HNUE JOURNAL OF SCIENCE DOI: 10.18173/2354-1059.2023-0017
Natural Science, 2023, Volume 68, Issue 2, pp. 13-23
This paper is available online at http://stdb.hnue.edu.vn

A NOTE ON L^1-L^1 ESTIMATES FOR SOLUTIONS TO VISCO-ELASTIC DAMPED σ-EVOLUTION MODELS

Dao Tuan Anh[1] and Ha Duc Thai[2]
[1]*School of Applied Mathematics and Informatics,*
Hanoi University of Science and Technology
[2]*Institute of Mathematics, Vietnam Academy of Science and Technology*

Abstract. This note is to conclude some $L^1 - L^1$ estimates for solutions to the following Cauchy problem for visco-elastic damped σ-evolution models:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\sigma u_t = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{0.1}$$

where $\sigma > 1$, in all space dimensions $n \geq 1$. The novelty of the paper is that we have succeeded in deriving the $L^1 - L^1$ estimates for solutions to (0.1) in general cases of $\sigma > 1$ without any constraint condition to space dimensions, i.e. our results are valid for any $n \geq 1$. To establish this, the application of the theory of modified Bessel functions linked to Faà di Bruno's formula comes into play for small frequencies, meanwhile, for both large frequencies and middle frequencies we will rigorously carry out some useful estimates to obtain the exponential decay.
Keywords: L^1 estimates, σ-evolution models, visco-elastic damping.

1. Introduction

During the last decades, the Cauchy problem for the σ-evolution equations has achieved great attention from many authors (for example, see [1-7]) because of their wide applications in various disciplines such as physics, chemistry, mechanics and so on. One of the most typical important equations is the wave model, i.e. $\sigma = 1$, which not only arises in many fields of applied sciences including acoustics, electromagnetics, and fluid dynamics but also describes mechanical waves (e.g. vibrating string with $n = 1$, vibrating membrane with $n = 2$, or vibrating elastic solid with $n = 3$) as well as light waves. Considering $\sigma = 2$ we recall the fourth-order evolution equations concerning problems of solid mechanics (e.g. beams with $n = 1$ or thin plates with $n = 2$) and vibration of

Received April 26, 2023. Revised June 20, 2023. Accepted June 27, 2023.
Contact Dao Tuan Anh, e-mail address: anh.daotuan@hust.edu.vn

an elastic surface. At present, there has been very little work on the question of getting $L^1 - L^1$ estimates for solutions to (0.1). Back in 2000, let us first recall the pioneering paper of Shibata [8] devoting to the study of one of the most well-known equations of (0.1) in the case $\sigma = 1$, the so-called visco-elastic damped wave equation. In the cited paper, relying on the very special structure of fundamental solutions to the wave equation he succeeded in obtaining the following $L^1 - L^1$ estimates:

$$\|u(t,\cdot)\|_{L^1} \lesssim \begin{cases} (1+t)^{\frac{n}{4}}\|u_0\|_{L^1} + (1+t)^{\frac{n+2}{4}}\|u_1\|_{L^1} & \text{if } n \geq 2 \text{ is even,} \\ (1+t)^{\frac{n-1}{4}}\|u_0\|_{L^1} + (1+t)^{\frac{n+1}{4}}\|u_1\|_{L^1} & \text{if } n \geq 3 \text{ is odd,} \end{cases}$$

by taking into consideration the connection to Fourier multipliers appearing for wave models. Quite recently, regarding a different interesting model of (0.1), namely that with $\sigma = 2$, D'Abbicco and his collaborators [9] have employed a different technique in comparison with that used in [8] to derive $L^1 - L^1$ estimates for solutions to the strongly damped plate equation as follows:

$$\|u(t,\cdot)\|_{L^1} \lesssim (1+t)^{\frac{n}{4}}\|u_0\|_{L^1} + (1+t)^{\frac{n+2}{4}}\|u_1\|_{L^1} \qquad \text{for any } n \geq 5.$$

Here this limitation of space dimensions comes from the technical difficulty. More precisely, the authors used Bernstein inequality to estimate Fourier multipliers for small frequencies. To apply this technique, it is necessary to require the above restriction to space dimensions n as well as the assumption of integer number for σ. Independently from the above mentioned results, Dao-Reissig in the recent paper [10] has considered the more general cases of (0.1) for any $\sigma > 1$ to achieve $L^1 - L^1$ estimates for solutions localized to small frequencies by applying the theory of modified Bessel functions linked to Faà di Bruno's formula. Unfortunately, this strategy fails in the treatment of large frequencies. For this reason, the present paper is to fill this lack and report some $L^1 - L^1$ estimates for solutions to (0.1) as well.

In order to state our main result, we introduce the following notations used in this paper:

- We write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leq Cg$, and $f \sim g$ when $g \lesssim f \lesssim g$.
- The spaces H^a_1 and $\dot{H}^a_1$, with $a \geq 0$, stand for Bessel and Riesz potential spaces based on L^1 spaces, where $\big\langle D \big\rangle^a$ and $|D|^a$ denote the pseudo-differential operators with symbols $\big\langle \xi \big\rangle^a$ and $|\xi|^a$, respectively.
- For a given number $s \in \mathbb{R}$, we denote $[s] := \max\big\{k \in \mathbb{Z} \ : \ k \leq s\big\}$ and $[s]^+ := \max\{s, 0\}$ as its integer part and its positive part, respectively.

The main purpose of this note is to prove the following result.

Theorem 1.1 (Main result). *Let $\sigma > 1$. Then, the Sobolev solutions to* (0.1) *satisfy the following $L^1 - L^1$ estimates:*

$$\begin{aligned}\big\||D|^a u(t,\cdot)\big\|_{L^1} &\lesssim (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_0\|_{H^a_1} \\ &\qquad + (1+t)^{1+\frac{1}{2}(1+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_1\|_{H_1^{[a-\sigma]^+}}, \\ \big\||D|^a u_t(t,\cdot)\big\|_{L^1} &\lesssim (1+t)^{\frac{1}{2}(1+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_0\|_{H_1^{2\sigma+[a-\sigma]^+}} \\ &\qquad + (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_1\|_{H_1^{2\sigma+[a-\sigma]^+}},\end{aligned}$$

where $a \geq 0$ and for all space dimensions $n \geq 1$.

Remark 1.1. Here we want to underline that at first glance, the decay estimates for solutions produced from the results of [8] or [9] are somehow better than those of Theorem 1.1 when we choose $\sigma = 1$ formally or $\sigma = 2$, respectively. The fact is that this comes from the very special structure of solutions to (0.1) in the cases of $\sigma = 1$ and $\sigma = 2$ coupled with some used techniques under the suitable restriction to space dimensions. However, these obtained results from Theorem 1.1 imply an important recognition that we may conclude the desired $L^1 - L^1$ estimates for solutions to (0.1) in general cases of $\sigma > 1$ without any constraint condition to space dimensions, i.e. our main result is valid for any $n \geq 1$.

2. Proof of main result

At first, using partial Fourier transformation to (0.1) we obtain the Cauchy problem for $\widehat{u}(t,\xi) := \mathfrak{F}(u(t,x))$, $\widehat{u}_0(\xi) := \mathfrak{F}(u_0(x))$ and $\widehat{u}_1(\xi) := \mathfrak{F}(u_1(x))$ as follows:

$$\widehat{u}_{tt} + |\xi|^{2\sigma}\widehat{u}_t + |\xi|^{2\sigma}\widehat{u} = 0, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi), \quad \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi). \tag{2.1}$$

The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-|\xi|^{2\sigma} \pm \sqrt{|\xi|^{4\sigma} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (2.1) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\begin{aligned}\widehat{u}(t,\xi) &= \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u}_0(\xi) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{u}_1(\xi) \\ &=: \widehat{\mathcal{K}}_0(t,\xi)\widehat{u}_0(\xi) + \widehat{\mathcal{K}}_1(t,\xi)\widehat{u}_1(\xi).\end{aligned}$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_{1,2} = \lambda_{1,2}(\xi) = -\dfrac{1}{2}\Big(|\xi|^{2\sigma} \mp i\sqrt{4|\xi|^{2\sigma} - |\xi|^{4\sigma}}\Big)$
   and $\lambda_{1,2} \sim -|\xi|^{2\sigma} \pm i|\xi|^{\sigma}, \quad \lambda_1 - \lambda_2 \sim i|\xi|^{\sigma} \quad$ for $|\xi| \in (0, 4^{-\frac{1}{\sigma}})$,

2. $\lambda_{1,2} = \lambda_{1,2}(\xi) = -\frac{1}{2}\Big(|\xi|^{2\sigma} \mp \sqrt{|\xi|^{4\sigma} - 4|\xi|^{2\sigma}}\Big)$
and $\lambda_1 \sim -1, \quad \lambda_2 \sim -|\xi|^{2\sigma}, \quad \lambda_1 - \lambda_2 \sim |\xi|^{2\sigma} \quad$ for $|\xi| \in (4^{\frac{1}{\sigma}}, \infty)$.

Let $\chi_k = \chi_k(|\xi|)$ with $k = 1, 2, 3$ be smooth cut-off functions having the following properties:

$$\chi_1(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \leq 4^{-\frac{1}{\sigma}}, \\ 0 & \text{if } |\xi| \geq 3^{-\frac{1}{\sigma}}, \end{cases} \qquad \chi_3(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \geq 4^{\frac{1}{\sigma}}, \\ 0 & \text{if } |\xi| \leq 3^{\frac{1}{\sigma}}, \end{cases}$$

and $\chi_2(|\xi|) = 1 - \chi_1(|\xi|) - \chi_3(|\xi|)$.

We note that $\chi_2(|\xi|) = 1$ if $3^{-\frac{1}{\sigma}} \leq |\xi| \leq 3^{\frac{1}{\sigma}}$ and $\chi_2(|\xi|) = 0$ if $|\xi| \leq 4^{-\frac{1}{\sigma}}$ or $|\xi| \geq 4^{\frac{1}{\sigma}}$. Let us now decompose the solutions to (0.1) into three parts localized individually to small, middle and large frequencies, that is,

$$u(t,x) = u_{\chi_1}(t,x) + u_{\chi_2}(t,x) + u_{\chi_3}(t,x),$$

where

$$u_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\widehat{u}(t,\xi)\big) \quad \text{with } k = 1, 2, 3.$$

Here $\mathfrak{F}^{-1}$ stands for the inverse Fourier transform. For this reason, we shall divide our considerations into three cases as follows.

2.1. Estimates for small frequencies

We follow the statements from Corollary 3.3 in the paper of Dao-Reissig [10] to obtain the following estimates for small frequencies.

Proposition 2.1. *Let $\sigma > 1$, $n \geq 1$ and $a \geq 0$. The Sobolev solutions to ((0.1)) satisfy the $L^1 - L^1$ estimates*

$$\big\||D|^a u_{\chi_1}(t,\cdot)\big\|_{L^1} \lesssim (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_0\|_{L^1} + (1+t)^{1+\frac{1}{2}(1+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_1\|_{L^1},$$
$$\big\||D|^a \partial_t u_{\chi_1}(t,\cdot)\big\|_{L^1} \lesssim (1+t)^{\frac{1}{2}(1+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_1\|_{L^1}.$$

2.2. Estimates for large frequencies

First, let us represent the characteristic roots in the form

$$\lambda_1(\xi) = -1 - \phi(\xi) \text{ and } \lambda_2(\xi) = -|\xi|^{2\sigma} + 1 + \phi(\xi), \tag{2.2}$$

where

$$\phi(\xi) = -1 + \int_0^1 \Big(1 - \frac{4}{|\xi|^{2\sigma}}s\Big)^{-\frac{1}{2}} ds. \tag{2.3}$$

For the sake of transparent representation for large frequencies, we introduce the following four notations:

$$\begin{aligned}
\mathcal{K}^1_{u_0}(t,x) &:= \mathfrak{F}^{-1}\Big(\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u}_0(\xi)\chi_3(|\xi|)\Big)(t,x),\\
\mathcal{K}^2_{u_0}(t,x) &:= \mathfrak{F}^{-1}\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u}_0(\xi)\chi_3(|\xi|)\Big)(t,x),\\
\mathcal{K}^1_{u_1}(t,x) &:= \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_1(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u}_1(\xi)\chi_3(|\xi|)\Big)(t,x),\\
\mathcal{K}^2_{u_1}(t,x) &:= \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_2(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u}_1(\xi)\chi_3(|\xi|)\Big)(t,x).
\end{aligned}$$

Then, our main goal in this section is to show the following assertions.

Lemma 2.1. *Let $\sigma > 1$ and $n \geq 1$. Let $a \geq 0$. The following estimates hold:*

$$\begin{aligned}
\big\|\partial_t^j|D|^a\mathcal{K}^1_{u_0}(t,\cdot)\big\|_{L^1} &\lesssim e^{-ct}\|u_0\|_{H^a_1},\\
\big\|\partial_t^j|D|^a\mathcal{K}^2_{u_0}(t,\cdot)\big\|_{L^1} &\lesssim e^{-ct}\|u_0\|_{H_1^{2\sigma j+[a-\sigma]^+}},\\
\big\|\partial_t^j|D|^a\mathcal{K}^1_{u_1}(t,\cdot)\big\|_{L^1} &\lesssim e^{-ct}\|u_1\|_{H_1^{[a-\sigma]^+}},\\
\big\|\partial_t^j|D|^a\mathcal{K}^2_{u_1}(t,\cdot)\big\|_{L^1} &\lesssim e^{-ct}\|u_1\|_{H_1^{2\sigma j+[a-\sigma]^+}},
\end{aligned}$$

where c is a suitable positive constant and for all integer numbers $j \geq 0$.

In order to prove Lemma 2.1, let us recall the following auxiliary estimates from Lemma 3.5 in [10].

Lemma 2.2. *The following estimates hold in $\mathbb{R}^n$ for sufficiently large $|\xi|$:*

$$\big|\partial_\xi^\alpha|\xi|^{2p\sigma}\big| \lesssim |\xi|^{2p\sigma-|\alpha|}, \tag{2.4}$$

$$\big|\partial_\xi^\alpha\phi(\xi)\big| \lesssim |\xi|^{-2\sigma-|\alpha|}, \tag{2.5}$$

$$\left|\partial_\xi^\alpha\left(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\right)\right| \lesssim e^{-ct}|\xi|^{2\sigma j+b-2\sigma-|\alpha|} \tag{2.6}$$

$$\left|\partial_\xi^\alpha\left(\frac{e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\right)\right| \lesssim e^{-ct}|\xi|^{2\sigma j+b-2\sigma-|\alpha|} \tag{2.7}$$

$$\left|\partial_\xi^\alpha\left(\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\right)\right| \lesssim e^{-ct}|\xi|^{b-|\alpha|} \tag{2.8}$$

$$\left|\partial_\xi^\alpha\left(\frac{e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\right)\right| \lesssim e^{-ct}|\xi|^{b-2\sigma-|\alpha|} \tag{2.9}$$

for all multi-index α, for any p, $b \in \mathbb{R}$ and $j \geq 0$, where c is a suitable positive constant.

Proof of Lemma 2.1. In order to indicate some estimates for $\mathcal{K}^2_{u_0}(t,x)$, we may write

$$\begin{aligned}
&\partial_t^j |D|^a \mathcal{K}^2_{u_0}(t,x)\\
&= \mathfrak{F}^{-1}\left(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^{\min\{a,\sigma\}-2\sigma j}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)|\xi|^{2\sigma j+[a-\sigma]^+}\widehat{u}_0(\xi)\right)(t,x)\\
&= \mathfrak{F}^{-1}\left(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^{\min\{a,\sigma\}-2\sigma j}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)\right)(t,x) * |D|^{2\sigma j+[a-\sigma]^+}u_0(x)\\
&=: \mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)(t,x) * |D|^{2\sigma j+[a-\sigma]^+}u_0(x).
\end{aligned}$$

By choosing $b=\min\{a,\sigma\}-2\sigma j$ in (2.6), we get

$$\Big|\partial_\xi^\alpha\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)\Big| \lesssim e^{-ct}|\xi|^{\min\{a,\sigma\}-2\sigma-|\alpha|} \lesssim e^{-ct}|\xi|^{-\sigma-|\alpha|},$$

where c is a suitable positive constant. Since

$$e^{ix\xi} = \sum_{k=1}^n \frac{x_k}{i|x|^2}\partial_{\xi_k}e^{ix\xi}, \tag{2.10}$$

carrying out m steps of partial integration we derive

$$\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)(t,x) = C\sum_{|\alpha|=m}\frac{(ix)^\alpha}{|x|^{2|\alpha|}}\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)\Big)(t,x).$$

For this reason, we obtain the following estimates:

$$\begin{aligned}
\big|\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)(t,x)\big| &\lesssim |x|^{-m}\Big\|\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)\Big)(t,\cdot)\Big\|_{L^\infty}\\
&\lesssim |x|^{-m}\big\|\partial_\xi^\alpha\big(\mathcal{L}^2_0(t,\xi)\big)\big\|_{L^1}\\
&\lesssim |x|^{-m}e^{-ct}\int_1^\infty |\xi|^{-\sigma-m+n-1}d|\xi|\\
&\lesssim e^{-ct}\begin{cases}|x|^{-(n-1)} & \text{if } 0<|x|\le 1 \text{ and } m=n-1,\\ |x|^{-(n+1)} & \text{if } |x|\ge 1 \text{ and } m=n+1,\end{cases}
\end{aligned}$$

where the assumption $\sigma>1$ comes into play. Hence, we arrive at

$$\begin{aligned}
\big\|\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)(t,\cdot)\big\|_{L^1} &\lesssim \int_{|x|\le 1}\big|\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)(t,x)\big|dx\\
&\quad+\int_{|x|\ge 1}\big|\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}^2_0(t,\xi)\big)(t,x)\big|dx\\
&\lesssim e^{-ct}\int_0^1 d|x| + e^{-ct}\int_1^\infty |x|^{-2}d|x| \lesssim e^{-ct}.
\end{aligned}$$

Then, employing Young's convolution inequality we have proved the second statement in Lemma 2.1. In the same way, we may also conclude the last statement and the third statement in Lemma 2.1, respectively, by using (2.7) and (2.9). Let us come back to estimate the first statement. Indeed, we can see that

$$\begin{aligned}\partial_t^j|D|^a\mathcal{K}_{u_0}^1(t,x) &= \partial_t^j|D|^a\mathfrak{F}^{-1}\Big(e^{\lambda_1(\xi)t}\chi_3(|\xi|)\widehat{u}_0(\xi)\Big)(t,x)\\ &\qquad + \partial_t^{j+1}|D|^a\mathfrak{F}^{-1}\left(\frac{e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)}\chi_3(|\xi|)\widehat{u}_0(\xi)\right)(t,x), \end{aligned}\tag{2.11}$$

by using the relation

$$\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)} = e^{\lambda_1(\xi)t} + \partial_t\left(\frac{e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)}\right).$$

In an analogous treatment to get the third statement, we derive the following estimate for the second term:

$$\left\|\partial_t^{j+1}|D|^a\mathfrak{F}^{-1}\left(\frac{e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)}\chi_3(|\xi|)\widehat{u}_0(\xi)\right)(t,\cdot)\right\|_{L^1} \lesssim e^{-ct}\|u_0\|_{H_1^{[a-\sigma]^+}}. \tag{2.12}$$

In order to control the first term, using the relation $\lambda_1(\xi) = -1-\phi(\xi)$ we write

$$e^{\lambda_1(\xi)t} = e^{-t}e^{-\phi(\xi)t} = e^{-t} - te^{-t}\phi(\xi)\int_0^1 e^{-\phi(\xi)tr}dr.$$

Hence, we obtain

$$\begin{aligned}&\mathfrak{F}^{-1}\Big(e^{\lambda_1(\xi)t}\chi_3(|\xi|)\widehat{u}_0(\xi)\Big)(t,x)\\ &\quad = e^{-t}\mathfrak{F}^{-1}\big(\widehat{u}_0(\xi)\big)(x) - e^{-t}\mathfrak{F}^{-1}\big((1-\chi_3(|\xi|))\widehat{u}_0(\xi)\big)(x)\\ &\qquad - te^{-t}\mathfrak{F}^{-1}\left(\phi(\xi)\chi_3(|\xi|)\widehat{u}_0(\xi)\int_0^1 e^{-\phi(\xi)tr}dr\right)(t,x).\end{aligned}\tag{2.13}$$

Obviously, we have

$$\left\|\partial_t^j|D|^a\Big(e^{-t}\mathfrak{F}^{-1}\big(\widehat{u}_0(\xi)\big)\Big)(t,\cdot)\right\|_{L^1} = e^{-t}\big\||D|^a u_0\big\|_{L^1} \lesssim e^{-t}\|u_0\|_{H_1^a}. \tag{2.14}$$

Moreover, due to $1-\chi_3 \in C_0^\infty$, we derive

$$\left\|\partial_t^j|D|^a\Big(e^{-t}\mathfrak{F}^{-1}\big(1-\chi_3(|\xi|)\big)\Big)(t,\cdot)\right\|_{L^1} \lesssim e^{-t}.$$

By using again Young's convolution inequality we obtain

$$\left\|\partial_t^j|D|^a\Big(e^{-t}\mathfrak{F}^{-1}\big((1-\chi_3(|\xi|))\widehat{u}_0(\xi)\big)\Big)(t,\cdot)\right\|_{L^1} \lesssim e^{-t}\|u_0\|_{L^1}. \tag{2.15}$$

Now, we re-write

$$
\begin{aligned}
&\partial_t^j |D|^a \left(te^{-t}\mathfrak{F}^{-1}\left(\phi(\xi)\chi_3(|\xi|)\widehat{u}_0(\xi)\int_0^1 e^{-\phi(\xi)tr}dr\right)\right)(t,x)\\
&\quad = \sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\,\partial_t^{\ell}|D|^a\mathfrak{F}^{-1}\left(\phi(\xi)\chi_3(|\xi|)\widehat{u}_0(\xi)\int_0^1 e^{-\phi(\xi)tr}dr\right)(t,x)\\
&\quad = \sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\,\mathfrak{F}^{-1}\left(\phi^{\ell+1}(\xi)|\xi|^{\min\{a,\sigma\}}\chi_3(|\xi|)\int_0^1 e^{-\phi(\xi)tr}(-r)^{\ell}\,dr\right)(t,x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad * |D|^{[a-\sigma]^+}u_0(x)\\
&\quad =: \sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\,\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}_0^1(t,\xi)\big)(t,x) * |D|^{[a-\sigma]^+}u_0(x).
\end{aligned}
$$

Thanks to (2.4) and (2.5), by using the Leibniz rule we have

$$
\big|\partial_\xi^\alpha\big(\widehat{\mathcal{L}}_0^1(t,\xi)\big)\big| \lesssim e^{\frac{t}{2}}|\xi|^{-2\sigma\ell-2\sigma+\min\{a,\sigma\}-|\alpha|} \lesssim e^{\frac{t}{2}}|\xi|^{-\sigma-|\alpha|}.
$$

Using again (2.10), and carrying out $n-1$ and $n+1$ steps of partial integration imply

$$
\big|\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}_0^1(t,\xi)\big)(t,x)\big| \lesssim e^{\frac{t}{2}}\begin{cases}|x|^{-(n-1)} & \text{if } 0<|x|\le 1,\\ |x|^{-(n+1)} & \text{if } |x|\ge 1.\end{cases}
$$

It follows

$$
\left|\sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\,\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}_0^1(t,\xi)\big)(t,x)\right| \lesssim e^{-ct}\begin{cases}|x|^{-(n-1)} & \text{if } 0<|x|\le 1,\\ |x|^{-(n+1)} & \text{if } |x|\ge 1,\end{cases}
$$

where c is a suitable positive constant. Therefore, we derive

$$
\left\|\sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\,\mathfrak{F}^{-1}\big(\widehat{\mathcal{L}}_0^1(t,\xi)\big)(t,\cdot)\right\|_{L^1} \lesssim e^{-ct}.
$$

Applying Young's convolution inequality gives

$$
\begin{aligned}
&\left\|\partial_t^j |D|^a \left(te^{-t}\mathfrak{F}^{-1}\left(\phi(\xi)\chi_3(|\xi|)\widehat{u_0}(\xi)\int_0^1 e^{-\phi(\xi)tr}dr\right)\right)(t,\cdot)\right\|_{L^1}\\
&\qquad \lesssim e^{-ct}\|u_0\|_{H_1^{[a-\sigma]^+}}. \qquad (2.16)
\end{aligned}
$$

Combining from (2.11) to (2.15) we may conclude the first statement in Lemma 2.1. Summarizing, the proof of Lemma 2.1 is completed. □

From the statements in Lemma 2.1 we obtain immediately the following result.

Proposition 2.2. *Let $\sigma > 1$ and $n \geq 1$. Let $a \geq 0$. The Sobolev solutions to (0.1) satisfy the $L^1 - L^1$ estimates*

$$\big\|\partial_t^j |D|^a u_{\chi_3}(t,\cdot)\big\|_{L^1} \lesssim e^{-ct}\Big(\|u_0\|_{H_1^{2\sigma j+[a-\sigma]^+}} + \|u_0\|_{H_1^a} + \|u_1\|_{H_1^{2\sigma j+[a-\sigma]^+}}\Big),$$

where c is a suitable positive constant and for all integer numbers $j \geq 0$.

2.3. Estimates for middle frequencies

Now let us turn to consider some estimates for middle frequencies, where $3^{-\frac{1}{\sigma}} \leq |\xi| \leq 3^{\frac{1}{\sigma}}$. Our goal is to clarify the exponential decay for solutions and some of their derivatives to (0.1) localized to middle frequencies, which were neglected or not well-studied in the references.

Proposition 2.3. *Let $\sigma > 1$ and $n \geq 1$. Let $a \geq 0$ and $j = 0, 1$. The Sobolev solutions to (0.1) satisfy the $L^1 - L^1$ estimates*

$$\big\|\partial_t^j |D|^a u_{\chi_2}(t,\cdot)\big\|_{L^1} \lesssim e^{-ct}\big(\|u_0\|_{L^1} + \|u_1\|_{L^1}\big),$$

where c is a suitable positive constant.

Proof. At first, with $3^{-\frac{1}{\sigma}} \leq |\xi| \leq 3^{\frac{1}{\sigma}}$ we use Cauchy's integral formula to re-write the above Fourier multipliers in the following form:

$$\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|) = \frac{1}{2\pi i}\left(\int_\Gamma \frac{(z+|\xi|^{2\sigma})e^{zt}}{z^2+|\xi|^{2\sigma}z+|\xi|^{2\sigma}}dz\right)\chi_2(|\xi|), \tag{2.17}$$

$$\widehat{\mathcal{K}}_1(t,\xi)\chi_2(|\xi|) = \frac{1}{2\pi i}\left(\int_\Gamma \frac{e^{zt}}{z^2+|\xi|^{2\sigma}z+|\xi|^{2\sigma}}dz\right)\chi_2(|\xi|), \tag{2.18}$$

where Γ is a closed curve containing the two characteristic roots $\lambda_{1,2}$. It is clear that $\lambda_1 = \lambda_2$ when $|\xi| = 2^{\frac{1}{\sigma}}$. Additionally, the set $\{\xi \in \mathbb{R}^n \;:\; |\xi| = 2^{\frac{1}{\sigma}}\}$ is not a singular set because we may give equivalent formulas as follows:

$$\widehat{\mathcal{K}}_0(t,\xi) = e^{\lambda_2 t} - \lambda_2 e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds \text{ and } \widehat{\mathcal{K}_1}(t,\xi) = e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds.$$

Therefore, it is reasonable to assume $\lambda_1 \neq \lambda_2$. Since $3^{-\frac{1}{\sigma}} < |\xi| < 3^{\frac{1}{\sigma}}$, the curve is also contained in the set $\{z \in \mathbb{C} \;:\; \text{Re}\, z \leq -c_0\}$, where c_0 is a positive constant. In order to verify (**??**) we express

$$\frac{(z+|\xi|^{2\sigma})e^{zt}}{z^2+|\xi|^{2\sigma}z+|\xi|^{2\sigma}} = \frac{(z+|\xi|^{2\sigma})e^{zt}}{(z-\lambda_1)(z-\lambda_2)} = -\frac{\lambda_2}{\lambda_1-\lambda_2}\frac{e^{zt}}{z-\lambda_1} + \frac{\lambda_1}{\lambda_1-\lambda_2}\frac{e^{zt}}{z-\lambda_2}.$$

For this reason, applying Cauchy's integral formula we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{(z+|\xi|^{2\sigma})e^{zt}}{z^2+|\xi|^{2\sigma}z+|\xi|^{2\sigma}}dz &= -\frac{\lambda_2}{\lambda_1-\lambda_2}\Big(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{e^{zt}}{z-\lambda_1}dz\Big)\\ &\quad+\frac{\lambda_1}{\lambda_1-\lambda_2}\Big(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{e^{zt}}{z-\lambda_2}dz\Big)\\ &= -\frac{\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 t}+\frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2 t}\chi_2(|\xi|)=\widehat{\mathcal{K}}_0(t,\xi)\end{aligned}$$

for middle frequencies. Here we split the curve Γ into two closed sub-curves separately Γ_1 and Γ_2 containing λ_1 and λ_2, respectively. In the same way we may conclude the relation (2.18). Now, taking account of estimates for $\widehat{\mathcal{K}}_0(t,\xi)$ we get

$$\begin{aligned}\mathfrak{F}^{-1}\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big) &= \int_{\mathbb{R}^n} e^{ix\xi}|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)d\xi\\ &= \sum_{k=1}^n \frac{x_k}{i|x|^2}\int_{\mathbb{R}^n}\partial_{\xi_k}\big(e^{ix\xi}\big)|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)d\xi,\end{aligned}$$

where we used (2.10). By induction argument, carrying out m steps of partial integration we derive

$$\mathfrak{F}^{-1}\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big) = C\sum_{|\alpha|=m}\frac{(ix)^\alpha}{|x|^{2|\alpha|}}\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)\Big),$$

for any non-negative integer m and C is a suitable constant. Hence, we arrive at the following estimates:

$$\begin{aligned}\big|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)\big| &\lesssim |x|^{-m}\Big\|\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)\Big)\Big\|_{L^\infty}\\ &\lesssim |x|^{-m}\big\|\partial_\xi^\alpha\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)\big\|_{L^1}\lesssim |x|^{-m}e^{-ct},\end{aligned}$$

where c is a suitable positive constant, since $3^{-\frac{1}{\sigma}}<|\xi|<3^{\frac{1}{\sigma}}$. This estimate immediately implies

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim e^{-ct}.$$

In an analogous way we may also conclude

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{\mathcal{K}}_1(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim e^{-ct}.$$

Similarly, we may arrive at the exponential decay for the following estimates:

$$\begin{aligned}\big\|\mathfrak{F}^{-1}\big(|\xi|^a\partial_t\widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}&\lesssim e^{-ct},\\ \big\|\mathfrak{F}^{-1}\big(|\xi|^a\partial_t\widehat{\mathcal{K}}_1(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}&\lesssim e^{-ct},\end{aligned}$$

where we notice that

$$\partial_t \widehat{\mathcal{K}}_0(t,\xi) = -|\xi|^{2\sigma}\widehat{\mathcal{K}}_1(t,\xi) \quad \text{and} \quad \partial_t \widehat{\mathcal{K}}_1(t,\xi) = \widehat{\mathcal{K}}_0(t,\xi) - |\xi|^{2\sigma}\widehat{\mathcal{K}}_1(t,\xi).$$

Therefore, applying Young's convolution inequality we get

$$\begin{aligned} \big\|\partial_t^j |D|^a u_{\chi_2}(t,\cdot)\big\|_{L^1} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{\mathcal{K}}_0(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^1} \\ &\quad + \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{\mathcal{K}}_1(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_1\|_{L^1} \\ &\lesssim e^{-ct}\big(\|u_0\|_{L^1} + \|u_1\|_{L^1}\big). \end{aligned}$$

Summarizing, the proof of Proposition 2.3 is completed. □

Finally, we give the proof of our main result as follows.

Proof of Theorem 1.1. We combine the statements from Propositions 2.1, 2.2 and 2.3 to conclude immediately all the desired estimates. This completes our proof. □

Acknowledgments. This research was partly supported by the Vietnam Ministry of Education and Training under grant number B2023-BKA-06.

REFERENCES

[1] T.A. Dao, 2020. Global in time existence of Sobolev solutions to semi-linear damped σ-evolution equations in L^q scales. PhD thesis, TU Bergakademie Freiberg, Germany.

[2] M. D'Abbicco, M.R. Ebert, 2017. A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations. *Nonlinear Anal.*, 149, pp. 1-40.

[3] T.A. Dao, H. Michihisa, 2020. Study of semi-linear σ-evolution equations with frictional and visco-elastic damping. *Commun. Pure Appl. Anal.*, 19, pp. 1581-1608.

[4] M. D'Abbicco, M. Reissig, 2014. Semilinear structural damped waves. *Math. Methods Appl. Sci.*, 37, pp. 1570-1592.

[5] T.A. Dao, M. Reissig, 2019. An application of L^1 estimates for oscillating integrals to parabolic like semi-linear structurally damped σ-evolution models. *J. Math. Anal. Appl.* 476, pp. 426-463.

[6] M.R. Ebert, M. Reissig, 2018. Methods for partial differential equations, qualitative properties of solutions, phase space analysis, semilinear models, *Birkhäuser*.

[7] R. Ikehata, 2021. A note on optimal L^2-estimates of solutions to some strongly damped σ-evolution equations. *Asymptot. Anal.*, 121, pp. 59-74.

[8] Y. Shibata, 2000. On the rate of decay of solutions to the linear viscoelastic equation. *Math. Methods Appl. Sci.*, 23, pp. 203-226.

[9] M. D'Abbicco, G. Girardi, J. Liang, 2019. $L^1 - L^1$ estimates for the strongly damped plate equation. *J. Math. Anal. Appl.*, 478, pp. 476-498.

[10] T.A. Dao, M. Reissig, 2019. L^1 estimates for oscillating integrals and their applications to semi-linear models with σ-evolution like structural damping. *Discrete Contin. Dyn. Syst. A*, 39, pp. 5431-5463.