

## SOME PROBLEMS WITH THE NET PRESENT VALUES AND THE INTERNAL RATE OF RETURNS UNDER DETERMINATE AND INDETERMINATE CONDITIONS

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**Abstract.** The Net Present Value (NPV) and the Internal Rate of Return (IRR) are the golden standards in evaluating the potential profitability of an investment. This paper points out several properties of the solutions of the equation determining the IRR, which includes the equivalence of this equation with an algebraic equation whose degree is less than or equal to  $n$ , the existence of solutions, the number of real solutions, the relative position of real roots with  $-1$  and  $0$  and some illustrations for that. Mokhtari et al. calculated the economic probability of the project in the case when the cash flow has exactly one random component with a normal, exponential, or uniform distribution by using the moment-generating function. We show that we can obtain their results by much simpler probabilistic transformations. At the same time, we generalize the problem to the case when the cash flow is a sequence of independent and normally distributed random variables. Simulation studies confirm the given results.

**Keywords:** net present value; internal rate of return; economic probability; normal, exponential, and uniform distributions.

### 1. Introduction

A project before its execution, after its completion, or even during its implementation process needs to be assessed holistically to see whether or not it has met the given requirements. Those requirements are relatively diverse and include assessments on environmental impact, risk level, socio-economic impact, or payback period. Regarding space, the project needs to be evaluated internally (performed by the organization administering the project, such as a representative of the contractor or company) and externally (performed by external people and agencies). In terms of time, pre-project evaluation (pre-implementation, start-up), mid-term evaluation, closing evaluation, and post-project evaluation are required (see [1]).

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In the economic evaluation of projects or investments, the net present value (NPV), and the internal rate of return (IRR) are always considered critical financial metrics. Researchers continuously increase the effectiveness of these tools. Yassin et al. [2] have shown an improved method to calculate NPV, and simultaneously proposed an alternative formula for this value. In [3], Sedliacikova used the final value of the project and modified the internal rate of return to evaluate investment projects. Daskovski et al. [4] recommended the system of six ratios (ROSS, ROIS, RTC, GETS, GES, and  $\beta$ ) used in estimates supporting the selection of investment projects. Percoco et al. [5] were interested in sensitivity analysis, they suggested using the differential importance measure to determine the most important exogenous variables in the case of evaluating the project by calculating IRR. However, these improvements are still very little-used.

What other mathematical and practical problems arise during calculation, especially regarding the properties of solutions of IRR equations? In this paper, we propose some properties of the real solutions of the equation determining IRR, including the comparison between the real solutions with two meaningful values  $-1$  and  $0$ . A further research direction is to consider the case when data obtained are only predictive, subjective, and therefore uncertain. To overcome this challenge, a flow of returns is assumed as random. Mokhtari et al. [6] calculated the economic probability  $P(NPV \geq 0)$  when only one term of the sequence of returns is a random variable with a normal, exponential, or uniform distribution. Their arguments are based on moment-generating functions. Here, we show the results in [6] can be proven by using much simpler probabilistic transformations. We also extend the work for a rather typical case, where cash flows form a sequence of independent and normally distributed random variables. Simulation studies show the difference between the economic probability according to the exact formula and according to the simulation is small and can be ignored.

## 2. Content

### 2.1. The NPV and IRR methods for evaluating the economic efficiency of investment projects

The net present value (NPV) represents the difference between the present value of net revenues over the project life and the initial cost of implementing the project. Suppose a project needs an initial outlay  $C$  and receives the net income (or balance, i.e. income minus costs)  $a_1, \dots, a_n$  at the end of the first period, ..., the  $n$ th period of the project, respectively. The period here can be a year, quarter, or month but usually in a year. If the interest rate  $i$  is fixed, NPV is given by

$$NPV = -C + \sum_{k=1}^n \frac{a_k}{(1+i)^k}, \quad (1)$$

where  $n$  is the number of periods, which is the project's duration.

A positive NPV means that the project or investment will be profitable. When NPV is zero, the project or investment breaks even. In contrast, a negative NPV will result in

a net loss. Therefore, only projects or investments with a non-negative NPV also known as viable projects are considered.

The internal rate of return (IRR) is the discount rate that makes the net present value NPV of the entire project's cash flow zero. Specifically, the IRR is the impact interest rate  $i$  that the following equation holds:

$$NPV = -C + \frac{a_1}{(1+i)^1} + \frac{a_2}{(1+i)^2} + \dots + \frac{a_n}{(1+i)^n} = 0. \quad (2)$$

In general, the higher the IRR, the more favored the investment. If the IRR is greater than or equal market discount rate, the company will accept the project as a good one. If the IRR is less than the discount rate, then it will be rejected. For projects with similar characteristics, the one with the highest IRR will usually be considered the best.

Many results on NPV and IRR can be found in [7-10].

NPV and IRR are mutually supportive in project evaluation. While NPV is always computable and uniquely determined, IRR is a solution of equation (2), so there can be cases where there is no solution, unique solution, or multiple solutions. IRR never takes on a value of  $-1$ . In the following section, we will consider the conditions under which IRR can be compared to two special values  $-1$  and  $0$ .

Without loss of generality, we always treat the project's duration as an integer,  $n \geq 1$ . Furthermore, we can assume that  $a_n \neq 0$ , since if  $a_n = 0$ , it is always possible to attain a shorter series of net revenues, i.e., a shorter project completion time. Now, we study some mathematical properties of IRR.

**Proposition 1.** Given the notation above, the following statements are true:

(i) The internal rate of return  $IRR = i$  is a solution of an algebraic equation whose degree is no more than  $n$ .

(ii) If  $-C \leq 0$ ,  $a_1 \leq 0$ , ...,  $a_{n-1} \leq 0$ ,  $a_n < 0$ , then all the real solutions of equation (2), if exists, must not exceed  $-1$ .

(iii) If  $-C < 0$ ,  $a_1 \leq 0$ , ...,  $a_{n-1} \leq 0$ ,  $a_n < 0$  and  $n$  is an odd number, then equation (2) has at least one real solution that is less than  $-1$ .

*Proof.* Put  $a_0 = -C$ , consider the sequence  $a_0, a_1, \dots, a_n$  with  $a_n \neq 0$ . Let  $a_{n_0}$  ( $0 \leq n_0 < n$ ) be the first non-zero term of this sequence, that is,  $a_0 = \dots = a_{n_0-1} = 0$ ,  $a_{n_0} \neq 0$ . Equation (2) becomes

$$\frac{a_{n_0}}{(1+i)^{n_0}} + \frac{a_{n_0+1}}{(1+i)^{n_0+1}} + \dots + \frac{a_n}{(1+i)^n} = 0.$$

The value of  $i$  needs to be different from  $-1$ . Multiplying both sides of the above equation with  $(1+i)^n \neq 0$ , we obtain

$$a_{n_0} (1+i)^{n-n_0} + a_{n_0+1} (1+i)^{n-n_0-1} + \dots + a_n = 0. \quad (3)$$

- Since  $a_{n_0} \neq 0$ , (3) is an algebraic equation of the degree  $n - n_0$  where  $0 < n - n_0 \leq n$ .

- With the assumption (b), for any  $i > -1$ , the left-hand side of (3) is a negative value. Thus, equation (3) cannot have real solutions that are greater than or equal to  $-1$ .

- If  $a_0 < 0$ , then  $n_0 = 0$ , so (3) is a polynomial equation in  $i$  of degree  $n$ .

$$f(i) \triangleq a_0(1+i)^n + a_1(1+i)^{n-1} + \dots + a_n = 0. \quad (4)$$

We observe that  $f(i)$  is continuous in  $(-\infty, -1]$  and  $f(-1) = a_n < 0$ . Furthermore, since  $n$  is odd,

$$f(-\infty) = \lim_{i \rightarrow -\infty} f(i) = \lim_{i \rightarrow -\infty} a_0(1+i)^n = +\infty.$$

According to the intermediate value theorem, there exists  $i_0 < -1$  such that  $f(i_0) = 0$ . Thus, (4) has a solution that is less than  $-1$ . Proposition 1 is fully proven.

**Proposition 2.** Assume that the sequence of net incomes from the beginning is a sequence of numbers with a single change in sign from non-positive to non-negative:

$$\exists k_0 \in \{1, \dots, n\} : a_0 = -C \leq 0, \dots, a_{k_0-1} \leq 0, a_{k_0} < 0, a_{k_0+1} \geq 0, \dots, a_n > 0.$$

Let  $C_d = -(a_0 + \dots + a_{k_0})$ ,  $T_s = a_{k_0+1} + \dots + a_n$  be the total expenditure in the first stage and the total revenue in the later stage of the project, respectively. Then

- Equation (2) has a unique solution (including multiples) in  $(-1, +\infty)$ .
- If  $C_d < T_s$ , then  $IRR > 0$ .
- If  $C_d = T_s$ , then  $IRR = 0$ .
- If  $C_d > T_s$ , then  $-1 < IRR < 0$ .

*Proof.* We rewrite equation (2) as follows:

$$\frac{a_0}{(1+i)^0} + \dots + \frac{a_{k_0}}{(1+i)^{k_0}} + \frac{a_{k_0+1}}{(1+i)^{k_0+1}} + \dots + \frac{a_n}{(1+i)^n} = 0. \quad (5)$$

Multiplying both sides of (5) with  $(1+i)^{k_0} \neq 0$  to get

$$p(i) \triangleq a_0(1+i)^{k_0} + \dots + a_{k_0} + \frac{a_{k_0+1}}{(1+i)^1} + \dots + \frac{a_n}{(1+i)^{n-k_0}} = 0. \quad (6)$$

- It is obvious that  $p(i)$ ,  $i \in (-1, +\infty)$  is a continuous function. Furthermore,

$$\lim_{i \rightarrow -1+0} p(i) = \lim_{i \rightarrow -1+0} \frac{a_n}{(1+i)^{n-k_0}} \in (0, +\infty],$$

$$\lim_{i \rightarrow +\infty} p(i) = \lim_{i \rightarrow +\infty} [a_0(1+i)^{k_0} + \dots + a_{k_0}] \leq a_{k_0} < 0.$$

Thus, equation (5) has solutions within the interval  $(-1, +\infty)$ .

The strict decreases in  $(-1, +\infty)$ , so the real solution will be unique in it.

- If  $C_d < T_s$ , then

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$$p(0) = a_0 + \dots + a_{k_0} + a_{k_0+1} + \dots + a_n = -C_d + T_s > 0$$

$$\lim_{i \rightarrow +\infty} p(i) = \lim_{i \rightarrow +\infty} \left[ a_0(1+i)^{k_0} + \dots + a_{k_0-1}(1+i) \right] + a_{k_0} \leq a_{k_0} < 0.$$

So equation (6) has solutions in the interval  $(0, +\infty)$  which means  $IRR \in (0, +\infty)$ . As stated in (a), this is the unique solution in  $(-1, +\infty)$ .

- If  $C_d = T_s$ , then

$$p(0) = a_0 + \dots + a_{k_0} + a_{k_0+1} + \dots + a_n = -C_d + T_s = 0.$$

So  $i = 0$  is a solution. It is unique in  $(-1, +\infty)$  also.

- If  $C_d > T_s$ , then

$$p(0) = a_0 + \dots + a_{k_0} + a_{k_0+1} + \dots + a_n = -C_d + T_s < 0,$$

$$p(-1+0) = \lim_{i \rightarrow -1+0} p(i) = \lim_{i \rightarrow -1+0} \frac{a_n}{(1+i)^{k_0-n}} \geq a_n > 0.$$

It follows that the equation has a solution in  $(-1, 0)$ , which is unique in  $(-1, +\infty)$ .

**Example 1.** Assume that the market interest rate is fixed at 7% for the upcoming years, and the cash flows are as follows:

- Initial cost 10, income 3, 3, 5, 4
- Initial cost 20, income -10, -3, -5
- Initial cost 30, income -10, -3, -5, 40
- Initial cost 40, income -10, -3, -5, 80
- Initial cost 10, income 20, -9.

Writing  $7\% = 0.07$  we can calculate NPVs as follows:

$$NPV_{(a)} = -10 + \frac{3}{1.07^1} + \frac{3}{1.07^2} + \frac{5}{1.07^3} + \frac{4}{1.07^4} \approx 2.557 > 0,$$

$$NPV_{(b)} \approx -36.048 < 0, \quad NPV_{(c)} \approx -15.532 < 0,$$

$$NPV_{(d)} \approx 4.984 > 0, \quad NPV_{(e)} = 0.831 > 0.$$

Thus, projects (b) and (c) are not feasible and have no economic value.

For IRR, we will examine each case separately.

- The sequence -10, 3, 3, 5, 4 has one change in sign from negative to positive.

According to Proposition 2, the equation for IRR has a unique solution in  $(-1, +\infty)$ .

Furthermore,  $C_d = 10 < 3 + 3 + 5 + 4 = T_s$ , so this solution is positive. We can check the equation

$$0 = -10 + \frac{3}{(1+i)^1} + \frac{3}{(1+i)^2} + \frac{5}{(1+i)^3} + \frac{4}{(1+i)^4}$$

has only two real distinct solutions  $i=0.1706$  and  $i=-1.6100$ , so the unique positive solution is  $i=i_{(a)} \approx 0.1706$ .

- The sequence  $-20, -10, -3, -5$  consists of only negative numbers and  $n=3$  is odd. According to Proposition 2, the IRR exists, is non-positive, and less than  $-1$ . The equation determining IRR is in fact

$$0 = -20 - \frac{10}{(1+i)^1} - \frac{3}{(1+i)^2} - \frac{5}{(1+i)^3}$$

$$\text{or } 20i^3 + 70i^2 + 83i + 38 = 0$$

and has a unique solution  $i=i_{(b)} \approx -1.7471 < -1$ .

- The sequence  $-30, -10, -3, -5, 40$  experiences one change in sign from negative to positive. We also see that  $C_d = 30+10+3+5 = 48 > 40 = T_s$ . According to Proposition 2, the real solution of (2) exists uniquely within the interval  $(-1, +\infty)$  and is negative. The equation determining IRR

$$0 = -30 - \frac{10}{(1+i)^1} - \frac{3}{(1+i)^2} - \frac{5}{(1+i)^3} + \frac{40}{(1+i)^4}$$

has exactly two real distinct solutions  $i=-2.1899$  and  $i=-0.0533$ .

Thus, the greater than  $-1$  solution is  $i=-0.0533$ .

- The sequence  $-40, -10, -3, -5, 80$  experiences one change in sign from negative to positive, and  $C_d = 40+10+3+5 = 58 < 80 = T_s$ . According to Proposition 2, the real solution exists uniquely in the interval  $(-1, +\infty)$  and is positive. Specifically, the equation

$$0 = -40 - \frac{10}{(1+i)^1} - \frac{3}{(1+i)^2} - \frac{5}{(1+i)^3} + \frac{80}{(1+i)^4}$$

has two real solutions  $i=-2.2626$  and  $i=0.0961$ . Thus, in this case, the positive solution is  $i=i_{(d)} = 0.0961$ .

- The sequence  $-10, 20, -9$  experiences two changes in sign, so there is no basis to assert the uniqueness of the solution of the equation for IRR. In this case

$$0 = -10 + \frac{20}{(1+i)^1} - \frac{9}{(1+i)^2} \Leftrightarrow i = \pm 0.3162.$$

Thus, the equation for IRR has two solutions in the interval  $(-1, +\infty)$ . Which solution should we take as the value for IRR? First, the total value of the cash flow is  $a_0 + a_1 + a_2 = -10 + 20 - 9 = 1$ , a positive number. Furthermore, we previously point out that  $NPV_{(e)} > 0$ . Therefore, to attain economic meaning, we choose the positive solution of the equation, which is  $i_{(e)} = 0.3162$ .

**Example 2.** There are  $n=11$  members in a ward, a group of people who financially support each other. Each member pays a basic amount of 10 million VND per month. During the duration of the ward's validity, each member can only take the ward (receives the money contributed by all other members) once. Each month, there is one person who takes the ward.

After the month of taking the ward, a member has to pay an additional amount of  $\delta=2$  million, which means he must pay 12 million VND per month. The ward begins when the first member takes the money and ends when everyone has taken turns.

For this ward, the cash flow of a certain member, for example, of the fourth is:

$$-10, -10, -10, 106, -12, -12, -12, -12, -12, -12, -12.$$

The above cash flow is a sequence that has two changes in sign, so Proposition 2 does not assert the uniqueness of the solution of the equation determining IRR

$$-10 - \frac{10}{(1+i)^1} - \frac{10}{(1+i)^2} + \frac{106}{(1+i)^3} - \frac{12}{(1+i)^4} - \dots - \frac{12}{(1+i)^{10}} = 0.$$

However, by using the approximation method, we obtain two real solutions:

$$i_1 = 0.0328, i_2 = 0.6444.$$

Which of the above solutions should we choose?

The current nominal annual interest rate applied by banks is around 7.2%, which means the monthly interest rate is  $i_m = 0.072/12 = 0.006$ . With this rate, the NPV of the cash flow is the following:

$$NPV = -10 + \frac{-10}{1.006^1} + \frac{-10}{1.006^2} + \frac{106}{1.006^3} + \frac{-12}{1.006^4} + \dots + \frac{-12}{1.006^{10}} = -6.2677 < 0.$$

As Schmidt [11] advises, when multiple IRRs exist for a single stream, we should take the IRR closest to the real cost of capital as the "true" IRR, which mean we will choose  $IRR = i_1 = 0.0328$ .

## 2.2. Project evaluation in indeterminate conditions

When evaluating the economic aspects of investment projects, meaningful parameters such as the price and quantity of goods handled are analyzed based on definite values previously known with perfect certainty. However, most of these parameters come from forecasts and are often uncertain. The source of uncertainty could be due to the customer, supplier, employee's behaviors, or technical problems in the processes. One way to deal with the challenge of uncertainty is to treat the cash flow  $a_1, \dots, a_n$  as a sequence of random variables. If the initial cost  $C$  is fixed, NPV can still be calculated by

$$NPV = -C + \sum_{k=1}^n \frac{a_k}{(1+i)^k}. \quad (7)$$

But now, the NPV is then a random variable. Therefore, the probability  $P(NPV \geq 0)$  is called the *economic probability* of the project. The economic probability

is of great interest to investors. The higher this probability, the more favored the project. A project with an economic probability of 1 is called a *certainly economical project*.

In what follows, we only consider the case when  $1+i > 0$ , and use the following functions:

The unit step function: 
$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The sign function: 
$$\text{sign}(x) = \begin{cases} -1, & x \leq 0 \\ 0, & x = 0 \\ 1, & x > 0, \end{cases}$$

The standard normal cumulative distribution function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{x^2}{2}\right) dx.$$

The following Proposition 3 summarizes the main results of [6], in which Mokhtari et al. also consider the cases of  $n=1,2$  and  $l=1,2$ .

**Proposition 3.** Assume that  $C \geq 0$  is a fixed number, and there is a period  $l$ , where the income  $a_l$  is random; for other periods,  $a_k$  is constant with  $k=1, \dots, n, k \neq l$ . Put

$$G_l = (1+i)^l \left( C - \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k} \right). \quad (8)$$

The economic probability can be expressed as follows:

i) If  $a_l$  has the normal distribution  $N(\mu_l, \sigma_l^2)$ ,  $\sigma_l > 0$ , then

$$P(NPV \geq 0) = 1 - \Phi\left(\frac{G_l - \mu_l}{\sigma_l}\right). \quad (9)$$

ii) If  $a_l$  has the exponential distribution  $E(\beta_l)$  with the density function  $(1/\beta_l)u(x)e^{-x/\beta_l}$ , then

$$P(NPV \geq 0) = \begin{cases} \exp\left(-\frac{G_l}{\beta_l}\right) & \text{if } C > \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k} \\ 1 & \text{if } C \leq \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k}. \end{cases} \quad (10)$$

iii) If  $a_l$  has the uniform distribution  $U(\alpha_l, \beta_l)$ ,  $\alpha_l < \beta_l$ , then

$$P(NPV \geq 0) = \begin{cases} 0 & \text{if } \beta_l \leq G_l \\ \frac{\beta_l - G_l}{\beta_l - \alpha_l} & \text{if } \alpha_l < G_l < \beta_l \\ 1 & \text{if } G_l \leq \alpha_l. \end{cases} \quad (11)$$

In [6], the above results are proved by the method of the moment-generating function. We will show below that Proposition 3 can be obtained by ordinary probabilistic transformations.

*Proof.* First, we rewrite the random variable NPV as

$$NPV = -C + \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k} + \frac{a_l}{(1+i)^l}.$$

Thus,

$$\begin{aligned} P(NPV \geq 0) &= P\left(-C + \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k} + \frac{a_l}{(1+i)^l} \geq 0\right) \\ &= P\left(a_l \geq (1+i)^l \left(C - \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k}\right)\right) = P(a_l \geq G_l). \end{aligned} \quad (12)$$

i) Assume that  $a_l \sim N(\mu_l, \sigma_l^2)$ ,  $\sigma_l > 0$ . Since  $Z = (a_l - \mu_l) / \sigma_l$  has the standard normal distribution  $N(0,1)$  (see [12]), the probability at (12) becomes

$$P\left(\frac{a_l - \mu_l}{\sigma_l} \geq \frac{G_l - \mu_l}{\sigma_l}\right) = 1 - \Phi\left(\frac{G_l - \mu_l}{\sigma_l}\right),$$

and this leads to (9).

ii) Now assume  $a_l \sim E(\beta_l)$ ,  $f_{a_l}(x) = (1/\beta_l)u(x)e^{-x/\beta_l}$ .

If  $G_l < 0$  or  $C < \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k}$ , then according to (12):

$$P(NPV \geq 0) = P(a_l \geq G_l) \geq P(a_l \geq 0) = 1.$$

If  $G_l \geq 0$  or  $C \geq \sum_{k=1, k \neq l}^n \frac{a_k}{(1+i)^k}$ , then again by (12)

$$P(NPV \geq 0) = P(a_l \geq G_l) = \frac{1}{\beta_l} \int_{G_l}^{\infty} e^{-\frac{x}{\beta_l}} dx = e^{-\frac{G_l}{\beta_l}}.$$

Thus, (10) is as follows:

iii) Now assume that  $a_l$  has a uniform distribution in  $(\alpha_l, \beta_l)$ , then

$$P(a_l \geq G_l) = \frac{\text{The length of the interval } (\alpha_l, \beta_l) \cap (G_l, \infty)}{\beta_l - \alpha_l}.$$

If  $G_l \geq \beta_l$ , then  $(\alpha_l, \beta_l) \cap (G_l, \infty) = \emptyset$ , so  $P(a_l \geq G_l) = 0$ .

If  $G_l \leq \alpha_l$ , then  $(\alpha_l, \beta_l) \cap (G_l, \infty) = (\alpha_l, \beta_l)$ , so  $P(a_l \geq G_l) = 1$ .

If  $\alpha_l < G_l < \beta_l$ , then  $(\alpha_l, \beta_l) \cap (G_l, \infty) = (G_l, \beta_l)$ , so

$$P(NPV \geq 0) = P(a_l \geq G_l) = \frac{\beta_l - G_l}{\beta_l - \alpha_l}.$$

Therefore, (11) is inferred, and Proposition 3 is proven.

In the following theorem, we suggest a formula for the economic probability in a comparatively typical case where the incomes are independent and normally distributed.

**Theorem 4.** Assuming the initial cost  $C > 0$  is fixed, and the cash flow  $a_1, \dots, a_n$  is a sequence of random variables that are mutually independent and normally distributed with  $a_k \sim N(\mu_k, \sigma_k^2)$  such that  $\Delta = \max(\sigma_1^2, \dots, \sigma_n^2) > 0$ . Put

$$NPV_* = -C + \sum_{k=1}^n \frac{\mu_k}{(1+i)^k}. \quad (13)$$

The following statements are true:

i) The economic probability of the project is given by:

$$P(NPV \geq 0) = 1 - \Phi \left[ -NPV_* \left( \sum_{k=1}^n \frac{\sigma_k^2}{(1+i)^{2k}} \right)^{-1/2} \right]. \quad (14)$$

ii) If  $n, i, \mu_k$ , and  $\sigma_k$  ( $k=1, \dots, n$ ) are constant, then as the initial cost  $C$  increases, the economic probability  $P(NPV \geq 0)$  decreases. Furthermore,

$$\text{sign}(NPV_*) = \text{sign} \left( P(NPV \geq 0) - \frac{1}{2} \right). \quad (15)$$

iii) If  $n, i, C$ , and  $\mu_k$  ( $k=1, \dots, n$ ) is fixed, then:

$$\lim_{\Delta \rightarrow 0} P(NPV \geq 0) = \begin{cases} 0 & \text{if } NPV_* < 0 \\ 0.5 & \text{if } NPV_* = 0 \\ 1 & \text{if } NPV_* > 0. \end{cases} \quad (16)$$

*Proof.* (i) From the assumptions, NPV is a normal random variable with the mean and variance as

$$E(NPV) = E \left( -C + \sum_{k=1}^n \frac{a_k}{(1+i)^k} \right) = -C + \sum_{k=1}^n \frac{\mu_k}{(1+i)^k} = NPV_*,$$

$$\text{Var}(NPV) = \sum_{k=1}^n \frac{\text{VAR}(a_k)}{(1+i)^{2k}} = \sum_{k=1}^n \frac{\sigma_k^2}{(1+i)^{2k}} > 0.$$

Thus  $Z = \frac{NPV - NPV_*}{\sqrt{\text{Var}(NPV)}} \sim N(0,1)$ . We obtain

$$P(0 \leq NPV) = P \left( \frac{-NPV_*}{\sqrt{\text{Var}(NPV)}} \leq \frac{NPV - NPV_*}{\sqrt{\text{Var}(NPV)}} \right) = 1 - \Phi \left( \frac{-NPV_*}{\sqrt{\text{Var}(NPV)}} \right)$$

$$= 1 - \Phi \left[ -NPV_* \left( \sum_{k=1}^n \frac{\sigma_k^2}{(1+i)^{2k}} \right)^{-1/2} \right].$$

Then (14) is as follows:

(ii) As  $C$  increases and the other parameters remain constant, the  $NPV_*$  decreases. The conclusion (ii) is deduced from (14), and the fact that  $\Phi(x)$  is a continuous and monotonically increasing function, and  $\Phi(0) = 0.5$ .

(iii) We know that  $\Phi(x)$  is a continuous and monotonically increasing function.

- If  $NPV_* < 0$ , then  $C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k} > 0$ , so

$$\frac{C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k}}{\left( \sum_{k=1}^n \frac{\sigma_k^2}{(1+i)^{2k}} \right)^{1/2}} \geq \frac{C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k}}{\left( \Delta^2 \sum_{k=1}^n \frac{1}{(1+i)^{2k}} \right)^{1/2}} \rightarrow +\infty \quad (\Delta \rightarrow 0).$$

From (14),  $\lim_{\Delta \rightarrow 0} P(NPV \geq 0) = 1 - \Phi(+\infty) = 0$ .

- If  $NPV_* = 0$ , then  $C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k} = 0$ , so  $\lim_{\Delta \rightarrow 0} P(NPV \geq 0) = 1 - \Phi(0) = 0.5$ .

- If  $NPV_* > 0$ , then  $C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k} < 0$ , so

$$\frac{C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k}}{\left( \sum_{k=1}^n \frac{\sigma_k^2}{(1+i)^{2k}} \right)^{1/2}} \leq \frac{C - \sum_{k=1}^n \frac{\mu_k}{(1+i)^k}}{\left( \Delta^2 \sum_{k=1}^n \frac{1}{(1+i)^{2k}} \right)^{1/2}} \rightarrow -\infty \quad (\Delta \rightarrow 0).$$

From (14),  $\lim_{\Delta \rightarrow 0} P(NPV \geq 0) = 1 - \Phi(-\infty) = 1$ .

Theorem 4 is completely proven.

**Remark.** In the case where  $\sigma_l^2 \neq 0$ ,  $\sigma_k^2 = 0 \quad \forall k \neq l$ , equation (14) becomes equation (9). Therefore, Theorem 4 is an extension of Proposition 3 in the case of normally distributed random variables.

If the initial cost  $C$  and incomes  $a_1 = \mu_1, \dots, a_n = \mu_n$  are fixed such that  $C < \sum_{k=1}^n \frac{a_k}{(1+i)^k} = \sum_{k=1}^n \frac{\mu_k}{(1+i)^k}$ , then NPV is a positive number. Conclusion (14) of Theorem 4 is a bit surprising.

**Corollary 5.** If the random variables  $a_1, \dots, a_n$  are independent and identically distributed random variables following  $N(\mu, \sigma^2)$ ,  $\sigma > 0$ , then

$$P(NPV \geq 0) = 1 - \Phi \left[ \frac{C - \mu \frac{1 - (1+i)^{-n}}{i}}{\sigma \sqrt{\frac{1 - (1+i)^{-2n}}{i(i+2)}}} \right]. \tag{17}$$

We just apply (14) to prove (17) with the remark

$$\sum_{k=1}^n \frac{\mu}{(1+i)^k} = \frac{\mu}{1+i} \frac{1 - \frac{1}{(1+i)^n}}{1 - \frac{1}{1+i}} = \mu \frac{1 - (1+i)^{-n}}{i}$$

$$\sum_{k=1}^n \frac{\sigma^2}{(1+i)^{2k}} = \frac{\sigma^2}{(1+i)^2} \frac{1 - (1+i)^{-2n}}{1 - \frac{1}{(1+i)^2}} = \sigma^2 \frac{1 - (1+i)^{-2n}}{i(i+2)}.$$

### 2.3. Simulation study

Hereafter, we conduct simulation studies to calculate the economic probabilities mentioned in Theorem 4. For convenience, we will consider such probabilities only in simpler cases as outlined in Corollary 5. To do that, we need to generate  $N$  sets of numbers, each consisting of  $n$  independent random variables with the same normal distribution  $N(\mu, \sigma^2)$ . In order to reduce the non-randomness of the obtained sets of numbers, we first generate  $N+100$  sets of  $2n$  independent and identically distributed random variables  $a_1^*, \dots, a_{2n}^*$  from the distribution  $N(\mu, \sigma^2)$ . Next, we discard the first 100 sets of numbers, leaving  $N$  ones. Then we can use the Kolmogorov-Smirnov test for normality, or more simply, remove sequences with large kurtosis or skewness and retain only  $N$  sets of  $n$  numbers  $a_1, \dots, a_n$ .

At each iteration, with a preselected value  $C$  and  $a_1, \dots, a_n$ , calculate the NPV according to (7). Repeat this process  $N$  times. The probability  $P(NPV \geq 0)$  approximately equal to the number of these inequalities  $NPV \geq 0$  that occur divided by  $N$ .

**Table 1. The economic probability  $P(NPV \geq 0)$  by (17) and by simulation**

$i$	$C$	$NPV^*$	$\sigma = 0.5$		$\sigma = 0.2$	
			By (16)	Simulation	By (16)	Simulation
0.05	1	3.3295	0.9997	1.0000	1.0000	1.0000
	2	2.3295	0.9918	0.9945	1.0000	1.0000
	3	1.3295	0.9147	0.9290	0.9997	0.9995
	4	0.3295	0.6329	0.6455	0.8020	0.8030
	5	-0.6705	0.2448	0.2540	0.0420	0.0420
0.10	1	2.7908	0.9994	1.0000	1.0000	1.0000
	2	1.7908	0.9819	0.9870	1.0000	1.0000
	3	0.7908	0.8224	0.8225	0.9896	0.9915
	4	-0.2092	0.4034	0.3895	<b>0.2704</b>	<b>0.2520</b>
	5	-1.2092	0.0787	0.0800	0.0002	0.0005

0.15	1	2.3522	0.9990	0.9985	1.0000	1.0000
	2	1.3522	0.9616	0.9695	1.0000	1.0000
	3	0.3522	0.6776	0.6865	0.8754	0.8770
	4	-0.6478	0.1982	0.1995	0.0170	0.0150
	5	-1.6478	0.0155	0.0150	0.0000	0.0005

Table 1 shows the results of the economic probability  $P(NPV \geq 0)$  calculated by the formula (17) and by simulation in cases

$$N = 2000; n = 5; \mu = 1; \sigma = 0.5, 0.2; i = 0.05, 0.10, 0.15; C = 1, 2, 3, 4, 5.$$

For convenience,  $NPV^*$ s are placed in the third column of Table 1.

According to Table 1, the largest difference between the economic probability by the exact formula and the one by the simulation is  $|0.2704 - 0.2520| = 0.0184$ , which occurs when  $\sigma = 0.2, i = 0.10, C = 4$ . Thus, the differences are quite small and can be ignored, and (17) is satisfactory.

The table also confirms that when  $NPV^* > 0$ , the economic probability exceeds 0.5.

### 3. Conclusions

The paper has shown that the equation for the net present value is actually an algebraic equation of degree less than or equal to  $n$  - the project completion time. Many properties of this equation's solutions have been suggested, including the existence, the uniqueness of real solutions, and the comparison of real solutions with  $-1$  and  $0$ . Illustrative examples (including Example 2 related to wards that often occur in many groups of residents nowadays) show some experience in choosing solutions from the internal rate of return equation to make this solution economically meaningful.

The paper also shows that the results in [6] can be neatly presented and proven by common probabilistic transformations. Finally, the probability that the project is economic is mentioned in the rather general case, where the cash flows form a sequence of independent and normally distributed random variables. When the related random variables are normally distributed, the obtained results here are more general than those in [6].

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