

**A NORMALITY CRITERION FOR FAMILIES OF HOLOMORPHIC
MAPPINGS UNDER A CONDITION OF UNIFORM BOUNDEDNESS
OF THEIR TANGENT MAPPINGS ON THE INVERSE IMAGES
OF HYPERSURFACE TARGETS**

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Abstract. In [1], the authors proved that there is a union D of a finite number of hypersurfaces in the complex projective space $P^n(\mathbb{C})$ such that for every entire curve f in $P^n(\mathbb{C})$ if the spherical derivative $f^\#$ of f is bounded on $f^{-1}(D)$, then $f^\#$ is bounded on the entire complex plane, and hence, f is a Brody curve. In this paper, we shall give the counterpart of their result on the normal family of holomorphic mappings.

Keywords: normal family, Nevanlinna theory.

1. Introduction

A classical normality criterion of Marty asserts that a family \mathcal{F} of meromorphic functions on a domain $D \subset \mathbb{C}$ is normal if and only if for each compact subset K of D , there is a positive constant $c(K)$ such that for all $z \in K$, and $f \in \mathcal{F}$

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq c(K).$$

The generalization of this criterion to the context of holomorphic functions on hyperbolic domains in \mathbb{C}^m with values in a complete complex Hermitian manifold of dimension appears in [2] and takes the following form:

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Theorem A. Let $D \subset \mathbb{C}^m$ be a hyperbolic domain. Let M be a complete complex Hermitian manifold of dimension with metric E_M . Let \mathcal{F} be a family of holomorphic mappings of D into M . If \mathcal{F} is a normal family then for each compact subset L of D , there is a constant $c(L) > 0$ such that for all $z \in L$ and $f \in \mathcal{F}$, it holds that

$$E_M(f(z), f^*, z(\xi)) \leq c(L) \cdot F_K^D(z, \xi). \quad (1.1)$$

(Here, as usual, F_K^D , denotes the infinitesimal Kobayashi metric.) Conversely, if (1.1) holds and if for some $p \in D$, we have that all $f(p)$ ($f \in \mathcal{F}$) are in some compact subset Q of M , then \mathcal{F} is a normal family.

In [3], Tan gave a normality criterion for families of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into the projective space $P^n(\mathbb{C})$ under a condition of uniform boundedness of their tangent mappings on intersections of compact subsets and inverse image of a finite union of hyperplanes (rather than on entire compact subsets). In the case of dimension one, this problem was studied by Lappan [4, 5] and Hinkkanen [6]. The main purpose of this paper is to extend the result of Tan in [3] to the case of hypersurface targets. Our result is also the counterpart of the result of Son and Tan in [1] on the normality criterion of holomorphic mappings.

Hypersurfaces D_1, \dots, D_q ($q \geq n + 1$) in $P^n(\mathbb{C})$ are said to be in general position if $\bigcap_{i=0}^n D_{j_i} = \emptyset$, for all $1 \leq j_0 < \dots < j_n \leq q$. Our main result can be stated as follow.

Theorem 1.1. *Let \mathcal{F} be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $P^n(\mathbb{C})$ ($n \geq 2$), and let D_1, \dots, D_q be hypersurfaces in general position in $P^n(\mathbb{C})$. Denote by d the smallest common multiple of $\deg D_1, \dots, \deg D_q$. Denote by E the length function corresponding to the Fubini-Study metric on $P^n(\mathbb{C})$. Assume that for each compact subset $K \subset D$ there is a positive constant $c(K)$ such that*

$$E(f(z), f_{*,z}(\xi)) \leq c(K) \cdot \|\xi\|$$

for all $z \in K \cap (\bigcup_{j=1}^q f^{-1}(H_j))$, $\xi \in T_z D \simeq \mathbb{C}^m$, and $f \in \mathcal{F}$.

If $q > 3n \binom{n+d}{n} - n$ then \mathcal{F} is normal on D .

2. Notations

Let ν be a nonnegative divisor on \mathbb{C} . For each positive integer (or $+\infty$) p , we define the counting function of ν (where multiplicities are truncated by p) by

$$N^{[p]}(r, \nu) := \int_1^r \frac{n_\nu^{[p]}}{t} dt \quad (1 < r < \infty)$$

where $n_\nu^{[p]}(t) = \sum_{|z| \leq t} \min\{\nu(z), p\}$. For brevity we will omit the character $[p]$ in the counting function if $p = +\infty$.

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For a meromorphic function φ on \mathbb{C} , we denote by $(\varphi)_0$ the divisor of zeros of φ . We have the following Jensen's formula for the counting function:

$$N(r, (\varphi)_0) - N(r, \left(\frac{1}{\varphi}\right)_0) = \frac{1}{2\pi} \int_0^{2\pi} \log |(\varphi(re^{i\theta}))| d\theta + O(1).$$

Let f be a holomorphic mapping of \mathbb{C} into $P^n(\mathbb{C})$ with a reduced representation $f = (f_0, \dots, f_n)$. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log \|f(e^{i\theta})\| d\theta, \quad r > 1,$$

where $\|f\| = \max_{i=0, \dots, n} |f_i|$.

Let D be a hypersurface in $P^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[x_0, \dots, x_n]$, $\deg Q = \deg D$. Assume that $f(\mathbb{C}) \not\subset D$, then the counting function of f with respect to D is defined by $N_f^{[p]}(r, D) := N^{[p]}(r, (Q(f_0, \dots, f_n))_0)$.

Let $V \subset P^n(\mathbb{C})$ be a projective variety of dimension k . Denote by $I(V)$ the prime ideal in $\mathbb{C}[x_0, \dots, x_n]$ defining V . Denote by $\mathbb{C}[x_0, \dots, x_n]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$ of degree m (including 0). Put $I(V)_m := \mathbb{C}[x_0, \dots, x_n]_m \cap I(V)$.

The Hilbert function H_V of V is defined by $H_V(m) := \dim \frac{\mathbb{C}[x_0, \dots, x_n]_m}{I(V)_m}$.

Consider two integer numbers q, N satisfying $q \geq N + 1$, $N \geq k$. Hypersurfaces D_1, \dots, D_q in $P^n(\mathbb{C})$ are said to be in N -subgeneral position with respect to V if $V \cap (\cap_{i=0}^N D_{j_i}) = \emptyset$, for all $1 \leq j_0 < \dots < j_N \leq q$.

Let f be an entire curve in the complex projective space $P^n(\mathbb{C})$. The spherical derivative $f^\#$ of f measures the length distortion from the Euclidean metric in \mathbb{C} to the Fubini-Study metric in $P^n(\mathbb{C})$. The explicit formula is

$$f^\# = (|f_0|^2 + \dots + |f_n|^2)^{-2} \cdot \sum_{0 \leq i < j \leq n} |f_i f_j' - f_j f_i'|^2,$$

where (f_0, \dots, f_n) is a reduced representation of f .

3. Proof of Theorem 1.1

Lemma 3.1 ([2], Lemma 3.1). *Let \mathcal{F} be a family of holomorphic mappings of \mathbb{C} into $P^n(\mathbb{C})$. If \mathcal{F} is not normal then there exist sequences $\{z_k\} \subset \mathbb{C}$ with $z_k \rightarrow z_0 \in \mathbb{C}$, $\{f_k\} \subset \mathcal{F}$, $\{\rho_k\} \subset \mathbb{R}$ with $\rho_k \rightarrow 0^+$, such that $g_k(\zeta) := f_k(z_k + \rho_k \zeta)$ converges uniformly on compact subsets of \mathbb{C} to a nonconstant holomorphic mapping g of \mathbb{C} into $P^n(\mathbb{C})$.*

Lemma 3.2 ([1], Theorem 1.1). *Let D_1, \dots, D_q be hypersurfaces in general position in $P^n(\mathbb{C})$, $n \geq 2$. Denote by d the smallest common multiple of $\deg D_1, \dots, \deg D_q$. Assume*

that there exists a non-constant entire curve g in $P^n(\mathbb{C})$ such that for each $j \in \{1, \dots, q\}$, either $g(\mathbb{C}) \subset D_j$ or $g^\# = 0$ on $g^{-1}(D_j)$. Then $q \leq 3n \binom{n+d}{n} - n$.

Proof of Theorem 1.1. Suppose that \mathcal{F} is not normal, by Lemma 3.1, there exist sequences $\{z_k\} \subset D$ with $z_k \rightarrow z_0 \in D$, $\{f_k\} \subset \mathcal{F}$, $\{\rho_k\} \subset \mathbb{R}$ with $\rho_k \rightarrow 0^+$, and Euclidean unit vectors $u_k \in \mathbb{C}^m$ such that $g_k(\zeta) := f_k(z_k + \rho_k u_k \zeta)$, where $\zeta \in \mathbb{C}$ satisfies $z_k + \rho_k u_k \zeta \in D$, converges uniformly on compact subsets of \mathbb{C} to a nonconstant holomorphic mapping g of \mathbb{C} into $P^n(\mathbb{C})$.

We take a reduced representation $\widehat{f}_k = (f_{0k}, \dots, f_{nk})$ of f_k . Take a closed ball $B(z_0, R) := \{z : \|z - z_0\| \leq R\} \subset D$. By the assumption, there is a positive constant c such that

$$E(f(z), f_{*,z}(\xi)) \leq c \cdot \|\xi\| \quad (3.1)$$

for all $z \in B(z_0, R) \cap (\cup_{j=1}^q f^{-1}(H_j))$, $\xi \in T_z D \simeq \mathbb{C}^m$, and $f \in \mathcal{F}$.

For each $j_0 \in \{1, \dots, q\}$ satisfying $g(\mathbb{C}) \not\subset D_{j_0}$, we now prove that $g^\#(\zeta) = 0$ for all $\zeta \in g^{-1}(D_{j_0})$. To do this, we consider an arbitrary point $\zeta_0 \in g^{-1}(D_{j_0})$. There are reduced representations $\widehat{g}_k = (g_{k0}, \dots, g_{kn})$ of g_k and $g = (g_0, \dots, g_n)$ of g in a neighbourhood U_{ζ_0} of ζ_0 such that g_{ki} converges uniformly on compact subsets of U_{ζ_0} to g_i , for all $i = 0, \dots, n$. By the definition of g_k , there are nowhere vanishing holomorphic functions $\{u_k\}$ on U_{ζ_0} such that $g_{ki}(\zeta) = u_k(\zeta) f_{ki}(z_k + \rho_k u_k \zeta)$, for all $\zeta \in U_{\zeta_0}$, $i \in \{0, \dots, n\}$, and all k sufficiently large. By Hurwitz's Theorem there are values $\{\zeta_k\}$ (for all k sufficiently large), $\zeta_k \rightarrow \zeta_0$ such that $\zeta_k \in g_k^{-1}(D_{j_0})$, and hence, $z_k + \rho_k u_k \zeta_k \in f_k^{-1}(D_{j_0})$. On the other hand, $z_k + \rho_k u_k \zeta \rightarrow z_0$, hence, $z_k + \rho_k u_k \zeta_k \in B(z_0, R) \cap f_k^{-1}(D_{j_0})$ for all k sufficiently large. Then by (3.1),

$$E(f_k(z_k + \rho_k u_k \zeta_k), f_{k*, z_k + \rho_k u_k \zeta_k}(\xi)) \leq c \cdot \|\xi\|,$$

for all k sufficiently large and $\xi \in T_{z_k + \rho_k u_k \zeta_k} D \simeq \mathbb{C}^m$. Then

$$\begin{aligned} E(g(\zeta_0), g_{*, \zeta_0}(t)) &= \lim_{k \rightarrow \infty} E(g_k(\zeta_0), g_{k*, \zeta_0}(t)) \\ &= \lim_{k \rightarrow \infty} E(f_k(z_k + \rho_k u_k \zeta_k), f_{k*, z_k + \rho_k u_k \zeta_k}(\rho_k u_k t)) \\ &\leq \lim_{k \rightarrow \infty} \rho_k c \|t u_k\| = 0 \end{aligned}$$

for each $t \in T_{\zeta_0} \simeq \mathbb{C}$. Hence, for each $j \in \{1, \dots, q\}$, either $g(\mathbb{C}) \subset D_j$ or $g^\# = 0$ on $g^{-1}(D_j)$. Therefore, since $q > 3n \binom{n+d}{n} - n$, Lemma 3.2 shows that g is a constant curve; this is impossible. \square

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