SOME MAIN OCCURRENCES OF CROSSED MODULES

Pham Thi Cuc

Received: 24 July 2017 / Accepted: 10 October 2017 / Published: November 2017

Abstract: *In this paper, we describe in detail some main occurrences of crossed modules. These are the common classes of crossed modules in algebra and topology. For every class of crossed modules, we also look for the analogues and show their special property.*

Keywords: *Crossed modules, morphisms of crossed modules.*

1. Introduction

Crossed modules (over groups) were invented almost 70 years ago by J. H. C. Whitehead in his work on combinatorial homotopy theory [10]. Whitehead's ideas on crossed modules and their applications were developed and explained in the book by R. Brown, P. J. Higgins, R. Sivera [1]. Some generalisations of the idea of crossed module were explained in the paper of G. Janelidze [3]. Recently, N. T. Quang and his co-workers have obtained some interesting concerning to extending the notion of crossed modules and solving the group extension problems of the type of a crossed module regards to the results of categorical theory [5, 6, 7, 8].

One can say that crossed modules have found important roles in many areas of mathematics including homotopy theory, homology and cohomology of groups, algebraic *K*-theory, cyclic homology, combinatorial group theory, differential geometry, etc. Possibly crossed modules should be considered one of the fundamental algebraic structures. A crossed module is a quadruple (B, D, d, θ) satisfying two given conditions, where $d: B \to D$, $\theta: D \to \mathrm{Aut}B$ are group homomorphisms. Giving a homomorphism $\theta: D \to \mathrm{Aut}B$ means giving an action of *D* on *B*. In the works on crossed modules ([1, 2, 4]), the authors mention some examples of crossed modules, but they do not explain in detail the homomorphism $\theta: D \to \mathrm{Aut}B$ (the action of *D* on *B*), in which in many cases this homomorphism is built not-natural or in a flexible way.

In this article we give an account of some of the main occurrences and uses of crossed modules in algebra and in topology in particular we give a detailed description of the homomorphisms $d: B \to D$, $\theta: D \to \mathrm{Aut}B$. These are: inclusion crossed modules, crossed modules of a module over a group ring, automorphism crossed modules of a group, semi-direct product crossed modules, pullback crossed modules of a crossed module along a group homomorphism, crossed modules constructed from a pointed topological space. We also give the equivalent forms or properties of each class of crossed modules.

Pham Thi Cuc
Faculty of Natural Sciences, Hong Duc Univesity
Email: Phamthicuc@hdu.edu.vn (✉)

2. Crossed modules

Definition. [10] A *crossed module* is a quadruple (B, D, d, θ) in which $d : B \to D$, $\theta : D \to \mathrm{Aut}B$ are goup homomorphisms (the homomorphism θ we will conceive of as a map $\theta : D \times B \to B$, analogously to the adjoint action $\mu : G \times G \to G$ of a group on itself) such that the two following diagrams commute:

$$\begin{tikzcd} B \times B \arrow[rr, "d \times Id"] \arrow[dr, "\mu"'] & & D \times B \arrow[dl, "\theta"] \\ & B & \end{tikzcd}$$

$$\begin{tikzcd} D \times B \arrow[r, "\theta"] \arrow[d, "d \times Id"'] & B \arrow[d, "d"] \\ D \times B \arrow[r, "\mu"] & D \end{tikzcd}$$

The two diagrams can be translated into equations, which may often be helpful.

$$\theta(d(b))(b') = \mu_b(b'),$$

$$d(\theta_x(b)) = \mu_x(d(b)),$$

Where $x \in D, b, b' \in B$.

The second equation is known as the *Peiffer identity*.

If $(B, D, d, \theta), (B', D', d', \theta')$ are crossed modules, a *morphism*,

$$(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$$

of crossed modules consists of group homomorphisms $f_1 : B \to B'$ and $f_0 : D \to D'$ such that the following diagram (of group homomorphisms) commutes

$$\begin{tikzcd} B \arrow[r, "f_1"] \arrow[d, "d"'] & B' \arrow[d, "d'"] \\ D \arrow[r, "f_0"] & D' \end{tikzcd}$$

and f_1 is an operator homomorphism, that is,

$$f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b)$$

for all $x \in D, b \in B$.

Crossed modules and their morphisms form a category denoted by **Cross**.

For a fixed group *D*, there is a subcategory **Cross**$_D$ of **Cross** whose objects are those crossed modules with *D* as the “base”, i.e., all crossed module (B, D, d, θ) for this fixed *D*, whose morphisms are morphisms (f_1, f_0) from (B,D,d,θ) to (B',D,d',θ') just those (f_1, f_0) in **Cross** in which $f_0 : D \to D$ is the identity homomorphism on *D*.

Below, we give some well known situations of crossed modules.

3. Inclusion crossed modules

Let *B* be a normal subgroup of a group *D* and $i : B \to D$ the inclusion, then we will say (B,D,i) is a *normal subgroup pair*. The homomorphism $\theta^0 : D \to \mathrm{Aut}B$ is given by conjugation. Then, the quadruple (B,D,i,θ^0) is called a *inclusion crossed module* [4].

Conversely, it is easy to prove the following lemma.

Lemma 1. [4] *If* (B,D,d,θ) *is a crossed module,* $d(B)$ *is a normal subgroup of D.*

The inclusion crossed module plays an important role in the group extension problem of the type of a crossed module [7]. It together with the general crossed module makes a homomorphism of crossed modules which is a constraint of this group extension problem.

4. Crossed modules of a module over a group ring

Let's recall that if *D* is a group, the free abelian group *ZD* generated by the elements of *D* is a ring. We call *ZD* the *group ring* of *D*.

Suppose *D* is a group and B is a left ZD-module, let $\mathbf{0}: B \to D$ be the trivial map sending everything in *B* to the identity element of *D*, $\theta : D \to \mathrm{Aut}B$ is given by module action. Then, (B,D,d,θ) is called a *crossed module of a module over a group ring.*

Again conversely:

Lemma 2. [4] *If* (B,D,d,θ) *is a crossed module,* Ker*d is central in B and inherits a natural D-module from the D-action on B. Moreover,* Im*d acts trivially on* Ker*d, so* Ker*d has a natural* Coker*d-module structure*.

As these two examples suggest, general crossed modules lie between the two extremes of normal subgroups and modules. Their structure bears a certain resemblance to both - they are "external" normal subgroups but also are "twisted" modules.

5. Crossed modules of a surjective group homomorphism

Let $p : B \to D$ be a surjective group homomorphism whose kernel lies in the center of *B*, the homomorphism $\theta^0 : D \to \mathrm{Aut}B$ is given by conjugation. Then, the quadruple (B,D,p,θ^0) is called a *crossed module of a surjective group homomorphism.*

Equivalently, given any central extension of groups

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{d} D \longrightarrow 1$$

that is, the above sequence is exact and A is in the center of B. Then, the surjective homomorphism $d : B \to D$ together with the action of D on B define a crossed module.

Thus, central extensions can be seen as special crossed modules. Conversely, a crossed module (B, D, d, θ) with the surjective boundary d defines a central extension. Because of this, one can use the results on crossed modules to solve the problems relating to central extensions.

Analogously to the inclusion crossed module, the crossed module plays an important role in the group extension problem of the co-type of a crossed module [9]. It together with the general crossed module makes a homomorphism of crossed modules which is a constraint of this group extension problem.

6. Automorphism crossed modules of a group

Let B be a group, then, as usual, let Aut(B) denote the group of automorphisms of B. The homomorphism $\mu : B \to \mathrm{Aut}(B)$ sends an element $b \in B$ to the inner automorphism of the group B. Then, the tuple $(B, \mathrm{Aut}(B), \mu, id)$ is a crossed module, called the *automorphism crossed module of the group B and* its own notation **Aut**(B) [4].

More generally, if A is some type of algebra, U(A) denotes the set of units of A, the homomorphism $\mu : A \to \mathrm{Aut}(A)$ sends a unit to the automorphism given by conjugation by it, then $(U(A), \mathrm{Aut}(A), \mu, id)$ is a crossed module.

This class of crossed modules has a very nice property with respect to general crossed modules. The homomorphism $\theta : D \to \mathrm{Aut}(B)$ of a general crossed module (B, D, d, θ) together with the homomorphism $\mu : B \to \mathrm{Aut}(B)$ gives a square:

$$
\begin{tikzcd}
B \arrow[r, "="] \arrow[d, "d"'] & B \arrow[d, "\mu"] \\
D \arrow[r, "\theta"'] & \mathrm{Aut}(B)
\end{tikzcd}
$$

We see that the first condition in the definition of a crossed module means the commutative square, i.e., the second condition in the definition of a morphism of crossed modules holds. Analogously, that the second condition in the definition of a crossed module means the second condition in the definition of a morphism of crossed modules holds. Thus, we do have a morphism of crossed modules $(id, \theta) : (B, D, d, \theta) \to (B, \mathrm{Aut}(B), \mu, id)$.

Moreover, in the group extension problem of the type of a crossed module [7], if we replace the general crossed module (B, D, d, θ) by the automorphism crossed module $(B, \mathrm{Aut}(B), \mu, id)$, we get the group extension problem as usual. Thus, the results of the group extension problem of the type of a crossed module cover those of the group extension problem.

7. Semi-direct product crossed modules

Let G be a group, $\alpha : M \to N$ be a morphism of left G-modules, and $N \rtimes G$ be the semi-direct product group. We define a homomorphism

$$d : M \to N \rtimes G \text{ by } d(m) = (\alpha(m), 1),$$

where 1 denotes the identity element of G. The homomorphism

$$\theta : N \rtimes G \to \mathrm{Aut}(M)$$

is defined via the projection from $N \rtimes G$ to G, that is,

$$\theta_{(n,g)}(m) = gm, \text{ for } n \in N, g \in G, m \in M.$$

Then, the quadruple is a crossed module.

In particular, if A and B are abelian groups, and B is considered to act trivially on A, the any homomorphism $f : A \to B$ gives a crossed module $(A, B, f, 0)$.

8. Pullback crossed modules of a crossed module along a group homomorphism

Suppose that we have a crossed module (B, D, d, θ), and a group homomorphism $\varphi : A \to D$, then we can form the "pullback group"

$$A \times_D B = \{(a,b) / \varphi(a) = d(b)\}$$

which is a subgroup of the product $A \times B$, where the multiplication in $A \times_D B$ is componentwise. The group homomorphism $d' : A \times_D B \to A$ is the restriction of the first projection morphism of the product, that is $d'(a,b) = a$.

The homomorphism $\theta' : A \to \mathrm{Aut}(A \times_D B)$ is defined by:

$$\theta'_{a'}(a,b) = (\mu_{a'}(a), \theta_{\phi(a')}(b)),$$

where $a' \in A, (a,b) \in A \times_D B, \mu$ is given by conjugation. Then, the tuple $(A \times_D B, A, d', \theta')$ is a crossed module. It is denoted by $\varphi^*(B, D, d, \theta)$ and called a *pullback crossed module of* (B, D, d, θ) *along* φ [4].

Moreover, there is a morphism of crossed module

$$(p, \varphi) : (A \times_D B, A, d', \theta') \to (B, D, d, \theta)$$

in which p is the second projection, i.e., $p : A \times_D B \to B, p(a,b) = b$.

In the above construction, if we replace groups by other algebraic structures, we obtain a class of pullback crossed modules of a given crossed module along a homomorphism.

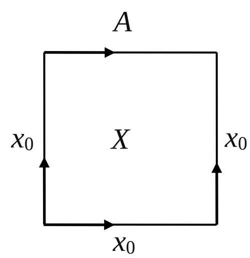
9. Crossed module constructed from a pointed topological space

Let X be a pointed topological space, that is a point x_0 has been chosen in X. Recall that *the fundamental group* $\pi_1(X, x_0)$ consists of all homotopy classes of continuous maps

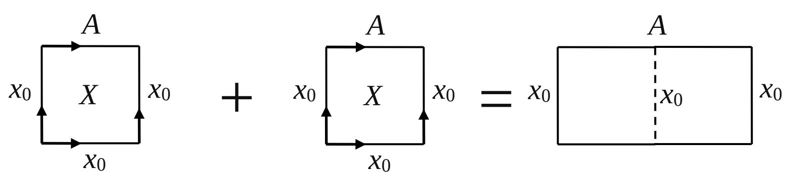
$f:[0,1]\to X$ with $f(0)=f(1)=x_0$. (Two such maps are *homotopic* if one can be continuously deformed into the other in such a way that the image of 0 and 1 remains x_0 throughout the deformation.) We think of these maps as paths in *X* beginning and ending at x_0.

The composition of paths yields a (not necessarily abelian) group structure on $\pi_1(X,x_0)$.

Now, if *A* is a subspace of *X* containing the point x_0 then we can consider *the second relative homotopy group* $\pi_2(X,A,x_0)$. This group consists of homotopy classes of continuous maps $g:[0,1]\times[0,1]\to X$ from the unit square into *X* which maps three edges of the square onto the point x_0 and the fourth edge into *A*. The appropriate picture of such a map *g* is



The juxtaposition of squares

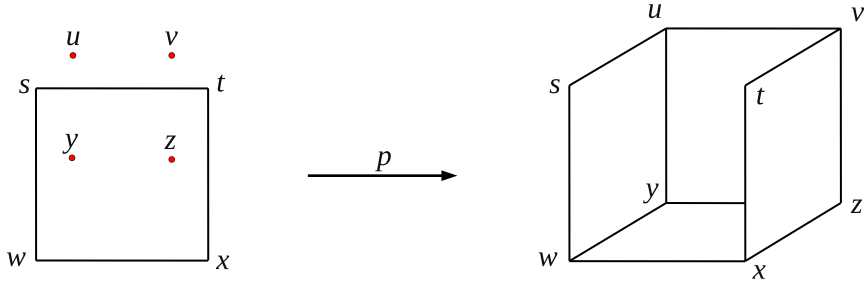


yields a (not necessarily abelian) group structure on $\pi_2(X,A,x_0)$.

By restricting to the fourth edge of the unit square we obtain a boundary homomorphism $\partial:\pi_2(X,A,x_0)\to\pi_1(A,x_0)$.

Moreover, we construct a homomorphism $\theta:\pi_1(A,x_0)\to\mathrm{Aut}(\pi_2(X,A,x_0))$ as follow.

First, there is a continuous map *p* from the unit square onto four faces of the unit cube:



Now, given a path $f:[0,1]\to A$ representing an element $[f]\in\pi_1(A,x_0)$, and a square $g:[0,1]\times[0,1]\to X$ representing an element $[g]\in\pi_2(X,A,x_0)$, we can construct a continuous map ${}^f g$ from the four faces of the unit cube to the space *X* by using g to map the face *uvyz* onto *X*, and mapping each horizontal line in the remaining three faces by *f* onto *A*. On composing ${}^f g$ with *p* we get a map which represents an element of $\pi_2(X,A,x_0)$. Thus, we have a homomorphism

$$\theta:\pi_1(A,x_0)\to \mathrm{Aut}(\pi_2(X,A,x_0)),\ \theta_{[f]}([g])=[{}^f g\circ p].$$

Therefore, the tuple $\left(\pi_2(X,A,x_0),\pi_1(A,x_0),\partial,\theta\right)$ is a crossed module. It is called a *fundamental crossed module* of the pair (X,A) [4].

Based on the fundamental crossed module of a pair of spaces, one can determine the second homotopy group of a CW-complex which is a free crossed module on 2-cells. Moreover, there is a functor from the category of pairs of pointed spaces to the category of crossed modules satisfying a form of the van Kampen theorem preserving the colimits.

References

[1] R. Brown, P. J. Higgins, R. Sivera (August 2011), *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids*, EMS Tracts in Mathematics, vol.15, 703 pages.

[2] R. Brown, O. Mucuk (1994), *Covering groups of non-connected topological groups revisited,* Math. Pro. Camb. Phil. Soc, 115, 997-110.

[3] G. Janelidze (2003), *Internal crossed modules*, Georgian Math. J., 10, no.1, 99-114.

[4] T. Porter (2010), *The crossed menagerie: an introduction to crossed gadgetry and cohomology in algebra and topology*, (ten chapters), Bangor and Ottawa.

[5] N. T. Quang, P. T. Cuc (2012), *Crossed bimodules over rings and Shukla cohomology*, Math. Commun, 17, no.2, 575-598.

[6] N. T. Quang, P. T. Cuc (2015), *Equivariant crossed modules and cohomology of groups with operators,* Bull. Korean Math. Soc, 52, no.4, 1077-1095.

[7] N. T. Quang, P. T. Cuc, N. T. Thuy (2014), *Crossed modules and strict Gr-categories,* Commun. Korean Math. Soc, vol.29, no.1, 9-22.

[8] N. T. Quang, C. T. K. Phung, P. T. Cuc (2014), *Braided equivariant crossed modules and cohomology of Γ-modules*, Indian Journal of Pure and Applied Mathematics, 953-975.

[9] N. T. Quang (2015), *Group extensions of the co-type of a crossed module and strict categorical groups,* arXiv: 1503.04379v1.

[10] J. H. C. Whitehead (1949), *Combinatorial homotopy II*, Bull. Amer. Math. Soc., 55, 453-439.