \mid \alpha \in A \rangle$.

Lemma 2.1. Let $I = \langle \mathbf{x}^\alpha \mid \alpha \in A \rangle$ be a monomial ideal. Then a monomial \mathbf{x}^β lies in I if and only if \mathbf{x}^β is divisible by \mathbf{x}^α for some $\alpha \in A$.

This lemma is easy to check from the definition. We have known that all monomial ideals of $K[x_1, \dots, x_n]$ are finitely generated. Moreover, we have the following proposition:

Proposition 2.1. A monomial ideal $I \subseteq K[x_1, \dots, x_n]$ has a basis $\mathbf{x}^{\alpha(1)}, \dots, \mathbf{x}^{\alpha(s)}$ with the property that $\mathbf{x}^{\alpha(i)}$ does not divide $\mathbf{x}^{\alpha(j)}$ for $i \neq j$. Furthermore, this basis is unique and is called the minimal basis of I .

For J is a monomial ideal in $K[x_1, \dots, x_n]$. We define the radical ideal $\sqrt{J} : \mathbf{x}^{\mathbf{u}}$ as follow:

Definition 2.1. $\sqrt{J} : \mathbf{x}^{\mathbf{u}}$ is a monomial ideal which is generated by a set of monomial $v \in K[x_1, \dots, x_n]$ such that there exist a positive number m so that $v^m \mathbf{x}^{\mathbf{u}}$ lies in J .

Theorem 2.1 (see [2]). *Let J be a monomials ideal in $K[x_1, \dots, x_n]$. Then J is Cohen-Macaulay if and only if for all vector \mathbf{u} in \mathbb{N}^n , the radical ideal $\sqrt{J} : \mathbf{x}^{\mathbf{u}}$ is Cohen-Macaulay.*

Theorem 2.2. *Let J be a monomial ideal in $K[x_1, \dots, x_n]$ and let $\{v_1, \dots, v_t\}$ is the minimal basis of J then*

$$\sqrt{J} : \mathbf{x}^{\mathbf{u}} = \left\langle \sqrt{\frac{v_i}{\gcd(v_i, \mathbf{x}^{\mathbf{u}})}} \mid i = 1, 2, \dots, t \right\rangle.$$

Proof. For $i = 1, 2, \dots, n$, we denote

$$S_{\mathbf{u}}(v_i) = \frac{v_i}{\gcd(v_i, \mathbf{x}^{\mathbf{u}})}$$

Then, $S_{\mathbf{u}}(v_i) = x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ where $\alpha_j > 0$ for all $j = 1, 2, \dots, k$. Let $M_i = \max\{\alpha_j \mid 1 \leq j \leq k\}$. Thus, $(\sqrt{S_{\mathbf{u}}(v_i)})^{M_i} = x_{i_1}^{M_i} \cdots x_{i_k}^{M_i}$. Hence, $S_{\mathbf{u}}(v_i)$ divides $(\sqrt{S_{\mathbf{u}}(v_i)})^{M_i}$. This implies that $(\sqrt{S_{\mathbf{u}}(v_i)})^{M_i} \mathbf{x}^{\mathbf{u}}$ is divisible by v_i . By Lemma 2.1, $(\sqrt{S_{\mathbf{u}}(v_i)})^{M_i} \mathbf{x}^{\mathbf{u}}$ is in J . By the Definition 2.1, $(\sqrt{S_{\mathbf{u}}(v_i)})^{M_i}$ is a generator of $\sqrt{J} : \mathbf{x}^{\mathbf{u}}$. Therefore,

$$\sqrt{J} : \mathbf{x}^{\mathbf{u}} \supseteq \left\langle \sqrt{\frac{v_i}{\gcd(v_i, \mathbf{x}^{\mathbf{u}})}} \mid i = 1, 2, \dots, t \right\rangle.$$

In converse, for v is a generator of $\sqrt{J} : \mathbf{x}^{\mathbf{u}}$, then there exist a positive number m so that $v^m \mathbf{x}^{\mathbf{u}}$ lies in J . By Lemma 2.1, $v^m \mathbf{x}^{\mathbf{u}}$ is divisible by v_j for some j . Wlog, assume that $v^m \mathbf{x}^{\mathbf{u}}$ is divisible by v_1 . Thus, $S_{\mathbf{u}}(v_1)$ divides $v^m \mathbf{x}^{\mathbf{u}}$. Since $\gcd(S_{\mathbf{u}}(v_1), \mathbf{x}^{\mathbf{u}}) = 1$, we have $S_{\mathbf{u}}(v_1)$ divides v^m . Hence, $\sqrt{S_{\mathbf{u}}(v_1)}$ divides v and so $v \in \langle \sqrt{S_{\mathbf{u}}(v_j)} \mid j = 1, 2, \dots, t \rangle$. Therefore,

$$\sqrt{J} : \mathbf{x}^{\mathbf{u}} \subseteq \left\langle \sqrt{\frac{v_i}{\gcd(v_i, \mathbf{x}^{\mathbf{u}})}} \mid i = 1, 2, \dots, t \right\rangle.$$

□

For an edge-weighted graph G_w and $\mathbf{u} = [u_1, u_2, u_3, \dots, u_n] \in \mathbb{N}^n$. We set $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$. Let

$$S_{\mathbf{u}}(x_i x_j) = \sqrt{\frac{(x_i x_j)^{w_{ij}}}{\gcd((x_i x_j)^{w_{ij}}, \mathbf{x}^{\mathbf{u}})}}.$$

Thus we have the following lemma:

Lemma 2.2. *With the above notations, we have*

- (i) $S_{\mathbf{u}}(x_i x_j) = x_i x_j$ if and only if $u_i < w_{ij}$ and $u_i < w_{ij}$.
- (ii) $S_{\mathbf{u}}(x_i x_j) = x_i$ if and only if $u_i < w_{ij} \leq u_j$.
- (iii) $S_{\mathbf{u}}(x_i x_j) = 1$ if and only if $u_i \geq w_{ij}$ and $u_i \geq w_{ij}$.

The proof of Lemma 2.2 is straight forward. Edge-ideal $I(G_w)$ of an edge-weighted graph is a monomial ideal in $R = K[x_1, \dots, x_n]$. Then, we have the following lemma:

Lemma 2.3. *Let G_w be an edge-weighted graph. Then:*

$$\sqrt{I(G_w) : \mathbf{x}^{\mathbf{u}}} = I(G(w, \mathbf{u})) + \langle V_{\mathbf{u}} \rangle$$

where $V_{\mathbf{u}}$ is a set of variables and $G(w, \mathbf{u})$ is a subgraph of the induced graph of G on the vertex set $V(G) \setminus V_{\mathbf{u}}$.

Proof. Let $G[V \setminus V_{\mathbf{u}}]$ denote a graph with the vertex set $V(G) \setminus V_{\mathbf{u}}$ and the edge set $E' \subset E(G)$. By Theorem 2.2, $\sqrt{I(G_w) : \mathbf{x}^{\mathbf{u}}} = \langle S_{\mathbf{u}}(x_i x_j) \mid x_i x_j \in E(G) \rangle$. Hence, $\sqrt{I(G_w) : \mathbf{x}^{\mathbf{u}}}$ is generated by a set of square-free monomials. From this basis, we find the minimal basis of $\sqrt{I(G_w) : \mathbf{x}^{\mathbf{u}}}$ and let $V_{\mathbf{u}}$ be the set of all variables in the minimal basis, the remain form an edge ideal of a graph $G(w, \mathbf{u})$, an induced graph of $G[V \setminus V_{\mathbf{u}}]$. \square

Corollary 2.1. *G_w is Cohen-Macaulay if and only if $G(w, \mathbf{u})$ is Cohen-Macaulay for each \mathbf{u} in \mathbb{N}^n .*

Now we will state the theorem about the Cohen-Macaulayness of cycle and tree graph (non-weighted), which will be used later in this paper.

Theorem 2.3 (see [4]). *Let C^d be a cycle and T be a tree (both are not weighted). Then,*

- (i) C^d is Cohen-Macaulay if and only if $d \in \{3; 5\}$.
- (ii) T is Cohen-Macaulay if and only if it is a suspension of a tree.

Here suspension of a graph G is a graph H whose vertex set is $V(H) = V(G) \cup \{y_1, \dots, y_n\}$ and whose edge set is $E(H) = E(G) \cup \{x_1 y_1, \dots, x_n y_n\}$. In this case, y_1, \dots, y_n are called leaf vertex.

Example 2.1. *Let P_w^4 be a trivially weighted 4-path where each edge has weight a .*

$$x_1 \xrightarrow{a} x_2 \xrightarrow{a} x_3 \xrightarrow{a} x_4 \xrightarrow{a} x_5$$

Let $\mathbf{u} = 0$ in \mathbb{N}^5 , then $\sqrt{I(P_w^4) : \mathbf{x}^{\mathbf{u}}} = \langle x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5 \rangle$. In this case, the variables set $V_{\mathbf{u}} =$ and the induced graph is P^4 . We have known that a d -path graph P^d is Cohen-Macaulay if and only if $d \in \{1, 3\}$. Thus, P_w^4 is not Cohen-Macaulay.

Remark 2.1. *If for some u , for some $x_i x_j$ in $E(G)$, $S_{\mathbf{u}}(x_i x_j) = 1$ then $\sqrt{I(G_w)} : \mathbf{x}^{\mathbf{u}} = K[x_1, \dots, x_n]$. In this case, it is trivial that $G(w, \mathbf{u})$ is Cohen-Macaulay. Therefore, throughout the rest of this paper, we will only consider vector $\mathbf{u} \in \mathbb{N}^n$ such that $S_{\mathbf{u}}(x_i x_j) \neq 1$ for all $x_i x_j \in E(G)$.*

3. Edge-weighted cycle

In this section, we prove Theorem 1.1 from the introduction.

Proposition 3.1. *If C_w^d is Cohen-Macaulay then $d \in \{3, 5\}$.*

Proof. Assume that C_w^d is Cohen-Macaulay. By Corollary 2.1, we know that $C^d(w, \mathbf{u})$ is Cohen-Macaulay for all $u \in N^d$. So $C^d(w, \mathbf{0}) = C^d$ implies that the unweighted cycle C^d is Cohen-Macaulay. Thus, we have $d \in \{3, 5\}$ by Theorem 2.3. \square

Proposition 3.2. *C^3 is always Cohen-Macaulay.*

Proof. We list cases of $C^3(w, \mathbf{u})$.

Case 3.1. *$C^3(w, \mathbf{u})$ is a null graph.*

Case 3.2. *$C^3(w, \mathbf{u})$ is a path of length 1.*

Case 3.3. *$C^3(w, \mathbf{u})$ is 3-cycle.*

Thus $C^3(w, \mathbf{u})$ is always Cohen-Macaulay. Therefore, by Theorem 2.3, C_w^3 is always Cohen-Macaulay. \square

Proposition 3.3. *C_w^5 is not Cohen-Macaulay if and only if $V_{\mathbf{u}}$ consists of two adjacent vertices of C_w^5 for some $u \in \mathbb{N}^5$.*

Proof. We list all the cases of $C^5(w, \mathbf{u})$.

Case 3.4. *$C^5(w, \mathbf{u})$ is a null graph.*

Case 3.5. *$C^5(w, \mathbf{u})$ is a path of length 1.*

Case 3.6. *$C^5(w, \mathbf{u})$ is a path of length 2.*

Case 3.7. *$C^5(w, \mathbf{u})$ is a path of length 3.*

Case 3.8. *$C^5(w, \mathbf{u})$ is 5-cycle.*

Thus $C^5(w, \mathbf{u})$ is not Cohen-Macaulay if and only if $C^5(w, \mathbf{u})$ has the form in Case 3.6. Since $C^5(w, \mathbf{u})$ is a path of length 2 if and only if $V_{\mathbf{u}}$ consists of two adjacent vertices of C_w^5 , we have C_w^5 is not Cohen-Macaulay if and only if $V_{\mathbf{u}}$ consists of two adjacent vertices of C_w^5 . \square

Proposition 3.4. *Assume $C^5(w, \mathbf{u})$ is not a null graph, $x_i \notin V_{\mathbf{u}}$, $x_j \in V_{\mathbf{u}}$ and x_i adjacent to x_j .*

(i) If $u_i < w_{ij}$, then $u_j < w_{ij}$.

(ii) If $u_j \geq w_{ij}$, then $u_i < w_{ij}$.

Proof. (i) If $u_i < w_{ij} \leq u_j$ then $S_{\mathbf{u}}(x_i x_j) = x_i$. So $x_i \in V_{\mathbf{u}}$, which is a contradiction.

(ii) If $u_j \geq w_{ij}$ and $u_i \geq w_{ij}$ then $S_{\mathbf{u}}(x_i x_j) = 1$. So $\sqrt{I(C_w^5)} : \mathbf{x}^{\mathbf{u}} = K[x_1, x_2, x_3, x_4, x_5]$. Thus $C_w^5(w, \mathbf{u})$ is a null graph, which is a contradiction. \square

Proposition 3.5. Let C_w^5 be a Cohen-Macaulay graph. Consider a path of length 3 of C_w^5 has the following form

$$x_1 \xrightarrow{a} x_2 \xrightarrow{b} x_3 \xrightarrow{c} x_4$$

Then

(i) If $a \geq b$, then $a \geq b \leq c$ or $a = b > c$.

(ii) If $a \leq b$, then $a \leq b \geq c$ or $a = b < c$.

Proof. We will prove that if $a < b < c$ or $a > b > c$ then there exists $u \in N^5$ such that $C_w^5(w, \mathbf{u})$ has the form in Case 3.6 (see Proposition 3.3), so C_w^5 is not Cohen-Macaulay.

Case 3.9. If $a > b > c$. Choose $\mathbf{x}^{\mathbf{u}} = x_2^b x_3^c$, we get

$$\sqrt{I(C_w^5)} : \mathbf{x}^{\mathbf{u}} = \langle x_1 x_2, x_3, x_4, x_4 x_5, x_5 x_1 \rangle = \langle x_3, x_4 \rangle + \langle x_1 x_2, x_5 x_1 \rangle.$$

So that $C_w^5(w, \mathbf{u})$ has the form in Case 3.6.

Case 3.10. If $a < b < c$. Chose $\mathbf{x}^{\mathbf{u}} = x_2^a x_3^b$, we get

$$\sqrt{I(C_w^5)} : \mathbf{x}^{\mathbf{u}} = \langle x_1, x_2, x_3 x_4, x_4 x_5, x_5 x_1 \rangle = \langle x_1, x_2 \rangle + \langle x_3 x_4, x_4 x_5 \rangle.$$

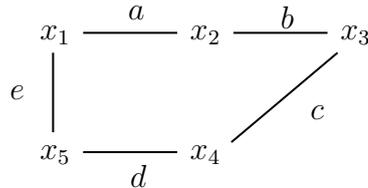
So that $C_w^5(w, \mathbf{u})$ has the form in Case 3.6. \square

Proof of Theorem 1.1.

(i) This is Proposition 3.1.

(ii) This is Proposition 3.2.

(iii) We prove that if C_w^5 is Cohen-Macaulay then it can be written in the form



such that $a \leq b \geq c \leq d \geq e = a$.

If all edges of C_w^5 have the same weight then it fits our hypothesized conclusion. Otherwise, by symmetry, we may assume that $a < b$. By Proposition 3.5, we get $b \geq c$. Then, we consider two cases as follows:

Case 3.11. $b \geq c \leq d$. By Proposition 3.5, we consider two following cases:

Case 3.11.1. $c \leq d \geq e$.

If $d \geq e$, then $e \leq a$. On the other hand, $a < b$ implies $e \geq a$. Therefore, $e = a$ and $a \leq b \geq c \leq d \geq e = a$.

If $d = e$, then $e \geq a$ (since $a < b$). We have $a < b \geq c \leq d = e \geq a$. If $e > a$, choose $\mathbf{x}^u = x_1^a x_4^c$, we get

$$\sqrt{J : \mathbf{x}^u} = \langle x_2, x_3 \rangle + \langle x_4 x_5, x_5 x_1 \rangle.$$

Thus, $C^5(w, \mathbf{u})$ is not Cohen-Macaulay. So $e = a < b \geq c \leq d = e$.

Case 3.11.2. $c = d > e$.

Similar to the Case 3.11.1, we have $e = a$ so that $e = a < b \geq c = d > e$.

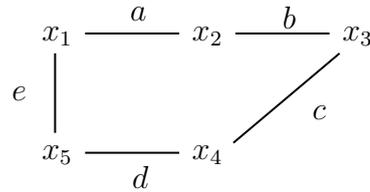
Case 3.12. $b = c > d$.

By Proposition 3.5, we have $d \leq e$, so $a < b = c > d \leq e$. Choose $\mathbf{x}^u = x_2^a x_4^d$, we get

$$\sqrt{J : \mathbf{x}^u} = \langle x_1, x_5 \rangle + \langle x_2 x_3, x_3 x_4 \rangle.$$

Thus $C^5(w, \mathbf{u})$ is not Cohen-Macaulay.

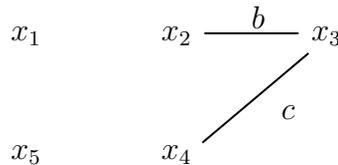
In conclusion, If C_w^5 is Cohen-Macaulay then it can be written in the form we mentioned in (c). Conversely, we prove that a weighted 5-cycle has the form



such that $a \leq b \geq c \leq d \geq e = a$ is Cohen-Macaulay.

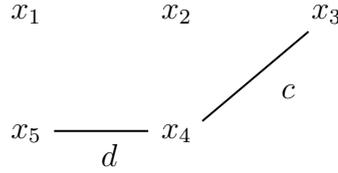
By Proposition 3.3, there are 5 cases of $V_{\mathbf{u}}$ that $C^5(w, \mathbf{u})$ is not Cohen-Macaulay as follow:

Case 3.13.



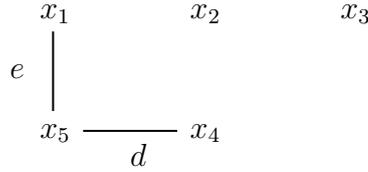
Assume that $V_{\mathbf{u}} = \{x_1, x_5\}$. Then $\{x_2x_3, x_3x_4\} = E(C^5(w, \mathbf{u}))$, it implies that $S_{\mathbf{u}}(x_2x_3) = x_2x_3, S_{\mathbf{u}}(x_3x_4) = x_3x_4$. So $u_3, u_4 < c; u_2, u_3 < b$. Note that $c \leq d$, so $u_4 < d$. Thus, by Proposition 3.4, we have $u_5 < d$. Then, since $x_5 \in V_{\mathbf{u}}$, we conclude that $S_{\mathbf{u}}(x_1x_5) = x_5$, so $u_5 < a \leq u_1$. Thus $u_2 < a$ by Proposition 3.4. So $S_{\mathbf{u}}(x_1x_2) = x_2$. It follows that $x_2 \in V_{\mathbf{u}}$, which contradicts the assumption.

Case 3.14.



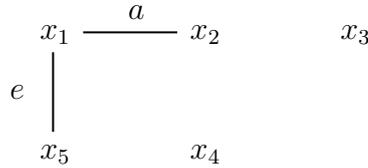
Assume that $V_{\mathbf{u}} = \{x_1, x_2\}$. Then $\{x_3x_4, x_4x_5\} = E(C^5(w, \mathbf{u}))$, it implies that $S_{\mathbf{u}}(x_3x_4) = x_3x_4, S_{\mathbf{u}}(x_4x_5) = x_4x_5$. So $u_3, u_4 < c; u_4, u_5 < d$. Note that $c \leq b$, so $u_3 < b$. Thus, by Proposition 3.4, we have $u_2 < b$. Then, since $x_2 \in V_{\mathbf{u}}$, we conclude that $S_{\mathbf{u}}(x_1x_2) = x_2$, so $u_2 < a \leq u_1$. Thus $u_5 < a$ by Proposition 3.4. So $S_{\mathbf{u}}(x_1x_5) = x_5$. It follows that $x_5 \in V_{\mathbf{u}}$, which contradicts the assumption.

Case 3.15.



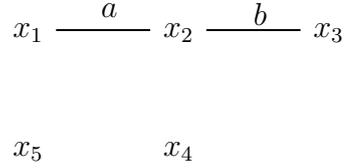
Assume that $V_{\mathbf{u}} = \{x_2, x_3\}$. Then $\{x_1x_5, x_4x_5\} = E(C^5(w, \mathbf{u}))$, it implies that $S_{\mathbf{u}}(x_1x_5) = x_1x_5, S_{\mathbf{u}}(x_4x_5) = x_4x_5$. So $u_5, u_1 < a; u_4, u_5 < d$. Thus, by Proposition 3.4, we have $u_2 < a$. Then, since $x_2 \in V_{\mathbf{u}}$, we conclude that $S_{\mathbf{u}}(x_2x_3) = x_2$, so $u_2 < b \leq u_3$. Note that $c \leq d$, so $c \leq u_3$. Thus $u_4 < c$ by Proposition 3.4. So $S_{\mathbf{u}}(x_4x_5) = x_5$. It follows that $x_5 \in V_{\mathbf{u}}$, which contradicts the assumption.

Case 3.16.



Assume that $V_{\mathbf{u}} = \{x_3, x_4\}$. Then $\{x_1x_2, x_1x_5\} = E(C^5(w, \mathbf{u}))$, it implies that $S_{\mathbf{u}}(x_1x_2) = x_1x_2, S_{\mathbf{u}}(x_1x_5) = x_1x_5$. So $u_1, u_2, u_5 < a$. Note that $a \leq b$, so $u_2 < b$. Thus, by Proposition 3.4, we have $u_3 < b$. Then, since $x_3 \in V_{\mathbf{u}}$, we conclude that $S_{\mathbf{u}}(x_3x_4) = x_3$, so $u_3 < c \leq u_4$. Thus $u_5 < c$ by Proposition 3.4. So $S_{\mathbf{u}}(x_4x_5) = x_5$. It follows that $x_5 \in V_{\mathbf{u}}$, which contradicts the assumption.

Case 3.17.



Assume that $V_{\mathbf{u}} = \{x_4, x_5\}$. Then $\{x_2x_3, x_1x_2\} = E(C^5(w, \mathbf{u}))$, it implies that $S_{\mathbf{u}}(x_2x_3) = x_2x_3, S_{\mathbf{u}}(x_1x_2) = x_1x_2$. So $u_1, u_2 < a; u_2, u_3 < b$. Thus, by Proposition 3.4, $u_5 < a$. Then, since $x_5 \in V_{\mathbf{u}}$, we conclude that $S_{\mathbf{u}}(x_4x_5) = x_5$, so $u_5 < d \leq u_4$. Note that $d \geq c$, so $u_4 \geq c$. Thus $u_3 < c$ by Proposition 3.4. So $S_{\mathbf{u}}(x_3x_4) = x_3$. It follows that $x_3 \in V_{\mathbf{u}}$, which contradicts the assumption.

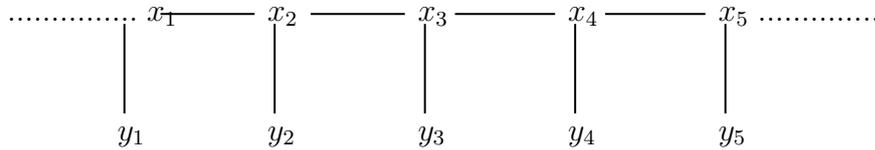
It follows that C_w^5 is Cohen-Macaulay, as claimed. □

The above theorem provides the explicit characterization of the Cohen-Macaulay weighted cycle.

4. Edge-weighted tree

First, we recall the definition of suspension of a graph. Let G be a graph with vertex set is $V(G) = \{x_1, \dots, x_n\}$. A suspension of G is a graph H whose vertex set is $V(H) = V(G) \cup \{y_1, \dots, y_n\}$ and whose edge set is $E(H) = E(G) \cup \{x_1y_1, \dots, x_ny_n\}$. In this case, y_1, \dots, y_n are called leaf vertex.

Graphically, this says that H has the form



where the above "row" is the graph G . We also know that for T is a tree, then T is Cohen-Macaulay if and only if T is a suspension of a tree.

Definition 4.1. Let G_w and H_λ be weighted graphs. Then H_λ is a weighted suspension of G_w if the underlying graph H is a suspension of G and we have $\lambda(x_ix_j) = w(x_ix_j)$ for all $x_ix_j \in E(G)$.

Remark 4.1. If the weighted tree T_w is Cohen-Macaulay, then the underlying tree T is Cohen-Macaulay, hence T is a suspension of a tree.

By Corollary 2.1, it is clear that when T_w is Cohen-Macaulay, then $T(w, \mathbf{0})$ is Cohen-Macaulay. In other words, the underlying tree T is Cohen-Macaulay, hence T is a suspension of a tree. Let H be the graph which T is a suspension of.

Since T is a suspension of H , $|V(T)|$ is even. Let $|V(T)| = 2m$ and $V(H) = \{x_1, \dots, x_m\}$. Then $V(T) = V(H) \cup \{y_1, \dots, y_m\}$ where y_k are leaf vertex. Here, for each

$$\mathbf{u} = [u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m] \in \mathbb{N}^{2m},$$

we set

$$\mathbf{x}^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \dots x_m^{u_m} y_1^{v_1} y_2^{v_2} \dots y_m^{v_m}.$$

Remark 4.2. *If the weighted tree T_w is Cohen-Macaulay, then for all vector \mathbf{u} , $T(w, \mathbf{u})$ is a suspension of a forest.*

Note that $T(w, \mathbf{u})$ is a forest. Moreover, by Corollary 2.1, $T(w, \mathbf{u})$ is Cohen-Macaulay, hence $T(w, \mathbf{u})$ is a suspension of a forest.

Theorem 4.1. *Let T_w be a weighted tree. Then T_w is Cohen-Macaulay, if and only if one of the following holds:*

- (i) $|V(T_w)| \leq 2$ or;
- (ii) T_w is a weighted suspension of a weighted tree H_w such that $w(x_i x_j) \leq w(x_i y_i)$ and $w(x_i x_j) \leq w(y_j x_j)$ for each $x_i x_j \in E(H)$, where y_k are the leaf vertex adjacent to x_k .

Proof. Let J be the edge ideal of T_w . Consider the case when $|V(T_w)| \leq 2$, then $T(w, \mathbf{u})$ will always Cohen-Macaulay. Indeed, if $|V(T_w)| = 1$, then it is Cohen-Macaulay. If $|V(T_w)| = 2$, then for each $\mathbf{u} \in \mathbb{N}^2$, $T(w, \mathbf{u})$ can be null graph, a vertex or a P^1 . In every situation, $T(w, \mathbf{u})$ is Cohen-Macaulay.

In the case when $|V(T_w)| > 2$, assume T_w is Cohen-Macaulay. With the notation as in Remark 4.1, suppose there exist a vertex x_{i^*} adjacent to x_i such that $w(x_i x_{i^*}) > w(x_i y_i)$. In this case, let $\mathbf{x}^{\mathbf{u}} = x_i^{w(x_i y_i)}$. Let $X_i = \{x_{i_1}, \dots, x_{i_r}, y_i \mid w(x_i x_{i_k}) \leq w(x_i y_i) \text{ for all } 1 \leq k \leq r\}$. We have the following claim:

Claim 4.1. The set $V = \{x_t, y_t \mid x_t \notin X_i, t \neq i\} \cup \{x_i\}$ is the vertex set of $T(w, \mathbf{u})$.

Proof of Claim 4.1. In one hand, we have $S_{\mathbf{u}}(x_i y_i) = y_i$ and $S_{\mathbf{u}}(x_i x_{i_k}) = x_{i_k}$ for all $k = 1, \dots, r$. Thus, $X_i \subseteq V_{\mathbf{u}}$. Moreover, since $x_{i_k} \in V_{\mathbf{u}}$, y_{i_k} is not a vertex of $T(w, \mathbf{u})$.

On the other hand, suppose that there exists a vertex x_t is in $V_{\mathbf{u}}$, $x_t \notin X_i$. Hence, there exist an adjacent vertex x_j of x_t such that $S_{\mathbf{u}}(x_t x_j) = x_t$. By Lemma 2.2, $u_t < w_{tj} \leq u_j$. If $t \neq i$ then $0 < w_{tj} \leq 0$. If $t = i$ then $w(x_i y_i) < w(x_i x_j) \leq 0$. Both cases are contradiction since the weight must be positive. Therefore, $x_t \notin V_{\mathbf{u}}$ for all $x_t \notin X_i$. Moreover, $S_{\mathbf{u}}(x_t y_t) = x_t y_t$ for all $x_t \notin X_i, t \neq i$, hence y_t is not in $V_{\mathbf{u}}$ and $x_t y_t \in E(T(w, \mathbf{u}))$ which means x_t, y_t are vertex of $T(w, \mathbf{u})$. We also have that $S_{\mathbf{u}}(x_i x_{i^*}) = x_i x_{i^*}$. Since $x_i, x_{i^*} \notin V_{\mathbf{u}}$, $x_i x_{i^*}$ is an edge in $T(w, \mathbf{u})$ which implies that x_i, x_{i^*} is a vertex of $T(w, \mathbf{u})$. Therefore, the set $V = \{x_t, y_t \mid x_t \notin X_i, t \neq i\} \cup \{x_i\}$ is the vertex set of $T(w, \mathbf{u})$.

By this claim, we have that $|V(T(w, \mathbf{u}))| = 2n - 2r + 1$. Thus, $T(w, \mathbf{u})$ can not be a suspension of a forest. This contradicts Remark 4.2.

Conversely, assume T_w is a weighted graph satisfies (ii). We have the following claim:

Claim 4.2. For all vector \mathbf{u} , for all vertex x_i in H , $x_i \in V(T(w, \mathbf{u}))$ if and only if $y_i \in V(T(w, \mathbf{u}))$.

Proof of Claim 4.2. Since y_i is the leaf vertex which adjacent to x_i , the adjacent vertex to y_i in $T(w, \mathbf{u})$ must be x_i . Thus, if $y_i \in V(T(w, \mathbf{u}))$, then $x_i \in V(T(w, \mathbf{u}))$. Conversely, if $x_i \in V(T(w, \mathbf{u}))$, then there exists j such that $x_i x_j \in E(T(w, \mathbf{u}))$. Thus, $S_{\mathbf{u}}(x_i x_j) = x_i x_j$. By Lemma 2.2, we have $w(x_i x_j) > \max\{u_i, u_j\}$. We also have $w(x_i y_i) \geq w(x_i x_j)$, then $w(x_i y_i) > u_i$. If $w(x_i y_i) \leq v_i$ then $S_{\mathbf{u}}(x_i y_i) = x_i$. This is a contradiction since $x_i \in V(T(w, \mathbf{u}))$. Hence, $w(x_i y_i) > v_i$ and so $S_{\mathbf{u}}(x_i y_i) = x_i y_i$. Therefore, $y_i \in V(T(w, \mathbf{u}))$.

We note that for every vertex of $T(w, \mathbf{u})$, its degree in $T(w, \mathbf{u})$ is less than or equal to its degree in T . Let x_i be a vertex in $T(w, \mathbf{u})$ such that the degree of x_i in $T(w, \mathbf{u})$ is greater than 1. By Claim 4.2, we have $y_i \in V(T(w, \mathbf{u}))$. Suppose there exists another vertex of degree 1 in $T(w, \mathbf{u})$ adjacent to x_i . Thus, this vertex has a degree greater than 1 in T . Suppose x_j is that vertex, then by Claim 4.2, we have y_j as a vertex of $T(w, \mathbf{u})$. Then, x_j is adjacent to both x_i and y_j . This is a contradiction.

It follows that T_w is Cohen-Macaulay, as claimed. \square

From here, we can now give the explicit characterization of the Cohen-Macaulay weighted Tree.

REFERENCES

- [1] C. Paulsen and S. Sather-Wagstaff, 2013. Edge ideals of weighted graphs. *J. Algebra Appl.*, Vol. 12, No. 5, 1250223, pp. 24.
- [2] M. Hochster, 1977. *Cohen-Macaulay rings, combinatorics, and simplicial complexes*. Proc. 2nd Oklahoma Ring Theory Conf. (B. McDonald and R. Morris, ed.), Dekker, New York, pp. 171-223.
- [3] E. Miller and B. Sturmfels, 2004. *Combinatorial commutative Algebra*. Springer, Berlin/Heidelberg/New York/Tokyo.
- [4] R. H. Villarreal, 2015. *Monomial algebras*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, second edition.