JUNG'S THEOREM AND EXTREMAL SETS IN A HEMI-SPHERE OF S^∞

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Abstract. The paper deals with the unit sphere S^∞ of a Hilbert space endowed with natural spherical metric. In the paper, we give a geometric characterization of extremal sets contained in a hemi-sphere of S^∞ that generalizes previously known results with respect to the classical Jung theorem.

Keywords: Extremal sets; Jung's constant; Kuratowski and Hausdorff measures of noncompactness.

1. Introduction

Let (M, d) be a metric space. For a non-empty bounded subset A of M we denote by $d(A)$ its diameter and by $r(A)$ the Chebyshev radius of A, *i.e.* $r(A) = \inf\{r_x(A) : x \in M\}$, where $r_x(A) = \sup\{d(y, x) : y \in A\}$ is the radius of A with respect to a point $x \in M$. A point $c \in M$ is called a Chebyshev center of A if $r_c(A) = r(A)$.

For a Banach space $(X, \|\cdot\|)$ with metric naturally generated by its norm $\|\cdot\|$ the Jung constant of X is defined by

$$J(X) = \sup\left\{\frac{r(A)}{d(A)} : A \text{ is a bounded subset of } X \text{ with } d(A) > 0\right\}.$$

A bounded subset A of X with $d(A) > 0$ is said to be extremal if $r(A) = J(X).d(A)$ ([1]). Since X possesses a linear structure one sees that in the above definition the supremum can be taken over all subsets A of X with $r(A) = 1$.

In the case of a metric space (M, d) without linear structure we consider for $R > 0$ the Jung function $J_M(R)$ of M defined as follows:

$$J_M(R) = \sup\left\{\frac{R}{d(A)} : A \subset M \text{ with } r(A) = R\right\}.$$

A bounded subset A of M with $d(A) > 0$ is said to be R-extremal, if $r(A) = R$ and $J_M(R) = \dfrac{R}{d(A)}$, and A is said to be extremal, if it is R-extremal for some $R > 0$.

For an n-dimensional Euclidean space E^n the Jung theorem asserts ([2]) that

$$J(E^n) = \sqrt{\frac{n}{2(n+1)}}.$$

Furthermore a bounded subset A of E^n is an extremal if and only if A contains all vertices of a regular n-simplex with edges of length $d(A)$. For a Hilbert space H (the infinite-dimensional case) the Jung constant of H were determined in [3] (*c.f.* [4]) $J(H) = \dfrac{1}{\sqrt{2}}$. The main result of our previous paper [1] gives a geometric characterization of extremal sets in a Hilbert space which is an infinite-dimensional version of classical Jung's theorem.

Our next aim is to treat the case of S^∞ which is the unit sphere of a Hilbert space endowed with the spherical metric. We give a geometric characterization of R-extremal sets in a hemi-sphere of S^∞ via descriptive investigation of Jung's function $J_{S^\infty}(R)$. The main result of this paper is a generalization of [1] for the case S^∞, which is also infinite-dimensional extension of the result of [5]. It should be noted that in [5] Dekster gave a version of the classical Jung theorem for unit sphere S^n of $(n+1)$- dimensional Euclidean space E^{n+1} with spherical metric. He partially extended his result in [6] for Alexandrov spaces of curvature bounded above . Also Lang and Schoreder [7] gave an upper bound for the radius of a bounded subset of a $CAT(\kappa)$ space in terms of its diameter.

Our main results are Theorem 1.1, Theorem 1.2 and Theorem 1.3 stated below.

Theorem 1.1. *For every $R \in (0, \dfrac{\pi}{2})$ we have*

$$J_{S^\infty}(R) = \frac{R}{D(R)}$$

where

$$D(R) = 2\sin^{-1}(\frac{\sin R}{\sqrt{2}}).$$

Theorem 1.2. *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev radius $r(A) = R \in (0, \dfrac{\pi}{2})$ and diameter $d(A) = d$. Then A is extremal if and only if for every $\varepsilon \in (0, d)$, for every positive integer m there exists a subset $A_{\varepsilon,m} = \{x_0, x_1, \ldots, x_m\}$ of A_ε such that $d_S(x_i, x_j) > d - \varepsilon$ for all $i \neq j$, $i, j = 0, 1..., m$, where $d_S(.,.)$ denotes the natural spherical metric.*

Theorem 1.3. *Let A be an extremal set in a hemi-sphere of S^∞ with $r(A) = R$. Then we have $\alpha(A) = D(R)$ and $\chi(A) = R$.*

Here $\alpha(A)$ and $\chi(A)$ denote the Kuratowski and Hausdorff measures of non-compactness of A which are defined as inf $\{d > 0 : A$ can be covered by finitely

many sets of diameter $\leqslant d\}$ and inf $\{\varepsilon > 0 : A$ can be covered by finitely many balls of radius $\leqslant \varepsilon\}$, respectively.

The paper is organized as follows. In section 2 we prove several lemmas related to the properties of Chebyshev centers in metric space S^∞ that we shall need in what follows. In particular we note that the existence and uniqueness of Chebyshev centers for subsets of a hemi-sphere of S^∞ are immediate from Proposition 3.1 of [7]. The proof of Theorems 1.1, 1.2 will be given in section 3. There we investigate Jung's function via the spherical technique exposed in section 2 and by using our previous result for the case of Hilbert spaces ([1]). In the final paragraph we give a proof of Theorem 1.3. As a corollary, we derive an extension of Gulevich's result for the hemi-sphere case of S^∞.

Notations. Throughout the paper, unless otherwise mentioned, we shall use the following notations.

- For $x, y \in S^\infty$ by $\widehat{xy}$ we mean the geodesic arc joining x to y and by $d_S(x, y)$ the length of $\widehat{xy}$. We also denote by $\overline{xy}$ the linear segment joining x to y and $d(x, y)$ is the distance between x and y in H.

- For each $c \in S^\infty$ we denote by S_c^∞ the hemi-sphere of S^∞ with pole at c, by L_c the closed tangent hyperplane of S^∞ at c. In most of our arguments later L_c will be considered as a Hilbert space with origin at c. By P_c we mean the orthogonal projection S_c^∞ to L_c. The image $P_c(B)$ of a subset B of S_c^∞ under operator P_c is denoted by B'.

- For a subset A of S^∞ with Chebyshev center c and radius $R \in (0, \frac{\pi}{2})$, by A_ε (for each $\varepsilon \in (0, R)$) we denote the set $A \setminus B(c, R - \varepsilon)$ and A_ε' denotes the image $P_c(A_\varepsilon)$ under P_c.

2. Chebyshev center and Mushroom Lemma

Lemma 2.1 ([7], Proposition 3.1). *Let A be a subset of a hemi-sphere of S^∞. Then there exists a unique point $c \in S^\infty$ such that $B(c, r(A)) \supset A$.*

Lemma 2.2. *Let A be a subset of a hemi-sphere of S^∞ and c is the Chebyshev center of A. Then $c \in \overline{co}A'$, where A' is the image of A under P_c and $\overline{co}A'$ denotes the closed convex hull of A' in the tangent space L_c.*

Proof. Assume on the contrary that $c \notin \overline{co}A'$. Then by the Hahn-Banach Theorem there exists a closed hyperplane Q of the tangent space L_c separating c from $\overline{co}A'$. Let c_1' be the point of Q closest to c and $c_1 \in S_c^\infty$ be its preimage under the mapping P_c. For each point $x' \in A'$ let us choose $x \in A$ such that $P_c(x) = x'$ and put

$$r = d_S(c, x), \ t = d_S(c, c_1), \ s = d_S(c_1, x), \ \alpha = \angle(\widehat{cx}, \widehat{cc_1})$$

Figure 1.

We have (see Fig.1)

$$d(c, x') = \sin r, \ d(c, c_1') = \sin t, \ \angle(\overline{cx'}, \ \overline{cc_1'}) = \alpha.$$

It is easy to see that

$$d(c, x') > d(c_1', x'), \ d(c, x') > d(c, c_1'),$$

or

$$\sin r > s', \ \sin r > \sin t.$$

Therefore in view of cosine law for spherical triangles cc_1x, $cc_1'x'$ and by an easy computation one gets

$$\cos s > \cos r + \varepsilon,$$

where $\varepsilon = 2\sin^2 \dfrac{t}{2}$ is a positive number independence of x. Clearly

$$s < r - \delta \quad \text{or} \quad d_S(c_1, x) < d_S(c, x) - \delta$$

for some $\delta > 0$ and independence of x. As x can be taken arbitrarily in A one gets $r_{c_1}(A) < r(A)$. This contradicts to the fact that $r(A)$ is the Chebyshev radius of A. So $c \in \overline{co}A'$.

The proof of Lemma 2.2 is completed. $\square$

The next lemma is a variation of Lemma 2 in [1] and Lemma 4 in [4].

Lemma 2.3 (Mushroom Lemma). *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev center at c and radius $r(A) = R > 0$. Then for every $\varepsilon \in (0, R)$ we have*

(i) $r(A_\varepsilon) = r(A)$;

(ii) $c \in \overline{co}A'_\varepsilon$;

(iii) c is the Chebyshev center of A' in the tangent space L_c.

Proof. (i). Suppose on contrary that $r(A_\varepsilon) < r(A)$. Then c is not Chebyshev center of A_ε. Let c_1 be a Chebyshev center of A_ε then $c_1 \neq c$ and $r_{c_1}(A_\varepsilon) < r_c(A_\varepsilon)$. Denoting by c_2 the mid-point of $\widehat{cc_1}$ we shall prove that

$$r_{c_1}(A_\varepsilon) < r_{c_2}(A_\varepsilon) < r_c(A_\varepsilon). \tag{2.3}$$

Setting

$$r = r_c(A_\varepsilon), \ r_1 = r_{c_1}(A_\varepsilon), \ r_2 = r_{c_2}(A_\varepsilon),$$

Since x is arbitrarily taken in A_ε and by the cosine law for spherical triangles xcc_2, xc_1c_2, a direct computation shows that

$$r_1 < r_2 < r,$$

or

$$r_{c_1}(A_\varepsilon) < r_{c_2}(A_\varepsilon) < r_c(A_\varepsilon).$$

Let z be a point strictly inside $\widehat{cc_1}$. We apply the following process of successive divisions in half. Let us first divide the arc $\widehat{cc_1}$ in half, then choose the half-arc containing z, and again divide it in half, and then choose the half-arc containing z, *etc.* until we reach z as a mid-point at some steps. By the continuity of $r_x(A_\varepsilon)$ in x one obtains

$$r_{c'}(A_\varepsilon) < r_z(A_\varepsilon) < r_c(A_\varepsilon). \tag{2.4}$$

Now let us fix a point z_0 strictly inside $\widehat{cc_1}$ such that $0 < d_S(z_0, c) < \dfrac{\varepsilon}{2}$. By (2.4) we have

$$d_S(x, z_0) \leqslant r_{z_0}(A_\varepsilon) < r(A)$$

for $x \in A_\varepsilon$ and

$$d_S(x, z_0) \leqslant d_S(x, c) + d_S(z_0, c) \leqslant r(A) - \varepsilon + \frac{\varepsilon}{2} = r(A) - \frac{\varepsilon}{2}$$

for $x \in A \setminus A_\varepsilon$. Therefore

$$r_{z_0}(A) \leqslant \max\{r_{z_0}(A_\varepsilon), \ r(A) - \frac{\varepsilon}{2}\} < r(A).$$

This contradicts the definition of $r(A)$. Hence $r(A_\varepsilon) = r(A)$.

(ii). Suppose on contrary that $c \notin \overline{co}A'_\varepsilon$. By the Hahn- Banach Theorem there exists a closed hyperplane Q in tangent space L_c separating c from $\overline{co}A'_\varepsilon$. Let c'_1 be the point of

Q closest to c and $c_1 \in S_c^\infty$ be its preimage under the mapping P_c, *i.e.* $P_c(c_1) = c_1'$. As in the proof of Lemma 2.2 one gets

$$r_{c_1}(A_\varepsilon) < r_c(A_\varepsilon) = r(A).$$

Analysis similar to in the proof of (i) shows that there is a point $z_0 \neq c$ such that $r_{z_0}(A) < r(A)$. This is a contradiction. Hence $c \in \overline{co}A_\varepsilon'$.

(iii). Putting $R' = \sin R$, in view of (ii) we deduce

$$c \in \overline{co}(A' \setminus B(c, R' - \delta)), \ \forall \delta \in (0, R')$$

and

$$A' \subset B(c, R'),$$

where $\overline{co}(A' \setminus B(c, R' - \delta))$ denotes the closed convex hull of $A' \setminus B(c, R' - \delta)$ in tangent space L_c.

If c is not the Chebyshev center of A', then there exits a point $c_1 \neq c$ in L_c and a positive number $R" < R'$ such that

$$A' \subset B(c_1, R").$$

Let M be the closed hyperlane in tangent space L_c passing through the mid-point of line segment cc_1 and orthogonal to $c - c_1$. Then M divide L_c into two half-spaces. Let us denote by M_c and M_{c_1} the closed half- spaces containing c and c_1 respectively. For each $x \in L_c$ it is easy to see that

$$x \in M_{c_1} \ \text{if} \ \|x - c_1\| < \|x - c\|;$$

$$x \in M_c, \ \text{otherwise}.$$

Therefore

$$B(c_1, R'') \setminus B(c, R") \subset M_{c_1}$$

and

$$A' \setminus B(c, R") \subset M_{c_1},$$

since $A' \subset B(c_1, R")$. On the other hand from (ii) one gets

$$c \in \overline{co}(A' \setminus B(c, R''))$$

since $0 < R" < R'$. This implies that $c \in M_{c_1}$ is a contradiction. So c is the Chebyshev center of A' in L_c. The proof of Lemma 2.3 is completed. $\qquad \square$

Lemma 2.4. *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev center at c and radius $r(A) = R > 0$. For p, q in A let us denote by $p' = P_c(p)$ and $q' = P_c(q)$. Then*

(i) $\|p' - q'\| \leqslant 2 \sin \left(\dfrac{d_S(p, q)}{2} \right)$;

(ii) *If $p, q \in A_\varepsilon$ for $\varepsilon \in (0, R)$ then*

$$\|p' - q'\|^2 \geqslant 4\sin^2\left(\frac{d_S(p,q)}{2}\right) - 2\left(\sin^2 R - \sin^2(R - \varepsilon)\right).$$

The proof of Lemma 2.4 is immediate by using the cosine law for spherical triangles cpq, $cp'q'$, and some elementary inequalities. We shall omit it.

Combining Lemmas 2.3 and 2.4 one gets

Lemma 2.5. *Let A be a subset of a hemi-sphere of S^∞ with Chebyshev center at c and radius $r(A) = R > 0$. Then for every $\varepsilon \in (0, R)$ we have*

(i) $r(A') = r(A'_\varepsilon) = \sin R$;

(ii) $d(A') = d(A'_\varepsilon) = 2\sin\left(\dfrac{d(A)}{2}\right)$.

3. Jung's function and Jung's theorem for a hemi-sphere of S^∞

Proof of Theorem 1.1. Assume that A is a subset of S^∞ with Chebyshev center at c and radius $r(A) = R \in (0, \frac{\pi}{2})$. Putting $A' = P_c(A)$ one gets $d(A') \geqslant \sqrt{2}r(A')$ (since the Jung constant of a Hilbert space is $\dfrac{1}{\sqrt{2}}$). By Lemma 2.5

$$d(A) = 2\sin^{-1}\left(\frac{d(A')}{2}\right) \geqslant 2\sin^{-1}\left(\frac{r(A')}{\sqrt{2}}\right) = 2\sin^{-1}\left(\frac{\sin R}{\sqrt{2}}\right) = D(R).$$

Therefore

$$J_{S^\infty}(R) \leqslant \frac{R}{D(R)}.$$

Let $L_c = L + c$ be the closed tangent hyperplane of S^∞ at c. Consider the orthonormal sequence $\{e_n\}_{n=1}^\infty$ in the hyperplane L of Hilbert space H. Putting $u'_n = c + \sin R . e_n$ $\forall n \geqslant 1$ we see that sequence $\{u'_n\}_{n=1}^\infty$ lie in tangent hyperplane L_c. Let $\{u_n\}_{n=1}^\infty$ be its preimage in hemi-sphere S_c^∞ under P_c, *i.e.* $P_c(u_n) = u'_n$ $\forall n \geqslant 1$. Clearly

$$d_S(c, u_n) = R, \quad \forall n \geq 1;$$

$$d_S(u_m, u_n) = 2\sin^{-1}\left(\frac{\sin R}{\sqrt{2}}\right), \quad \forall m, n \geq 1, m \neq n.$$

Let $c_1 \neq c$ be a point in S_c^∞. Setting $c'_1 = P_c(c_1)$ by the cosine law for spherical triangles cc_1u_n, $cc'_1u'_n$ we have

$$\cos\left(d_S(c_1, u_n)\right) = \cos\left(d_S(c, c_1)\right)\cos R + \langle c'_1 - c, e_n \rangle. \tag{3.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product of L. Setting $A = \{u_n\}_{n=1}^\infty$ and $R_1 = r_{c_1}(A)$, since the sequence $\{e_n\}_{n=1}^\infty$ converges weakly to 0, from (3.1) one gets

$$\cos R_1 < \cos R \;\; (\Rightarrow R_1 > R).$$

So $r(A) = r_c(A) = R$ and

$$d(A) = 2 \sin^{-1} \left(\frac{\sin R}{\sqrt{2}} \right).$$

We conclude that $J_{S^\infty}(R) = \dfrac{R}{D(R)}$ and the proof of Theorem 1.1 is completed. $\qquad \square$

Proof of Theorem 1.2. Assume that A is an extremal set in S^∞ with Chebyshev center at c and radius $r(A) = R \in (0, \pi/2)$. Then we have $d(A) = D(R) = 2 \sin^{-1} \left(\dfrac{\sin R}{\sqrt{2}} \right)$. By Lemma 2.5 one has for each $\varepsilon \in (0, R)$

$$r(A') = r(A'_\varepsilon) = \sin R;$$

$$d(A') = d(A'_\varepsilon) = 2 \sin \left(\frac{d(A)}{2} \right) = \sqrt{2} \sin R.$$

Hence A'_ε is an extremal set in the Hilbert space L. We now apply the main result of [1] to A'_ε for every $\delta \in (0, r(A'_\varepsilon))$, for any positive integer number m we can choose $x'_0, x'_1, ..., x'_m \in A'_\varepsilon$ such that

$$\|x'_i - x'_j\| > d(A') - \sqrt{2}\delta = \sqrt{2}(\sin R - \delta) \quad \forall i, j = 0, 1, ..., m, i \neq j.$$

Let us choose $x_0, x_1, ..., x_m \in A_\varepsilon$ such that $P_c(x_i) = x'_i \;\; \forall i = 0, 1, ...m$. In view of Lemma 2.4

$$d_S(x_i, x_j) \geqslant 2 \sin^{-1} \left(\frac{\|x'_i - x'_j\|}{2} \right) \geqslant 2 \sin^{-1} \left(\frac{(\sin R - \delta)}{\sqrt{2}} \right) \quad \forall i, j = 0, 1, ..., m, i \neq j.$$

Let $\delta > 0$ be sufficiently small so that

$$2 \sin^{-1} \left(\frac{(\sin R - \delta)}{\sqrt{2}} \right) > 2 \sin^{-1} \left(\frac{\sin R}{\sqrt{2}} \right) - \varepsilon = d(A) - \varepsilon.$$

Putting

$$A_{\varepsilon, m} = \{x_0, x_1, \ldots, x_m\}$$

we see that this is a subset of A_ε satisfying

$$d_S(x_i, x_j) > d(A) - \varepsilon, \;\; \forall i, j = 0, 1..., m, i \neq j.$$

Conversely assume that for every $\varepsilon \in (0, R)$, for any positive integer number m there exists a subset $A_{\varepsilon,m} = \{x_0, x_1, \ldots, x_m\} \subset A_\varepsilon$ such that: $d_S(x_i, x_j) > d(A) - \varepsilon$ for all $i, j = 0, 1 \ldots, m, i \neq j$. Putting $x_i' = P_c(x_i) \, \forall i = 0, 1 \ldots, m$, from Lemma 2.4 one deduces

$$\|x_i' - x_j'\|^2 \geqslant 4 \sin^2 \left(\frac{d(A) - \varepsilon}{2} \right) - 2 \Big(\sin^2 R - \sin^2(R - \varepsilon) \Big), \, \forall i, j = 0, 1 \ldots, m, i \neq j.$$

For each $\delta \in (0, d(A')$ we choose $\varepsilon \in (0, R)$ sufficiently small so that

$$4 \sin^2 \left(\frac{d(A) - \varepsilon}{2} \right) - 2 \Big(\sin^2 R - \sin^2(R - \varepsilon) \Big) > \left(2 \sin(\frac{d(A)}{2}) - \delta \right)^2.$$

Then

$$\|x_i' - x_j'\| > 2 \sin \left(\frac{d(A)}{2} \right) - \delta = d(A') - \delta, \, \forall i, j = 0, 1 \ldots, m, i \neq j.$$

Hence, the main result of [1] A' is an extremal set in Hilbert space L. Therefore

$$d(A') = \sqrt{2} r(A') = \sqrt{2} \sin R.$$

Thus by Lemma 2.5

$$d(A) = 2 \sin^{-1} \left(\frac{d(A')}{2} \right) = 2 \sin^{-1} \left(\frac{\sin R}{\sqrt{2}} \right) = D(R).$$

So A is R-extremal set in S^∞. The proof of the Theorem 1.2 is completed. $\square$

4. Measures of non-compactness of extremal sets

Proof of Theorem 1.3. First we prove that $\alpha(A) = D(R)$. Obviously $\alpha(A) \leqslant D(R)$ since $d(A) = D(R)$. Assume on the contrary that $\alpha(A) < D(R)$. Then one can choose $\varepsilon > 0$ satisfying $\alpha(A) < D(R) - \varepsilon$, and so subsets $A_1, A_2, \ldots, A_m$ of A such that

$$A = \bigcup_{i=1}^m A_i,$$

$$d(A_i) \leqslant D(R) - \varepsilon, \, \forall i = 1, 2, \ldots, m.$$

such that

$$x_i x_j > D(R) - \varepsilon, \, \forall i, j \in \{0, 1, \ldots, m\}, i \neq j.$$

Clearly, there exists at least one set among $A_1, A_2, \ldots, A_m$, say A_1 such that A_1 consisting at least two points of A_ε. Hence $d(A_1) > D(R) - \varepsilon$. This contradicts to the choice of $A_1, A_2, \ldots, A_m$ above. So $\alpha(A) = D(R)$.

Now we show that $\chi(A) = R$. Clearly $\chi(A) \leqslant R$. Let R_1 be a number satisfying $R_1 > \chi(A)$. Then there exist a finite number of balls of radius R_1 say $B_1, B_2, \ldots, B_m$ of S^∞ such that

$$A \subset \bigcup_{i=1}^m B_i.$$

By Theorem 1.2 for each positive integer number n there exists a subset C_n consisting $n.m + 1$ points of A such that the distance between arbitrary two points of C_n not less than $d(A) - \dfrac{1}{n}$. Denoting by

$$J_i = \{n \in \mathbb{N} : B_i \text{ consisting at least } n+1 \text{ point of } C_n\}, \quad i = 1, 2, ..., m.$$

one sees that there exists at least one set among J_1, J_2, . . . , J_m, say J_1 such that $|J_1| = \infty$, where $|J_1|$ denotes the cardinality of J_1. Now for each number $\varepsilon \in (0, R)$ and each positive integer number k, we choose a positive integer number $n \in J_1$ sufficiently large so that $n \geqslant k$ and $\dfrac{1}{n} < \varepsilon$. Then $A \cap B_1$ contains $k + 1$ points of A satisfying the property that distance between any two points of which is not less than $d(A) - \varepsilon$.

This implies

$$d(A \cap B_1) = d(A) = D(R),$$

so $A \cap B_1$ is also an extremal set in S^∞ by Theorem 1.2. Hence

$$2\sin^{-1}\left(\frac{\sin R}{\sqrt{2}}\right) = d(A) = d(A \cap B_1) = 2\sin^{-1}\left(\frac{\sin(r(A \cap B_1))}{\sqrt{2}}\right).$$

It follows that $r(A \cap B_1) = R$, so $R_1 \geqslant R$. Hence $\chi(A) \geqslant R$ and one concludes $\chi(A) = R$.

The proof of Theorem 1.3 is completed. $\qquad\square$

From Theorem 1.3 one deduces that there are no relatively compact sets in a hemi-sphere of S^∞ which are extremal. We obtain the following extension of Gulevich's result in [8] for our case.

Corollary 4.1. *Suppose that A is a relatively compact set in a hemi-sphere of S^∞ with $d(A) > 0$. Then $r(A) < \dfrac{R}{D(R)}\, d(A)$, where $D(R)$ is as above.*

Remark 4.1. *For any bounded subset A of S^∞ with radius $r(A) = R \in (0, \dfrac{\pi}{2})$ we can easily check that the following inequality holds*

$$\chi(A) \leqslant J_{S^\infty}(R)\alpha(A) \tag{4.1}$$

Indeed, if $d > \alpha(A)$ one can be choose subsets A_1, A_2,..., A_m of A such that

$$A = \bigcup_{i=1}^{m} A_i,$$

$$d(A_i) \leqslant d, \quad \forall i = 1, 2, ..., m.$$

Setting

$$R_1 = \max\{r(A_i) : i = 1, 2..., m\} = r(A_{i_0})$$

for some $i_0 \in \{1, 2, ..., m\}$ we have

$$R \geqslant R_1 \geqslant \chi(A),$$

and

$$d \geqslant d(A_{i_0}).$$

Therefore

$$\chi(A) \leqslant R_1 \leqslant J_{S^\infty}(R_1)d(A_{i_0}) \leqslant J_{S^\infty}(R)d.$$

Since d is arbitrarily greater than $\alpha(A)$ one gets (4.1).

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