NON-AUTONOMOUS STOCHASTIC EVOLUTION EQUATIONS, INERTIAL MANIFOLDS AND CHAFEE-INFANTE MODELS

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Abstract: *Consider a stochastic evolution equation containing Stratonovich-multiplicative white noise of the form* $\frac{du}{dt} + Au = f(t,u) + u^\circ \dot{W}$ *where the partial differential operator* A *is positive definite, self-adjoint with a discrete spectrum; and the nonlinear part* f *satisfies the* φ *-Lipschitz condition with* φ *belonging to an admissible function space. We prove the existence of a (stochastic) inertial manifold for the solutions to the above equation. Our method relies on the Lyapunov-Perron equation in a combination with the admissibility of function spaces. An application to the non-autonomous Chafee - Infante equations is given to illustrate our results.*

Keywords: *Stochastic inertial manifold;* φ *- Lipschitz; Admissibility, Lyapunov - Perron equation, nonautonomous Chafee - Infante equations.*

1. Introduction

In the present paper, we study the existence of an inertial manifold for a class of stochastic partial differential equations (SPDE) in which the nonlinear part is assumed to be φ -Lipschitz. Concretely, we will prove the existence of an inertial manifold for the following stochastic evolution equation driven by linear multiplicative white noise in the sense of Stratonovich $\frac{du}{dt} + Au = f(t,u) + u^\circ \dot{W}$ (1.1)

where A is a positive definite, self-adjoint, closed linear operator with a discrete spectrum; f is φ - Lipschitz (see Definition 2.3); and $u^\circ \dot{W}$ is the noise.

There are two main difficulties when we transfer to the case of SPDE with φ -Lipschitz nonlinear term f : Firstly, since the nonlinear operator f is φ -Lipschitz, the existence and uniqueness theorem for solutions to (1.1) is not available. Secondly, the appearance of the white noise changes the formula of mild solutions for SPDE, and therefore changes the representation of Lyapunov-Perron equation used in the construction of the inertial manifold.

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To overcome such difficulties, we reformulate the definition of inertial manifolds suchthat it contains the existence and uniqueness theorem as a property of the manifold (see Definition 2.5 below). Furthermore, we construct the structure of the mild solutions to (1.1) using the white noise in such a way that it allows to combine the exponential estimates of the linear part of Eq. (1.1) with the existence and uniqueness of its bounded solutions (in negative direction) in the case of φ-Lipschitz nonlinear forcing terms. Consequently, we obtain the existence of an inertial manifold for semi-linear SPDE with φ -Lipschitz nonlinear term and general spectral gap conditions.

Our main result is contained in Theorem 2.8 which extends the results in [12] to the case of semilinear SPDE. Finally, we apply the obtained result to the nonautonomous Chafee - Infante equations (see Section 4).

2. Inertial Manifolds

Throughout this paper we assume that A is a positive definite, self-adjoint, closed and linear operator on a separable Hilbert space X with a discrete spectrum, say

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \text{each with finite multiplicity and} \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis in X consisted of the corresponding eigenfunctions of A (i.e., $Ae_k = \lambda_k e_k$).

Let then λ_N and λ_{N+1} be two successive and different eigenvalues with $\lambda_N < \lambda_{N+1}$, let further P be the orthogonal projection onto the first N eigenvectors of the operator A. Denote by $(e^{-tA})_{t\ge 0}$ the semigroup generated by $-A$.

Since ImP is finite dimension, we have that the restriction $(e^{-tA}P)_{t\ge 0}$ of the semigroup $(e^{-tA})_{t\ge 0}$ to ImP can be extended to the whole line $\mathbb{R}$.

For $0 \le \theta < 1/2$ we then recall the following dichotomy estimates (see [22]):

$$\begin{aligned}
&\| e^{-tA}P \| \le Me^{\lambda_N |t|}, \quad t \in \mathbb{R} \text{ for some constant } M \ge 1,\\
&\| A^{\theta} e^{-tA} P \| \le \lambda_N^{\theta} M e^{\lambda_N |t|}, \quad t \in \mathbb{R},\\
&\| e^{-tA}(I-P) \| \le Me^{-\lambda_{N+1} t}, \quad t \ge 0,\\
&\text{and}\\
&\| A^{\theta} e^{-tA}(I-P) \| \le M\left[\left(\frac{\theta}{t}\right)^{\theta} + \lambda_{N+1}^{\theta}\right] e^{-\lambda_{N+1} t}, \quad t > 0, \theta > 0.
\end{aligned} \tag{2.1}$$

Next, we recall some notions on function spaces and refer to Massera and Schaffer [19], Rabiger and Schnaubelt [20], and Huy [11] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$.

The space $L_{1,loc}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ (modulo λ -nullfunctions) becomes a Frechet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|\, dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{N}$.

We can now define Banach function spaces as follows

Definition 2.1. [12] A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ -nullfunctions) is called a *Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

(1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \le |\varphi(\cdot)|$, λ -a.e., then $\psi \in E$ and $\|\psi\|_E \le \|\varphi\|_E$,

(2) the characteristic functions χ_A belongs to E for all $A \in \mathcal{B}$ of finite measure,

$$\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty \qquad \text{and} \qquad \inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0,$$

(3) E $L_{1,loc}(\mathbb{R})$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \le \beta_{p_n} \|f\|_E$ for all $f \in E$.

We remark that condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}$ there exists a number $\beta_J \ge 0 \ne$ such that $\int_J |f(t|)|\, dt \le \beta_J \|f\|_E$ for all $f \in E$.

Definition 2.2. [12] The Banach function space E is called *admissible* if

(1) there is a constant $M \ge 1$ such that for every compact interval $[a,b] \subset \mathbb{R}$ we have

$$\int_a^b |\varphi(t)|\, dt \le \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{2.2}$$

(2) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1(t) := \int_{t-1}^{t} \varphi(\tau)d\tau$ belongs to E.

(3) E is T_τ^+ - invariant and T_τ^- - invariant, where T_τ^+ and T_τ^- are defined for $\tau \in \mathbb{R}_+$ by

$$\begin{aligned} T_\tau^+\varphi(t) &:= \varphi(t-\tau) \quad \text{for } t \in \mathbb{R}, \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{2.3}$$

Moreover, there are N_1, N_2 such that $\|T_\tau^+\| \le N_1, \|T_\tau^-\| \le N_2$ for all $\tau \in \mathbb{R}_+$.

Next, we introduce the notion of φ -Lipschitz function in the following definition.

Definition 2.3. For $\theta \in [0,1/2)$ put $X_\theta := D(A^\theta)$. Let E be an admissible Banach function space on $\mathbb{R}$ and φ be a positive function belonging to E. A function $f : \mathbb{R} \times X_\theta \to X$ is said to be φ - Lipschitz if f satisfies

$\|f(t,x)\| \le \varphi(t)\left(1+\|A^\theta x\|\right)$ for a.e. $t \in \mathbb{R}$ and all $x \in X_\theta$;

$\|f(t,x_1) - f(t,x_2)\| \le \varphi(t)\|A^\theta (x_1 - x_2)\|$ for a.e. $t \in \mathbb{R}$ and all $x_1, x_2 \in X_\theta$.

We can define the Green function as follows

$$G(t,s) = \begin{cases} e^{-(t-s)A}(I-P) & \text{for } t > s, \\ -e^{-(t-s)A}P & \text{for } t \le s. \end{cases} \tag{2.4}$$

Then, one can see that $G(t,s)$ maps X into X_θ. Also, by the dichotomy estimates and for $\gamma = (\lambda_N + \lambda_{N+1})/2$ we have

$$\left\|e^{\gamma(t-s)} A^\theta G(t,s)\right\| \le K(t,s)e^{-\alpha|t-s|} \quad \text{for all } t,s \in \mathbb{R} \tag{2.5}$$

where $\alpha = (\lambda_{N+1} - \lambda_N)/2$ and $K(t,s) = \begin{cases} M\left(\left(\dfrac{\theta}{t-s}\right)^\theta + \lambda_{N+1}^\theta\right) & \text{if } t > s \\ M\lambda_N^\theta & \text{if } t \le s \end{cases}.$

We then recall the definition of metric dynamical systems (MDS) associated with the Wiener process which will be used throughout this paper. For details on these notions we refer the reader to [1,4,9,17,18,21].

Definition 2.4. [1] A family of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *metric dynamical system* (MDS) if the following conditions are satisfied

(i) $\theta_0 = Id_\Omega$, and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t,s \in \mathbb{R}$;

(ii) The map $(t,\omega) \mapsto \theta_t\omega$ is $(\mathcal{B} \otimes \mathcal{F}; \mathcal{F})$ - measurable;

(iii) $\mathbb{P}$ is invariant respect to θ_t for all $t \in \mathbb{R}$;

In this paper, we deal with the MDS induced by the Wiener process. Precisely, let W_t be a two-sided Wiener process with trajectories in the space $C_0(\mathbb{R},\mathbb{R})$ of real continuous functions defined on $\mathbb{R}$, taking zero value at $t = 0$; $\mathcal{F}$ is the Borel σ - algebra associated with the Wiener process; $\mathbb{P}$ is the classical Wiener measure on Ω and for each $t \in \mathbb{R}$ the mapping $\theta_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t). \tag{2.6}$$

Moreover, we will consider a subset $\Omega \subset C_0(\mathbb{R},\mathbb{R})$, which is invariant under $\{\theta_t\}_{t\in\mathbb{R}}$, i.e., $\theta_t\Omega = \Omega$ for $t \in \mathbb{R}$.

Now, we make precisely the notion of a stochastic inertial manifold, and then prove its existence for solutions to SPDE (1.1). To do this, we first rewrite equation (1.1) in a more convenient form. To this purpose, let $z(\cdot)$ be a unique stationary solution to the following scalar equation

$$dz + zdt = dW_t \tag{2.7}$$

Then, by putting $v(t) = e^{-z(\theta_t\omega)}u(t)$ and using Ito formula, we arrive at

$$\begin{aligned} de^{-z(\theta_t\omega)} &= \left(z(\theta_t\omega)e^{-z(\theta_t\omega)} + \frac{1}{2}e^{-z(\theta_t\omega)} \right)dt - e^{-z(\theta_t\omega)}dW_t \\ &= z(\theta_t\omega)e^{-z(\theta_t\omega)}dt - e^{-z(\theta_t\omega)}{}_\circ dW_t, \end{aligned} \tag{2.8}$$

where the second equality above follows from the conversion between the Ito and Stratonovich integrals. Furthermore, we have that

$$dv = d\left(e^{-z(\theta_t\omega)}u\right) = u_\circ de^{-z(\theta_t\omega)} + e^{-z(\theta_t\omega)}{}_\circ du. \tag{2.9}$$

Hence, Eq. (1.1) becomes

$$\frac{dv}{dt} + Av = z(\theta_t\omega)v + e^{-z(\theta_t\omega)}f(t, e^{z(\theta_t\omega)}v). \tag{2.10}$$

Next, by a *mild solution* to equation (2.10) on an interval $\mathbb{J}$ we mean a strongly measurable function $v(\cdot)$ defined on $\mathbb{J}$ with the values on X_θ that satisfies the integral equation

$$v(t) = e^{-(t-s)A+\int_s^t z(\theta_r\omega)dr}v(s) + \int_s^t e^{-(\tau-s)A+\int_s^\tau z(\theta_r\omega)dr - z(\theta_\tau\omega)} f\left(\tau, e^{z(\theta_\tau\omega)}v(\tau)\right)d\tau \tag{2.11}$$

for a.e. $t \geq s, t, s \in \mathbb{J}$ and $\omega \in \Omega$.

We then give the notion of inertial manifolds in the following definition.

Definition 2.5. A stochastic inertial manifold for mild solutions to Eq. (2.10) is a collection of Lipschitz surfaces $\{M(\omega)\}_{\omega\in\Omega}$ in X such that

(i) for each $\omega \in \Omega, M(\omega)$ can be represented as the graph of a Lipschitz mapping $m(\omega): PX \to QX_\theta$, i.e., $M(\omega) = \{ x + m(\omega)x^{..} x \in PX \};$;

(ii) there exists a constant $\gamma > 0$ such that to each $x_0 \in M(\omega)$ there corresponds one and only one solution $v(\cdot)$ to Eq. (2.11) on $(-\infty, 0]$ such that $v(0) = x_0$ and

$$\sup_{t\leq 0}\left\| e^{\gamma t - \int_0^t z(\theta_r\omega)dr} A^\theta v(t) \right\| < \infty \tag{2.12}$$

(iii) $M(\omega)$ is positively invariant under Eq. (2.11), i.e., if a solution $v(t), t \geq 0$ of Eq. (2.11) satisfies $v(0) \in M(\omega)$, then we have $v(t) \in M(\theta_t\omega)$ for all $t > 0$;

(iv) $M(\omega)$ exponentially attracts all the solutions to Eq. (2.11), i.e., for any solution $v(\cdot)$ of Eq. (2.11) there exist a solution $\overset{*}{v}(\cdot)$ of Eq. (2.11) with $\overset{*}{v}(t) \in M(\omega)$ for all $t \geq 0$ and a constant $H(\omega)$ such that $\left\| A^{\theta}\left(\overset{*}{v}(t) - v(t)\right) \right\| \leq H(\omega) e^{-\gamma t}$ for $t > 0$.

Lemma 2.6. *Let* $f : \mathbb{R} \times X_{\theta} \to X$ *be* φ*-Lipschitz for a positive function* φ *belonging to an admissible space* E *such that*

$$R(\varphi, \theta) := \sup_{t \in \mathbb{R}} \left(\int_{t-1}^{t} \frac{\varphi(s)^{\frac{1+\theta}{2\theta}}}{(t-s)^{\frac{1+\theta}{2}}} ds \right) < \infty. \tag{2.13}$$

Let $v(t), t \leq 0$*, be a solution to* (2.11) *such that* $v(t) \in X_{\theta}$ *for* $t \leq 0$ *and*

$$\sup_{t \leq 0} \left\| e^{\gamma t - \int_0^t z(\theta_r \omega) dr} A^{\theta} v(t) \right\| < \infty. \tag{2.14}$$

Then, $v(t)$ satisfies

$$v(t) = e^{-tA + \int_0^t z(\theta_r \omega) dr} \mu + \int_{-\infty}^{0} G(t,s) e^{\int_s^t z(\theta_r \omega) dr - z(\theta_s \omega)} f\left(s, e^{z(\theta_s \omega)} v(s)\right) ds \tag{2.15}$$

where $\mu \in PX$, and $G(t,s)$ is the Green function defined as in (2.4).

Proof. Put $y(t) = \int_{-\infty}^{0} G(t,s) e^{\int_s^t z(\theta_r \omega) dr - z(\theta_s \omega)} f\left(s, e^{z(\theta_s \omega)} v(s)\right) ds.$ (2.16)

We have $y(t) \in X_{\theta}$ for $t \leq 0$, and

$$\sup_{t \leq 0} \left\| e^{\gamma t - \int_0^t z(\theta_r \omega) dr} A^{\theta} y(t) \right\| \leq k \left(1 + \sup_{t \leq 0} e^{\gamma t - \int_0^t z(\theta_r \omega) dr} \left\| A^{\theta} v(t) \right\| \right) < \infty \tag{2.17}$$

Where

$$k = \begin{cases} \dfrac{M\left(\theta^{\theta} N_1 + \lambda_{N+1}^{\theta} N_1 + \lambda_N^{\theta} N_2\right)}{1 - e^{-\alpha}} \left\| \Lambda_1 \varphi \right\|_{\infty} + M \theta^{\theta} R(\varphi, \theta) \left(\dfrac{1-\theta}{(1+\theta)\alpha} \right)^{\frac{1-\theta}{1+\theta}} & \text{for } 0 < \theta < \dfrac{1}{2} \\ \dfrac{M(N_1 + N_2)}{1 - e^{-\alpha}} \left\| \Lambda_1 \varphi \right\|_{\infty} & \text{for } \theta = 0. \end{cases} \tag{2.18}$$

By computing directly, one can verify that $y(\cdot)$ satisfies the integral equation

$$y(0) = e^{tA - \int_0^t z(\theta_r\omega)dr} y(t) + \int_t^0 e^{sA - \int_0^s z(\theta_r\omega)dr - z(\theta_s\omega)} f\left(s, e^{z(\theta_s\omega)} v(s)\right) ds. \tag{2.19}$$

On the other hand,

$$v(0) = e^{tA - \int_0^t z(\theta_r\omega)dr} v(t) + \int_t^0 e^{sA - \int_0^s z(\theta_r\omega)dr - z(\theta_s\omega)} f\left(s, e^{z(\theta_s\omega)} v(s)\right) ds.$$

Then

$$v(0) - y(0) = e^{tA - \int_0^t z(\theta_r\omega)dr} [v(t) - y(t)]. \tag{2.20}$$

We need to prove that $v(0) - y(0) \in PX$. To do this, applying the operator $A^\theta (I - P)$ to (2.20), we have

$$\left\| A^\theta (I-P)[v(0) - y(0)] \right\| = \left\| e^{tA - \int_0^t z(\theta_r\omega)dr} A^\theta (I-P)[v(t) - y(t)] \right\|$$

$$\leq M e^{(\lambda_{N+1} - \gamma)t} \| I - P \| \left\| e^{\gamma t - \int_0^t z(\theta_r\omega)dr} A^\theta [v(t) - y(t)] \right\|.$$

Since $\left\| e^{\gamma t - \int_0^t z(\theta_r\omega)dr} A^\theta [v(t) - y(t)] \right\| < \infty$, letting $t \to -\infty$ we obtain

$$\left\| A^\theta (I-P)[v(0) - y(0)] \right\| = 0,$$

Hence $A^\theta (I-P)[v(0) - y(0)] = 0.$

Since A^θ is injective, it follows that $(I-P)[v(0) - y(0)] = 0$.

Thus, $\mu = v(0) - y(0) \in PX$.

Since the restriction of e^{-tA} on PX, $t \geq 0$, is invertible with the inverse $e^{tA}P$, we have for $t \leq 0$ that

$$v(t) = e^{-tA + \int_0^t z(\theta_r\omega)dr} \mu + y(t)$$

$$= e^{-tA + \int_0^t z(\theta_r\omega)dr} \mu + \int_{-\infty}^0 G(t,s) e^{\int_s^t z(\theta_r\omega)dr - z(\theta_s\omega)} f\left(s, e^{z(\theta_s\omega)} v(s)\right) ds.$$

The proof is completed.

The following lemma describes the existence and uniqueness of solution belonging to weighted spaces.

Lemma 2.7. *Let $f:\mathbb{R}\times X_\theta \to X$ be φ -Lipschitz for φ satisfying the condition (2.13). Let the constant k be defined as in* (2.18). *Then, if $k<1$, there corresponds to each $\xi \in PX$ one and only one solution $v(t)=v(t,\omega,\xi)$ of Eq.(2.11) on $(-\infty,0]$ satisfying the condition $v(0)=\xi$ and* $\sup_{t\le 0} e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta v(t)\right\|<\infty.$

Proof. We denote

$$L_\infty^{\gamma,-}=L_\infty^{\gamma}\left((-\infty,0],X_\theta\right)$$

$$=\{h:(-\infty,0]\to X_\theta \mid h \text{ is strongly measurable}$$

$$\text{and } \sup_{t\le 0} e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta h(t)\right\|<\infty\}$$

endowed with the norm $\|h\|_{\gamma,-,\infty}=\sup_{t\le 0} e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta h(t)\right\|.$

For each $\xi\in PX$, we define the transformation T as

$$(Tv)(t)=e^{-tA+\int_0^t z(\theta_r\omega)dr}\xi+\int_{-\infty}^0 G(t,s)e^{\int_s^t z(\theta_r\omega)dr-z(\theta_s\omega)} f\left(s,e^{z(\theta_s\omega)}v(s)\right)ds \text{ for } t\le 0.$$

By the following estimates,

$$\|Tv\|_{\gamma,-,\infty}\le M\lambda_N^\theta\|\xi\|+k\left(1+\sup_{t\le 0} e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta v(t)\right\|\right),\forall v(\cdot)\in L_\infty^{\gamma,-},$$

and $\|Tu(\cdot)-Tv(\cdot)\|_{\gamma,-,\infty}\le k\| u(\cdot)-v(\cdot)\|_{\gamma,-,\infty},\ \forall\, v\in L_\infty^{\gamma,-}$.

We conclude that $T:L_\infty^{\gamma,-}\to L_\infty^{\gamma,-}$ is contraction since $k<1$. Thus, there exists a unique $v(\cdot)\in L_\infty^{\gamma,-}$ such that $Tv=v$.

By definition of T we have that $v(\cdot)$ is the unique solution in $L_\infty^{\gamma,-}$ of (2.11) for $t\le 0$.

Theorem 2.8. *Let φ belong to an admissible space E and satisfy condition* (2.13) *and let f be φ -Lipschitz. Suppose that*

$$k<1 \text{ and } \frac{M^3k\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_\infty+k<1 \tag{2.21}$$

where k is defined as in (2.18).

Then, there exists a stochastic inertial manifold for mild solutions to Eq. (2.10).

Proof. For each $\omega \in \Omega$ we define the map $m(\omega): PX \to QX_\theta$ by

$$m(\omega)x = \int_{-\infty}^{0} e^{sA - \int_0^s z(\theta_r\omega)dr - z(\theta_s\omega)} (I-P) f(s, e^{z(\theta_s\omega)} v(s)) ds = (I-P)v(0) \tag{2.22}$$

where $v(.)$ is the unique solution of Eq. (2.11) in $L_\infty^{\gamma,-}$ satisfying $Pv(0) = x$.

(Note that Lemma 2.7 guarantees the existence and uniqueness of such a v). Furthermore, for each $\omega \in \Omega$ we put $M(\omega) = \{x + m(\omega)x : x \in PX\}$.

From the definition of $m(\omega)$ it follows that

$$M(\omega) = \{v_0 \in PX \text{ there exists a solution } v = v(t,\omega,v_0) \in L_\infty^{\gamma,-}((-\infty,0], X_\theta) \text{ of (2.11) with } v(0) = v_0\}.$$

Then, $M(\omega)$ satisfies all the properties of an inertial manifold from Definition 2.5.

Firstly, we show that $m(\omega)$ is Lipschitz continuous. Indeed, for x_1, x_2 belonging to PX one has

$$\begin{aligned}
&\left\| A^\theta \left(m(\omega)x_1 - m(\omega)x_2 \right) \right\| \\
&\le \int_{-\infty}^{0} \left\| A^\theta e^{sA - \int_0^s z(\theta_r\omega)dr} (I-P) \right\| \left(e^{-z(\theta_s\omega)} \left\| f(s, e^{z(\theta_s\omega)} v_1(s)) - f(s, e^{z(\theta_s\omega)} v_2(s)) \right\| \right) ds \\
&\le \int_{-\infty}^{0} \left\| A^\theta e^{sA - \int_0^s z(\theta_r\omega)dr} (I-P) \right\| \varphi(s) \left\| A^\theta \left(v_1(s) - v_2(s) \right) \right\| ds \\
&\le \int_{-\infty}^{0} e^{-\gamma s} \left\| A^\theta G(0,s) \right\| \varphi(s) e^{\gamma s - \int_0^s z(\theta_r\omega)dr} \left\| A^\theta \left(v_1(s) - v_2(s) \right) \right\| \le k \left\| v_1(\cdot) - v_2(\cdot) \right\|_{\gamma.\theta,\infty}.
\end{aligned} \tag{2.23}$$

Next, we estimate $\left\| v_1(\cdot) - v_2(\cdot) \right\|_{\gamma.\theta,\infty}$. We have that

$$\begin{aligned}
&e^{\gamma t - \int_0^t z(\theta_r\omega)dr} \left\| A^\theta \left(v_1(t) - v_2(t) \right) \right\| = \Bigg\| e^{\gamma t - \int_0^t z(\theta_r\omega)dr} A^\theta \Bigg(e^{-tA + \int_0^t z(\theta_r\omega)dr} (x_1 - x_2) \\
&+ \int_{-\infty}^{0} G(t,s) e^{\int_s^t z(\theta_r\omega)dr - z(\theta_s\omega)} \left[f(s, e^{z(\theta_s\omega)} v_1(s)) - f(s, e^{z(\theta_s\omega)} v_2(s)) \right] \Bigg) \Bigg\| \\
&\le M \lambda_N^\theta \left\| A^\theta (x_1 - x_2) \right\| + k \left\| v_1(\cdot) - v_2(\cdot) \right\|_{\gamma,-,\infty} \quad \text{for all } t \le 0.
\end{aligned}$$

Hence, we obtain

$$\left\|v_1(\cdot)-v_2(\cdot)\right\|_{\gamma,-,\infty} \le M\lambda_N^{\theta}\left\|A^{\theta}(x_1-x_2)\right\| + k\left\|v_1(\cdot)-v_2(\cdot)\right\|_{\gamma,-,\infty}$$

and since $k<1$, we get $\left\|v_1(\cdot)-v_2(\cdot)\right\|_{\gamma,-,\infty} \le \dfrac{M\lambda_N^{\theta}}{1-k}\left\|A^{\theta}(x_1-x_2)\right\|.$

Substituting the above inequality into (2.23) we obtain

$$\left\|A^{\theta}\left(m(\omega)x_1-m(\omega)x_2\right)\right\| \le \frac{Mk\lambda_N^{\theta}}{1-k}\left\|A^{\theta}(x_1-x_2)\right\|.$$

Therefore, the property (i) in Definition 2.5 holds.

The property (ii) in Definition 2.5 follows from Lemma 2.7.

We then prove the property (iii). To do this, for each fixed $\omega\in\Omega, v_0\in M(\omega)$ and $t>0$, let $v(\cdot)$ be a mild solution of (2.10) on $[0,t]$ with initial datum v_0 (in the fiber ω).

Put $\xi(s,\theta_t\omega,v(t,\omega,v_0)) = v(s+t,\omega,v_0)$ for all $s\le 0$.

Then, to prove $v(t)=v(t,\omega,v_0)\in M(\theta_t\omega)$ we will show that $\xi\in L_{\infty}^{\gamma,-}(\theta_t\omega)$.

This claim follows from the fact that

$$\begin{aligned}
&\sup_{s\le 0} e^{\gamma s-\int_0^s z(\theta_{r+t}\omega)dr}\left\|A^{\theta}\xi(s,\theta_t\omega,v(t,\omega,v_0))\right\| \\
&= \sup_{s\le 0} e^{\gamma s-\int_t^{s+t} z(\theta_r\omega)dr}\left\|A^{\theta}v(s+t,\omega,v_0)\right\| \\
&= \sup_{\tau\le t} e^{\gamma(\tau-t)-\int_t^{\tau} z(\theta_r\omega)dr}\left\|A^{\theta}v(\tau,\omega,v_0)\right\| \\
&= e^{-\gamma t+\int_0^t z(\theta_r\omega)dr}\sup_{\tau\le t} e^{\gamma\tau-\int_0^{\tau} z(\theta_r\omega)dr}\left\|A^{\theta}v(\tau,\omega,v_0)\right\| < \infty.
\end{aligned}$$

Therefore, the property (iii) in Definition 2.5 holds.

Lastly, we prove the property (iv). To this end, denote

$$L_{\infty}^{\gamma,+} = \{v:[0,+\infty)\to X_{\theta} \text{ ¨ } v \text{ is strongly measurable and } \sup_{t\ge 0} e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^{\theta}v(t)\right\|<+\infty\}.$$

Assume that $v^*(\cdot), v(\cdot)$ are two solutions of (2.10). Let $w=v^*-v$, then w is a solution to the equation

$$\frac{dw}{dt} + Aw = z(\theta_t\omega)w + e^{-z(\theta_t\omega)}F(t,e^{z(\theta_t\omega)}w) \tag{2.24}$$

where $F(t,e^{z(\theta_t\omega)}w)=f(t,e^{z(\theta_t\omega)}(u+w))-f(t,e^{z(\theta_t\omega)}u)$.

One can see that if $w\in L_\infty^{\gamma,+}$ solve (2.24) then w can be expressed by

$$w(t)=e^{-tA+\int_0^t z(\theta_r\omega)dr}Qw(0)+\int_0^{+\infty}G(t,s)e^{\int_s^t z(\theta_r\omega)dr-z(\theta_s\omega)}F(s,e^{z(\theta_s\omega)}w(s))ds. \quad (2.25)$$

Since $u^*(0)=u(0)+w(0)\in M(\omega)$, and u^* lies on M iff $Qu^*(0)=m(\omega)\left(Pu^*(0)\right)$ we have $Qw(0)=-Qu(0)+m(\omega)\left(Pu(0)+Pw(0)\right)$.

Substituting this equality into (2.25) we obtain

$$w(t)=e^{-tA+\int_0^t z(\theta_r\omega)dr}\left[-Qu(0)+m(\omega)\left(Pu(0)+Pw(0)\right)\right]$$
$$+\int_0^{+\infty}G(t,s)e^{\int_s^t z(\theta_r\omega)dr-z(\theta_s\omega)}F(s,e^{z(\theta_s\omega)}w(s))ds. \quad (2.26)$$

We now prove the existence of solution $w(\cdot)\in L_\infty^{\gamma,+}$ to the Eq. (2.26).

To do this, we show that the transformation T defined by

$$(Tx)(t)=e^{-tA+\int_0^t z(\theta_r\omega)dr}\left[-Qu(0)+m(\omega)\left(Pu(0)+Px(0)\right)\right]$$
$$+\int_0^{+\infty}G(t,s)e^{\int_s^t z(\theta_r\omega)dr-z(\theta_s\omega)}F(s,e^{z(\theta_s\omega)}x(s))ds \text{ for } t\ge 0. \quad (2.27)$$

acts from $L_\infty^{\gamma,+}$ into itself and is a contraction.

Indeed, for $x(\cdot)\in L_\infty^{\gamma,+}$ we have $\left\|F(t,e^{z(\theta_t\omega)}x(t))\right\|\le\varphi(t)e^{z(\theta_t\omega)}\left\|A^\theta x(t)\right\|$

and by putting $q(x)=\left[-Qu(0)+m(\omega)\left(Pu(0)+Px(0)\right)\right]$ we can estimate

$$e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta(Tx)(t)\right\|\le e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta e^{-tA+\int_0^t z(\theta_r\omega)dr}q(x)\right\| \quad (2.28)$$
$$+e^{\gamma t-\int_0^t z(\theta_r\omega)dr}\left\|A^\theta\int_0^{+\infty}G(t,s)e^{\int_s^t z(\theta_r\omega)dr-z(\theta_s\omega)}F(s,e^{z(\theta_s\omega)}x(s))ds\right\|$$
$$\le e^{\gamma t}\left\|A^\theta e^{-tA}q(x)\right\|+k\|x\|_{L_\infty^{\gamma,+}}$$

$$e^{\gamma t}\left\|A^{\theta}e^{-tA}q(x)\right\| \le \left\|e^{\gamma t}A^{\theta}e^{-tA}\left(-Qu(0)+m(\omega)\left(Pu(0)\right)\right)\right\|$$
$$+\left\|e^{\gamma t}A^{\theta}e^{-tA}\left(m(\omega)\left((Pu(0)+Px(0)\right)-m(\omega)\left(Pu(0)\right)\right)\right\|$$

and
$$\le Me^{-(\lambda_{N+1}-\gamma)t}\left(\left\|A^{\theta}\left(-Qu(0)+m(\omega)\left(Pu(0)\right)\right)\right\|\right.$$
$$\left.+\left\|A^{\theta}\left(m(\omega)\left(Pu(0)+Px(0)\right)-m(\omega)\left(Pu(0)\right)\right)\right\|\right)$$
$$\le M\eta + M\left\|A^{\theta}\left(m(\omega)\left(Pu(0)+Px(0)\right)-m(\omega)\left(Pu(0)\right)\right)\right\|$$

Where $\eta = \left\|A^{\theta}\left(-Qu(0)+m(\omega)\left(Pu(0)\right)\right)\right\|$. By Lipschitz property of $m(\omega)$ we have

$$\left\|A^{\theta}\left(m(\omega)\left(Pu(0)+Px(0)\right)-m(\omega)\left(Pu(0)\right)\right)\right\| \le \frac{Mk\lambda_N^{\theta}}{1-k}\left\|A^{\theta}Px(0)\right\|$$
$$\le \frac{Mk\lambda_N^{\theta}}{1-k}\left\|\int_0^{\infty} A^{\theta}e^{sA-\int_0^s z(\theta_r\omega)dr-z(\theta_s\omega)}PF(s,e^{z(\theta_s\omega)}x(s))ds\right\|$$
$$\le \frac{M^2k\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_{\infty}\left\|x\right\|_{L_{\infty}^{\gamma,+}}.$$

where k is defined as in (2.18). Substituting these estimates into (2.27) we obtain $Tx \in L_{\infty}^{\gamma,+}$ and $\left\|Tx\right\|_{L_{\infty}^{\gamma,+}} \le M\eta + \left[\frac{M^3k\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_{\infty}+k\right]\left\|x\right\|_{L_{\infty}^{\gamma,+}}$.

Therefore, T acts from $L_{\infty}^{\gamma,+}$ into itself. Now, using the fact that $\left\|F(t,e^{z(\theta_t\omega)}w_1)-F(t,e^{z(\theta_t\omega)}w_2)\right\| \le \varphi(t)e^{z(\theta_t\omega)}\left\|A^{\theta}(w_1-w_2)\right\|$ and for $x,z \in L_{\infty}^{\gamma,+}$ we estimate

$$\left\|e^{\gamma t-\int_0^t z(\theta_r\omega)dr}A^{\theta}(Tx(t)-Tz(t))\right\|$$
$$\le \frac{Mk\lambda_N^{2}}{1-k}\left\|\int_0^{\infty} A^{\theta}e^{sA-\int_0^s z(\theta_r\omega)dr-z(\theta_s\omega)}P\left(F(s,e^{z(\theta_s\omega)}x)-F(s,e^{z(\theta_s\omega)}z)\right)\right\|ds$$
$$+\int_0^{\infty}\left\|e^{\gamma(t-s)}A^{\theta}G(t,s)\right\|\left\|e^{\gamma s-\int_0^s z(\theta_r)dr-z(\theta_s\omega)}\left(F(s,e^{z(\theta_s\omega)}x)-F(s,e^{z(\theta_s\omega)}z)\right)\right\|$$
$$\le \left[\frac{M^3k\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_{\infty}+k\right]\left\|x(\cdot)-z(\cdot)\right\|_{L_{\infty}^{\gamma,+}}.$$

Hence, if $\dfrac{M^3 k\lambda_N^{2\theta} N_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_\infty + k < 1$, then $T : L_\infty^{\gamma,+} \to L_\infty^{\gamma,+}$ is a contraction.

Thus, there exists a unique $w(\cdot) \in L_\infty^{\gamma,+}$ such that $v^*(0) = u(0) + w(0) \in M(\omega)$ and

$$\left\|A^\theta\left(v^*(t) - v(t)\right)\right\| = \left\|A^\theta w(t)\right\| \le \frac{M\eta}{1-L} e^{-\gamma t + \int_0^t z(\theta_r\omega)dr}, \forall t \ge 0$$

$$\le \frac{M\eta}{1-L}\kappa(\omega)e^{-\gamma t}, \ \forall t \ge 0$$

$$\le H(\omega)e^{-\gamma t}, \quad \forall t \ge 0.$$

where $H(\omega) = \dfrac{M\eta}{1-L}\kappa(\omega)$, $L = \dfrac{M^3 k\lambda_N^{2\theta} N_2}{(1-k)(1-e^{-\alpha})}\left\|\Lambda_1\varphi\right\|_\infty + k.$

Therefore, $M(\omega)$ exponentially attracts every solution u of (2.10).

Remark 2.9. By the determination the constant k we have that, for $0 \le \theta < \frac{1}{2}$, the condition (2.21) is fulfilled if the following two conditions hold.

(i) the difference $\lambda_{N+1} - \lambda_N$ is sufficiently large, and/or

(ii) the norm $\left\|\Lambda_1\varphi\right\|_\infty = \sup_{t\in\mathbb{R}} \int_{t-1}^t \varphi(s)ds$ is sufficiently small.

3. Application to Chafee-Infante Equation

In this section we will apply our results to non-autonomous Chafee-Infante equation with multiplicative noise which has the form

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + ru(t,x) - b(t)u^3(t,x) + u(t,x)\dot{W}(t), \ t > 0, 0 < x < \pi \\ u(t,0) = u(t,\pi) = 0, t \in \mathbb{R}; u(0,x) = \phi(x) \ \ 0 < x < \pi. \end{cases} \tag{3.1}$$

We choose the Hilbert space $X = L_2[0,\pi]$, consider the linear operator $A : D(A) \to X$ defined by

$$D(A) = \left\{y \in X : y \text{ and } y' \text{ are absolutely continuous } y' \in X, y(0) = y(\pi) = 0\right\},$$

$$A(y) = -y'' - ry \ \ \forall y \in D(A).$$

Without loss off generality, we can assume $r < 1$ then A is a positive operator with discrete point spectrum being $1^2 - r, 2^2 - r, \cdots, n^2 - r, \cdots$.

Now, (3.1) can be expressed as the following abstract Cauchy problem.

$$\begin{cases} \dfrac{du(t,\cdot)}{dt} + Au(t,\cdot) & = f(t,u(t,\cdot)) + u(t,\cdot)\dot{W}(t) \quad t > 0 \\ \qquad\quad u(0,\cdot) & = \phi(\cdot) \in X \end{cases} \tag{3.2}$$

where $f : \mathbb{R} \times X \to X$ is defined by $f(t,\phi)(x) := -b(t)\phi^3(x)\ \forall x \in (0,\pi)$.

By "cut-off" technique we next modify the equation (3.2) into modified one in which we can apply our results.

Concretely, for any fixed $\rho > 0$ we denote by $B_\rho := \{v \in X : \|v\| \le 1\}$ the ball with radius ρ in X. One can see that $\|f(t,u) - f(t,v)\| \le 3\rho^2 b(t)\|u - v\|\quad \forall u, v \in B_\rho$.

Now, let $\chi(s)$ be an infinitely differentiable function on $[0,+\infty)$ such that

$\chi(s) = 1,\ \ 0 \le s \le 1;\ \ \chi(s) = 0,\ \ s \ge 2;\ 0 \le \chi(s) \le 1,\ \ |\chi'(s)| \le 2,\ \ s \in [0,+\infty)$

and consider a mapping $G : \mathbb{R} \times X \to X$ such that

$$G(t,u)(x) = \chi\left(\frac{2}{\rho}\|u\|\right) f(t,u)(x)\ \ \forall u \in D(A).$$

Then G is φ - Lipschitz with $\varphi(t) = 3\rho(5\rho + 4)b(t)$. Indeed, for any $u, v \in D(A)$

If $u, v \in B_\rho$, we have

$$\begin{aligned}\|G(t,u) - G(t,v)\| &= \left\|\chi\left(\frac{2}{\rho}\|u\|\right) f(t,u) - \chi\left(\frac{2}{\rho}\|v\|\right) f(t,v)\right\| \\ &\le \left\|\chi\left(\frac{2}{\rho}\|u\|\right) f(t,u) - \chi\left(\frac{2}{\rho}\|v\|\right) f(t,u)\right\| + \left\|\chi\left(\frac{2}{\rho}\|v\|\right) f(t,u) - \chi\left(\frac{2}{\rho}\|v\|\right) f(t,v)\right\| \\ &\le 2\frac{2}{\rho}\big|\|u\| - \|v\|\big|.\|f(t,u)\| + \|f(t,u) - f(t,v)\| \\ &\le 3\rho^2 b(t)\left(\frac{4}{\rho}(1+\rho) + 1\right)\|u - v\| \\ &\le \varphi(t)\|u - v\|;\end{aligned}$$

if $u \in B_\rho$ and $v \notin B_\rho$, we have

$$\begin{aligned}\|G(t,u) - G(t,v)\| &= \left\|\chi\left(\frac{2}{\rho}\|u\|\right) f(t,u) - \chi\left(\frac{2}{\rho}\|v\|\right) f(t,v)\right\| \\ &\le \left\|\chi\left(\frac{2}{\rho}\|u\|\right) f(t,u) - \chi\left(\frac{2}{\rho}\|v\|\right) f(t,u)\right\| \\ &\le 2\frac{2}{\rho}\big|\|u\| - \|v\|\big|.\|f(t,u)\| \\ &\le 3\rho^2 b(t)\frac{4}{\rho}(1+\rho)\|u - v\| \\ &\le \varphi(t)\|u - v\|;\end{aligned}$$

If $u, v \notin B_\rho$, we have $\|G(t,u) - G(t,v)\| = 0 \le \varphi(t)\|u - v\|$.

Thus, G is φ - Lipschitz with $\varphi(t) = 3\rho(5\rho + 4)b(t)$.

Now, we consider the following stochastic semilinear differential equation of abstract form

$$\begin{cases} \dfrac{du(t,\cdot)}{dt} + Au(t,\cdot) & = G(t,u(t,\cdot)) + u(t,\cdot)\dot{W}(t) \quad t > 0 \\ u(0,\cdot) & = \phi(\cdot) \in X. \end{cases} \tag{3.3}$$

The equation (3.3) is called the modified equation of (3.1) which defines the asymptotic behavior of original one.

Furthermore, $\gamma = \dfrac{\lambda_{N+1} - \lambda_N}{2} = \dfrac{2N+1}{2}$ is large enough when N is sufficiently large. Theorem 2.8 implies that if the norm $\|\Lambda_1 \varphi\|_\infty = \sup\limits_{t\in\mathbb{R}} \int\limits_{t-1}^{t} \varphi(s)ds = \sup\limits_{t\in\mathbb{R}} \int\limits_{t-1}^{t} 3\rho(5\rho+4)b(s)ds$ is sufficiently small, then there exists an inertial manifold for mild solutions to Eq. (3.3). This is an inertial manifold for mild solutions of Eq. (3.1) which are staying in B_ρ as $t \to -\infty$.

4. Conclusion

By Theorem 2.8, we proved the existence of an inertial manifolds for a class of stochastic differential equations which relate to non-uniformly Lipschitzian nonlinearity. Furthermore, in Section 3 we presented an example to illustrate our results.

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