EXPONENTIAL STABILITY OF 2D DISCRETE SYSTEMS WITH MIXED TIME-VARYING DELAYS

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Abstract: *This paper deals with the problem of exponential stability of two-dimensional (2D) discrete-time systems with mixed directional time-varying delays. By constructing an improved 2D Lyapunov-Krasovskii functional candidate some new delay-dependent condition for the exponential stability of the system are derived in terms of linear matrix inequalities (LMIs).*

Keywords: 2D systems, Exponential stability, Lyapunov-Krasovskii function, Linear matrix inequalities (LMIs).

1. Introduction

Two-dimensional systems have many applications in different arears as geographical data processing, electrical circuit networks, power systems, energy exchange processes, multibody mechanics, process control, aerospace engineering and physical processes [18, 4, 9, 14]. In recent years, 2-D switched systems have attracted the attention of various scientists who have made the significant contributions in stability theory. Most commonly utilized state-space models of 2D systems are the Roesser model, the Fornasini-Marchesini (FM) local model and the Attasi model [18, 17, 5, 4]. Time-delay phenomena are frequently in various practical systems. The existence of time delay may lead to instability or poor performance of the system, so it is of significance to study time-delay systems. The exponential stability for 2D state delay systems has been studied. There have been many previous results on stability for 2D discrete systems with time-varying delays[3, 13, 6, 7, 12, 19]. However, to the best of our knowledge, the problem of stability 2D systems with state delays, especially for 2D systems with mixed delays, has not been fully investigated to date.

In this paper, we study the problem of exponential stability of a class of 2D discrete-time systems described by the Roesser model with mixed time-varying delays. Delay-range-dependent exponential stability criteria of 2D systems discrete-time with mixed time-varying delays are established in terms of linear matrix inequalities .

Notations: Z denotes the set of integers, $Z[a,b] \triangleq \{a, a+1, \ldots, b\}$ for $a, b \in Z$, $a \le b$. $R^{n\times m}$ denotes the set of $n\times m$ real matrices and $\mathrm{diag}(A,B) \triangleq \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for two matrices A, B. $\mathrm{Sym}(A) \triangleq A + A^{\mathrm{T}}$ for $A \in R^{n\times n}$. A matrix $M \in R^{n\times n}$ is semi-positive definite, $M \ge 0$, if $x^{\mathrm{T}}Mx \ge 0$, $\forall x \in R^{n}$; M is positive definite, $M > 0$, if $x^{\mathrm{T}}Mx > 0$, $\forall x \in R^{n}$, $x \ne 0$.

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2. Preliminaries

Consider a class of 2-D discrete-time systems with mixed directional time-varying delays described by the following Roesser model (2-D DRM)

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + A_\tau \begin{bmatrix} x^h(i-\tau_h(i),j) \\ x^v(i,j-\tau_v(j)) \end{bmatrix} + A_d \begin{bmatrix} \sum_{l=1}^{d_h(i)} x^h(i-l,j) \\ \sum_{t=1}^{d_v(j)} x^v(i,j-t) \end{bmatrix}, \quad i,j \in Z^+, \tag{1}$$

where $x^h(i,j) \in R^{n_h}$ and $x^v(i,j) \in R^{n_v}$ are the horizontal state vector and the vertical state vector, respectively. A, A_τ and A_d are constant matrices with appropriate dimensions. $\tau_h(i)$, $d_h(i)$ and $\tau_v(j)$, $d_v(j)$ are respectively the directional time-varying delays along the horizontal and vertical directions satisfying

$$\tau_{hm} \le \tau_h(i) \le \tau_{hM}, \quad \tau_{vm} \le \tau_v(j) \le \tau_{vM}, \tag{2}$$

$$d_{hm} \le d_h(i) \le d_{hM}, \quad d_{vm} \le d_v(j) \le d_{vM}, \tag{3}$$

where τ_{hm}, τ_{hM}, τ_{vm}, τ_{vM}, d_{hm}, d_{hM}, d_{vm} and d_{vM} are known nonnegative integers involving the upper and the lower bounds of delays. Denote $\mu_h = \max(\tau_{hM}, d_{hM})$ and $\mu_v = \max(\tau_{vM}, d_{vM})$. Initial condition of (1) is defined by

$$\begin{aligned} x^h(i,j) &= \phi(i,j), i \in Z[-\mu_h, 0], 0 \le j \le z_1, \\ x^h(i,j) &= 0, j > z_1 \\ x^v(i,j) &= \psi(i,j), j \in Z[-\mu_v, 0], 0 \le i \le z_2, \\ x^v(i,j) &= 0, i > z_2, \end{aligned} \tag{4}$$

where $\phi(k,.) \in l_2(Z^+), \forall k \in Z[-\mu_h, 0]$ and $\psi(.,l) \in l_2(Z^+), \forall l \in Z[-\mu_v, 0]$, $z_1 < \infty$ and $z_2 < \infty$.

Definition 1. *System (1) is said to be exponentially stable if there exist scalars* $N > 0$ *and* $0 < \beta < 10$ *such that any solution* $x(i,j)$ *of (1) satisfies*

$$\sum_{i+j=\Gamma} \| x(i,j) \|^2 \le N\beta^{(\Gamma-\kappa_0)} \sum_{i+j=\kappa_0} \| x(i,j) \|_C^2 \tag{5}$$

holds for all $i+j=\Gamma \ge \kappa_0 = i_\kappa + j_\kappa$, where

$$\sum_{i+j=\Gamma} \| x(i,j) \|_C^2 \triangleq \sup_{\substack{-\mu_v \le t \le 0 \\ -\mu_h \le s \le 0}} \sum_{i+j=\kappa_0} \{ \| x^h(i+s,j) \|^2 + \| x^v(i,j+t) \|^2, \| z^h(i+s,j) \|^2 + \| z^v(i,j+t) \|^2 \},$$

$$z^h(i+s,j) = x^h(i+s+1,j) - x^h(i+s,j), \; z^v(i,j+t) = x^v(i,j+t+1) - x^v(i,j+t).$$

Lemma 1. [3] *For any vector* $\omega(t) \in R^n$, *two positive integers* ℓ_1 *and* ℓ_2, *and a symmetric positive matrix* $H \in R^{n \times n}$, *the following inequality holds,*

$$-(\ell_2-\ell_1-1)\sum_{t=\ell_2}^{\ell_1}\omega^{\mathrm{T}}(t)H\omega(t)\leq\left[\sum_{t=\ell_2}^{\ell_1}\omega^{\mathrm{T}}(t)\right]H\left[\sum_{t=\ell_2}^{\ell_1}\omega(t)\right] \tag{6}$$

Lemma 2. [3] *For a symmetric positive definite matrix* $R\in R^{n\times n}$, *positive integers* h,v *and a function* $x:Z[i-h,i]\times[j-v,j]\to R^{n}$, $i,j\in Z^{+}$, *the following inequalities hold*

$$\sum_{l=i-h}^{i-1}\delta_1^{\mathrm{T}}(l,j)R\delta_1(l,j)\geq\frac{1}{h}\big[x(i,j)-x(i-h,j)\big]^{\mathrm{T}}R\big[x(i,j)-x(i-h,j)\big], \tag{7}$$

$$\sum_{s=j-v}^{j-1}\delta_2^{\mathrm{T}}(i,s)R\delta_2(i,s)\geq\frac{1}{v}\big[x(i,j)-x(i,j-v)\big]^{\mathrm{T}}R\big[x(i,j)-x(i,j-v)\big], \tag{8}$$

where $\delta_1(l,j)=x(l+1,j)-x(l,j)$ and $\delta_2(i,s)=x(i,s+1)-x(i,s)$.

3. Main results

We are now in a position to derive LMI-based conditions ensuring that system (1) exponential stable. For the brevity, in the following we denote the block matrix $I(\alpha,\beta)=diag(\alpha I_{n_h},\beta I_{n_v})$ for any scalars α,β.

Theorem 1. *For given nonnegative integers* τ_{hm},τ_{hM}, τ_{vm},τ_{vM}, d_{hm},d_{hM}, d_{vm} *and* d_{vM}, *if there exist symmetric positive definite matrices* $P=diag(P_h,P_v)$, $Q=diag(Q_h,Q_v)$, $R=diag(R_h,R_v)$, $X=diag(X_h,X_v)$, $Y=diag(Y_h,Y_v)$, $S=diag(S_h,S_v)$, $Z=diag(Z_h,Z_v)$ *and* $0<\beta<1$ *such that the following LMI holds*

$$\Phi=\begin{bmatrix}\Psi & A^{\mathrm{T}}P & D^{\mathrm{T}}\Upsilon\\ & -P & 0\\ & * & -\Pi\end{bmatrix}<0 \tag{9}$$

where $\Upsilon=\begin{bmatrix}X & Y & S\end{bmatrix}$, $\Pi=diag(X,Y,S)$, and

$$\Psi=\begin{bmatrix}\Psi_{11} & 0 & X & \overline{Y} & 0\\ & -(\overline{R}+2\overline{S}) & \overline{S} & \overline{S} & 0\\ & * & -(Q+X+\overline{S}) & 0 & 0\\ & * & * & -(\overline{X}+\overline{S}) & 0\\ & * & * & * & -Z\end{bmatrix},$$

$$\Psi_{11}=\overline{Q}+R+Z-P-X-\overline{Y}$$

$$X=I(\beta\tau_{hm}^2,\beta\tau_{vm}^2)X,\quad Y=I(\beta\tau_{hM}^2,\beta\tau_{vM}^2)Y,\quad S=I(\beta r_{\tau h}^2,\beta r_{\tau v}^2)S,$$

$$R=I(\beta(1+r_{\tau h}),\beta(1+r_{\tau v}))R,\quad Z=I(\beta r_{dh},\beta r_{dv})Z,\quad P=I(\beta,\beta)P,\overline{Q}=I(\beta,\beta)Q,$$

$$X = I(\beta^{1+\tau_{hm}}, \beta^{1+\tau_{vm}})X, \quad \overline{Y} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})Y, \quad \overline{R} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})R,$$
$$\overline{S} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})S, \quad Q = I(\beta^{1+\tau_{hm}}, \beta^{1+\tau_{vm}})Q, \quad \overline{X} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})X,$$
$$r_{\tau h} = \tau_{hM} - \tau_{hm}, \quad r_{dh} = \frac{d_{hM}(d_{hM} + d_{hm})(d_{hM} - d_{hm} + 1)}{2},$$
$$r_{\tau v} = \tau_{vM} - \tau_{vm}, \quad r_{dv} = \frac{d_{vM}(d_{vM} + d_{vm})(d_{vM} - d_{vm} + 1)}{2},$$
$$A = \begin{bmatrix} A & A_\tau & 0 & 0 & A_d \end{bmatrix}, \quad D = \begin{bmatrix} A - I & A_\tau & 0 & 0 & A_d \end{bmatrix},$$

then system (1) is exponentially stable.

Proof. For the brevity, in the following, we denote

$$x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \quad x(i+1, j+1) = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix},$$
$$x_\tau(i,j) = \begin{bmatrix} x^h(i - \tau_h(i), j) \\ x^v(i, j - \tau_v(j)) \end{bmatrix}, \quad x_{\tau M}(i,j) = \begin{bmatrix} x^h(i - \tau_{hM}), j) \\ x^v(i, j - \tau_{vM}) \end{bmatrix},$$
$$x_{\tau m}(i,j) = \begin{bmatrix} x^h(i - \tau_{hm}), j) \\ x^v(i, j - \tau_{vm}) \end{bmatrix}, \quad x_d(i,j) = \begin{bmatrix} \sum_{l=1}^{d_h(i)} x^h(i-l, j) \\ \sum_{t=1}^{d_v(j)} x^v(i, j-t) \end{bmatrix},$$
$$\delta^h(i,j) = x^h(i+1,j) - x^h(i,j), \quad \delta^v(i,j) = x^v(i,j+1) - x^v(i,j),$$
$$\eta(i,j) = \begin{bmatrix} x^{\mathrm{T}}(i,j) & x_\tau^{\mathrm{T}}(i,j) & x_{\tau m}^{\mathrm{T}}(i,j) & x_{\tau M}^{\mathrm{T}}(i,j) & x_d^{\mathrm{T}}(i,j) \end{bmatrix}^{\mathrm{T}}.$$

We consider the following Lyapunov-Krasovskii functional

$$V(i,j) = \underbrace{\sum_{q=1}^{8} V_q^h(x^h(i,j))}_{V^h(i,j)} + \underbrace{\sum_{q=1}^{8} V_q^v(x^v(i,j))}_{V^v(i,j)} \tag{10}$$

where
$$V_1^h(x^h(i,j)) = x^{h\mathrm{T}}(i,j) P_h x^h(i,j),$$
$$V_2^h(x^h(i,j)) = \sum_{l=i-\tau_{hm}}^{i-1} x^{h\mathrm{T}}(l,j) Q_h x^h(l,j) \beta^{i-l},$$
$$V_3^h(x^h(i,j)) = \sum_{l=i-\tau_h(i)}^{i-1} x^{h\mathrm{T}}(l,j) R_h x^h(l,j) \beta^{i-l},$$
$$V_4^h(x^h(i,j)) = \sum_{s=-\tau_{hM}+1}^{-\tau_{hm}} \sum_{l=i+s}^{i-1} x^{h\mathrm{T}}(l,j) R_h x^h(l,j) \beta^{i-l},$$

$$V_5^h(x^h(i,j)) = \tau_{hm} \sum_{s=-\tau_{hm}}^{-1} \sum_{l=i+s}^{i-1} \delta^{h\mathrm{T}}(l,j) X_h \delta^h(l,j) \beta^{i-l},$$

$$V_6^h(x^h(i,j)) = \tau_{hM} \sum_{s=-\tau_{hM}}^{-1} \sum_{l=i+s}^{i-1} \delta^h(l,j)^T Y_h \delta^h(l,j) \beta^{i-l},$$

$$V_7^h(x^h(i,j)) = r_{\tau h} \sum_{s=-\tau_{hM}}^{-\tau_{hm}-1} \sum_{l=i+s}^{i-1} \delta^{h\mathrm{T}}(l,j) S_h \delta^h(l,j) \beta^{i-l},$$

$$V_8^h(x^h(i,j)) = d_{hM} \sum_{s=d_{hm}}^{d_{hM}} \sum_{l=1}^{s} \sum_{p=i-l}^{i-1} x^{h\mathrm{T}}(p,j) Z_h x^h(p,j) \beta^{i-p},$$

and

$$V_1^v(x^v(i,j)) = x^{v\mathrm{T}}(i,j) P_v x^v(i,j),$$

$$V_2^v(x^v(i,j)) = \sum_{t=j-\tau_{vm}}^{j-1} x^{v\mathrm{T}}(i,t) Q_v x^v(i,t) \beta^{j-t},$$

$$V_3^v(x^v(i,j)) = \sum_{t=j-\tau_v(j)}^{j-1} x^{v\mathrm{T}}(i,t) R_v x^v(i,t) \beta^{j-t},$$

$$V_4^v(x^v(i,j)) = \sum_{k=-\tau_{vM}+1}^{-\tau_{vm}} \sum_{t=j+k}^{j-1} x^{v\mathrm{T}}(i,t) R_v x^v(i,t) \beta^{j-t},$$

$$V_5^v(x^v(i,j)) = \tau_{vm} \sum_{k=-\tau_{vm}}^{-1} \sum_{t=j+k}^{j-1} \delta^{v\mathrm{T}}(i,t) X_v \delta^v(i,t) \beta^{j-t},$$

$$V_6^v(x^v(i,j)) = \tau_{vM} \sum_{k=-\tau_{vM}}^{-1} \sum_{t=j+k}^{j-1} \delta^{v\mathrm{T}}(i,t) Y_v \delta^v(i,t) \beta^{j-t},$$

$$V_7^v(x^h(i,j)) = r_{\tau v} \sum_{k=-\tau_{vM}}^{-\tau_{vm}-1} \sum_{t=j+k}^{j-1} \delta^{v\mathrm{T}}(i,t) S_v \delta^v(i,t) \beta^{j-t},$$

$$V_8^v(x^v(i,j)) = d_{vM} \sum_{k=d_{vm}}^{d_{vM}} \sum_{t=1}^{k} \sum_{p=j-t}^{j-1} x^{v\mathrm{T}}(i,p) Z_v x^v(i,p) \beta^{j-p}.$$

Clearly, $V(i,j) \geq 0, \forall i,j \in Z^+$. With respect to 2-D DRM (1), the $\Delta V_\beta(i,j)$ is defined directionally as follows

$$\Delta V_\beta(i,j) \triangleq V^h(i+1,j) - \beta V^h(i,j) + V^v(i,j+1) - \beta V^v(i,j) \triangleq \Delta V_\beta^h(i,j) + \Delta V_\beta^v(i,j) \quad (11)$$

First, we have

$$\Delta V_{1\beta}^h(x^h(i,j)) = x^{h\mathrm{T}}(i+1,j) P_h x(i+1,j) - \beta x^{h\mathrm{T}}(i,j) P_h x^h(i,j)$$

$$\Delta V_{2\beta}^h((x^h(i,j)) = \beta x^{h\mathrm{T}}(i,j) Q_h x(i,j) - \beta^{1+\tau_{hm}} x^{h\mathrm{T}}(i-\tau_{hm},j) Q_h x^h(i-\tau_{hm},j)$$

$$\Delta V_{3\beta}^h((x^h(i,j)) = \sum_{l=i+1-\tau_h(i+1)}^{i} x^{h\mathrm{T}}(l,j) R_h x^h(l,j) - \sum_{l=i-\tau_h(i)}^{i-1} x^{h\mathrm{T}}(l,j) R_h x^h(l,j)$$

$$\leq \beta x^{h\mathrm{T}}(i,j)R_h x^h(i,j) - \beta^{1+\tau_{hM}} x^{h\mathrm{T}}(i-\tau_h(i),j)R_h x^h(i-\tau_h(i),j) + \sum_{l=i+1-\tau_{hM}}^{i-\tau_{hm}} x^{h\mathrm{T}}(l,j)R_h x^h(l,j)\beta^{i+1-l}$$

$$\Delta V_{4\beta}^h((x^h(i,j)) = \sum_{s=-\tau_{hM}}^{-\tau_{hm}} \left[\sum_{l=i+1+s}^{i} x^{h\mathrm{T}}(l,j)R_h x^h(l,j)\beta^{i+1-l} - \sum_{l=i+s}^{i-1} x^{h\mathrm{T}}(l,j)R_h x^h(l,j)\beta^{i+1-l}\right]$$

$$\leq \sum_{s=-\tau_{hM}}^{-\tau_{hm}} \left[\beta x^{h\mathrm{T}}(i,j)R_h x(i,j) - x^{h\mathrm{T}}(i+s,j)R_h x(i+s,j)\beta^{1-s}\right]$$

$$\leq \left[r_{\tau h}\beta x^{h\mathrm{T}}(i,j)R_h x(i,j) - \sum_{l=i+1-\tau_{hM}}^{i-\tau_{hm}} x^{h\mathrm{T}}(l,j)R_h x^h(l,j)\beta^{i+1-l}\right] \tag{12}$$

and $\Delta V_{n\beta}^h((x^h(i,j))$ ($n = 5,6,7$) are given as

$$\begin{aligned}
\Delta V_{5\beta}^h(x^h(i,j)) &= \tau_{hm} \sum_{s=-\tau_{hm}}^{-1} \left[\sum_{l=i+1+s}^{i} \delta^{h\mathrm{T}}(l,j)X_h\delta^h(l,j)\beta^{i+1-l} - \sum_{l=i+s}^{i-1} \delta^{h\mathrm{T}}(l,j)X_h\delta^h(l,j)\beta^{i+1-l}\right] \\
&\leq \tau_{hm}^2\beta\delta^{h\mathrm{T}}(i,j)X_h\delta^h(i,j) - \tau_{hm}\beta^{1+\tau_{hm}} \sum_{l=i-\tau_{hm}}^{i-1} \delta^{h\mathrm{T}}(l,j)X_h\delta^h(l,j), \\
\Delta V_{6\beta}^h(x^h(i,j)) &= \tau_{hM} \sum_{s=-\tau_{hM}}^{-1} \left[\sum_{l=i+1+s}^{i} \delta^{h\mathrm{T}}(l,j)Y_h\delta^h(l,j)\beta^{i+1-l} - \sum_{l=i+s}^{i-1} \delta^{h\mathrm{T}}(l,j)Y_h\delta^h(l,j)\beta^{i+1-l}\right] \\
&\leq \tau_{hM}^2\beta\delta^{h\mathrm{T}}(i,j)Y_h\delta^h(i,j) - \tau_{hM}\beta^{1+\tau_{hM}} \sum_{l=i-\tau_{hM}}^{i-1} \delta^{h\mathrm{T}}(l,j)Y_h\delta^h(l,j), \\
\Delta V_{7\beta}^h(x^h(i,j)) &= r_{\tau h} \sum_{s=-\tau_{hM}}^{\tau_{hm}-1} \left[\sum_{l=i+1+s}^{i} \delta^{h\mathrm{T}}(l,j)S_h\delta^h(l,j)\beta^{i+1-l} - \sum_{l=i+s}^{i-1} \delta^{h\mathrm{T}}(l,j)S_h\delta^h(l,j)\beta^{i+1-l}\right] \\
&\leq r_{\tau h}^2\beta\delta^{h\mathrm{T}}(i,j)S_h\delta^h(i,j) - r_{\tau h}\beta^{1+\tau_{hM}} \sum_{l=i-\tau_{hM}}^{i-\tau_{hm}-1} \delta^{h\mathrm{T}}(l,j)S_h\delta^h(l,j)
\end{aligned} \tag{13}$$

By Lemma 2, we have

$$\begin{aligned}
&-\tau_{hm}\beta^{1+\tau_{hm}} \sum_{l=i-\tau_{hm}}^{i-1} \delta^{h\mathrm{T}}(l,j)X_h\delta^h(l,j) \\
&\leq -[x^h(i,j) - x^h(i-\tau_{hm},j)]^{\mathrm{T}}\left(\beta^{1+\tau_{hm}}X_h\right)[x^h(i,j) - x^h(i-\tau_{hm},j)] \\
&= \begin{bmatrix} x^h(i,j) \\ x^h(i-\tau_{hm},j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\beta^{1+\tau_{hm}}X_h & \beta^{1+\tau_{hm}}X_h \\ & -\beta^{1+\tau_{hm}}X_h \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^h(i-\tau_{hm},j) \end{bmatrix}
\end{aligned} \tag{14}$$

and

$$-\tau_{hM}\beta^{1+\tau_{hM}}\sum_{l=i-\tau_{hM}}^{i-1}\delta^{h\mathrm{T}}(l,j)Y_h\delta^h(l,j)$$

$$\leq -[x^h(i,j)-x^h(i-\tau_{hM},j)]^{\mathrm{T}}\left(\beta^{1+\tau_{hM}}Y_h\right)[x^h(i,j)-x^h(i-\tau_{hM},j)]$$

$$=\begin{bmatrix} x^h(i,j) \\ x^h(i-\tau_{hM},j)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -\beta^{1+\tau_{hM}}Y_h & \beta^{1+\tau_{hM}}Y_h \\ & -\beta^{1+\tau_{hM}}Y_h\end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^h(i-\tau_{hM},j)\end{bmatrix} \quad (15)$$

$$-r_{\tau h}\beta^{1+\tau_{hM}}\sum_{l=i-\tau_{hM}}^{i-\tau_{hm}-1}\delta^{h\mathrm{T}}(l,j)S_h\delta^h(l,j)$$

$$\leq -(\tau_h(i)-\tau_{hm})\sum_{l=i-\tau_h(i)}^{i-\tau_{hm}-1}\delta^{h\mathrm{T}}(l,j)\beta^{1+\tau_{hM}}S_h\delta(l,j)-(\tau_{hM}-\tau_h(i))\sum_{l=i-\tau_{hM}}^{i-\tau_h(i)-1}\delta^{h\mathrm{T}}(l,j)\beta^{1+\tau_{hM}}S_h\delta(l,j)$$

$$\leq -[\sum_{l=i-\tau_h(i)}^{i-\tau_{hm}-1}\delta^{h\mathrm{T}}(l,j)]\beta^{1+\tau_{hM}}S_h[\sum_{l=i-\tau_h(i)}^{i-\tau_{hm}-1}\delta^h(l,j)]-[\sum_{l=i-\tau_{hM}}^{i-\tau_h(i)-1}z^{h\mathrm{T}}(l,j)]\beta^{1+\tau_{hM}}S_h[\sum_{l=i-\tau_{hM}}^{i-\tau_h(i)-1}\delta^h(l,j)]$$

$$\leq\begin{bmatrix} x^h(i-\tau_h(i),j) \\ x^h(i-\tau_{hm},j) \\ x^h(i-\tau_{hM},j)\end{bmatrix}^{\mathrm{T}} C_o\otimes\left(\beta^{1+\tau_{hM}}S_h\right)\begin{bmatrix} x^h(i-\tau_h(i),j) \\ x^h(i-\tau_{hm},j) \\ x^h(i-\tau_{hM},j)\end{bmatrix} \quad (16)$$

where $C_o=\begin{bmatrix} -2 & 1 & 1 \\ & -1 & 0 \\ & * & -1\end{bmatrix}$ and the symbol $\otimes$ denotes the Kronecker product of two matrices.

By Lemma 1 again, $\Delta V_{8\beta}^h((x^h(i,j))$ is given as

$$\Delta V_{8\beta}^h(x^h(i,j)) = d_{hM}\sum_{s=d_{hm}}^{d_{hM}}\sum_{l=1}^{s}(\sum_{p=i+1-l}^{i}x^{h\mathrm{T}}(p,j)Z_hx^h(p,j)\beta^{i+1-p}-\sum_{p=i-l}^{i-1}x^{h\mathrm{T}}(p,j)Z_hx^h(p,j)\beta^{i+1-p})$$

$$= r_{dh}\beta x^{h\mathrm{T}}(i,j)Z_hx^h(i,j)-d_{hM}\sum_{s=d_{hm}}^{d_{hM}}\sum_{l=1}^{s}x^{h\mathrm{T}}(i-l,j)Z_hx^h(i-l,j)\beta^{1-l}$$

$$\leq r_{dh}\beta x^{h\mathrm{T}}(i,j)Z_hx^h(i,j)-d_{hM}\sum_{l=1}^{d_h(i)}x^{h\mathrm{T}}(i-l,j)Z_hx^h(i-l,j)$$

$$\leq r_{dh}\beta x^{h\mathrm{T}}(i,j)Z_hx^h(i,j)-d_h(i)\sum_{l=1}^{d_h(i)}x^{h\mathrm{T}}(i-l,j)Z_hx^h(i-l,j)$$

$$\leq r_{dh}\beta x^{h\mathrm{T}}(i,j)Z_hx^h(i,j)-(\sum_{l=1}^{d_h(i)}x^h(i-l,j))^{\mathrm{T}}Z_h(\sum_{l=1}^{d_h(i)}x^h(i-l,j)) \quad (17)$$

From (12) to (17) we have

$$\begin{aligned}
\sum_{n=1}^{8}\Delta V_{n\beta}^{h}(x^{h}(i,j)) \le\; & x^{hT}(i+1,j)P_h x(i+1,j)+x^{hT}(i,j)(\beta Q_h+\beta(1+r_{\tau h})R_h+\beta r_{dh}Z_h-\beta P_h)x^h(i,j)\\
& -\beta^{1+\tau_{hm}}x^{hT}(i-\tau_{hm},j)Q_h x^h(i-\tau_{hm},j)-\beta^{1+\tau_{hM}}x^{hT}(i-\tau_h(i),j)R_h x^h(i-\tau_h(i),j)\\
& +\delta^{hT}(i,j)(\tau_{hm}^2\beta X_h+\tau_{hM}^2\beta Y_h+r_{\tau h}^2\beta S_h)\delta^h(i,j)\\
& +\begin{bmatrix} x^h(i,j)\\ x^h(i-\tau_{hm},j)\end{bmatrix}^{T}\begin{bmatrix}-\beta^{1+\tau_{hm}}X_h & \beta^{1+\tau_{hm}}X_h\\ & -\beta^{1+\tau_{hm}}X_h\end{bmatrix}\begin{bmatrix} x^h(i,j)\\ x^h(i-\tau_{hm},j)\end{bmatrix}\\
& +\begin{bmatrix} x^h(i,j)\\ x^h(i-\tau_{hM},j)\end{bmatrix}^{T}\begin{bmatrix}-\beta^{1+\tau_{hM}}Y_h & \beta^{1+\tau_{hM}}Y_h\\ & -\beta^{1+\tau_{hM}}Y_h\end{bmatrix}\begin{bmatrix} x^h(i,j)\\ x^h(i-\tau_{hM},j)\end{bmatrix}\\
& +\begin{bmatrix} x^h(i-\tau_h(i),j)\\ x^h(i-\tau_{hm},j)\\ x^h(i-\tau_{hM},j)\end{bmatrix}^{T} C_o\otimes\left(\beta^{1+\tau_{hM}}S_h\right)\begin{bmatrix} x^h(i-\tau_h(i),j)\\ x^h(i-\tau_{hm},j)\\ x^h(i-\tau_{hM},j)\end{bmatrix}\\
& -\Big(\sum_{l=1}^{d_h(i)}x^h(i-l,j)\Big)^{T}Z_h\Big(\sum_{l=1}^{d_h(i)}x^h(i-l,j)\Big)\\
\le\; & x^{hT}(i+1,j)P_h x(i+1,j)+\delta^{hT}(i,j)(\widehat{Xh+Yh+Sh})\delta^h(i,j)+\eta^{hT}(i,j)\Psi_h\eta^h(i,j) \quad (18)
\end{aligned}$$

where $Xh=\tau_{hm}^2\beta X_h$, $Yh=\tau_{hM}^2\beta Y_h$, $Sh=r_{\tau h}^2\beta S_h$ and

$$\eta^h(i,j)=\begin{bmatrix} x^{hT}(i,j) & x^{hT}(i-\tau_h(i),j) & x^{hT}(i-\tau_{hm},j) & x^{hT}(i-\tau_{hM},j) & \sum_{l=1}^{d_h(i)}x^{hT}(i-l,j)\end{bmatrix}^{T},$$

$$\Psi_h=\begin{bmatrix}\Psi_{h11} & 0 & \beta^{1+\tau_{hm}}X_h & \beta^{1+\tau_{hM}}Y_h & 0\\ & -\beta^{1+\tau_{hM}}R_h-2\beta^{1+\tau_{hM}}S_h & \beta^{1+\tau_{hM}}S_h & \beta^{1+\tau_{hM}}S_h & 0\\ & * & -\beta^{1+\tau_{hm}}(Q_h+X_h)-\beta^{1+\tau_{hM}}S_h & 0 & 0\\ & * & * & -\beta^{1+\tau_{hM}}(X_h+S_h) & 0\\ & * & * & * & -Z_h\end{bmatrix},$$

$$\Psi_{h11}=\beta Q_h+\beta(1+r_{\tau h})R_h+\beta r_{dh}Z_h-\beta P_h-\beta^{1+\tau_{hm}}X_h-\beta^{1+\tau_{hM}}Y_h.$$

Similarly, we have

$$\Delta V_{1\beta}^{v}(x^v(i,j))=x^{vT}(i,j+1)P_v x(i,j+1)-\beta x^{vT}(i,j)P_v x^v(i,j)$$

$$\Delta V_{2\beta}^{v}((x^v(i,j))=\beta x^{vT}(i,j)Q_v x(i,j)-\beta^{1+\tau_{vm}}x^{vT}(i,j-\tau_{vm})Q_v x^v(i,j-\tau_{vm})$$

$$\Delta V_{3\beta}^{v}((x^{v}(i,j)) \le \beta x^{v\mathrm{T}}(i,j)R_{v}x^{v}(i,j) - \beta^{1+\tau_{vM}} x^{v\mathrm{T}}(i,j-\tau_{v}(j))R_{v}x^{v}(i,j-\tau_{v}(j)) + \sum_{t=j+1-\tau_{vM}}^{j-\tau_{vm}} x^{v\mathrm{T}}(i,t)R_{v}x^{v}(i,t)\beta^{j+1-t}$$

$$\Delta V_{4\beta}^{v}((x^{v}(i,j)) \le \left[\beta(r_{\tau v})x^{v\mathrm{T}}(i,j)R_{v}x(i,j) - \sum_{t=j+1-\tau_{vM}}^{j-\tau_{vm}} x^{v\mathrm{T}}(i,t)R_{v}x^{v}(i,t)\beta^{j+1-t}\right]$$

$$\Delta V_{5\beta}^{v}(x^{v}(i,j)) \le \tau_{vm}^{2}\beta\delta^{v\mathrm{T}}(i,j)X_{v}\delta^{v}(i,j) - \begin{bmatrix} x^{v}(i,j) \\ x^{v}(i,j-\tau_{vm}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\beta^{1+\tau_{vm}}X_{v} & \beta^{1+\tau_{vm}}X_{v} \\ & -\beta^{1+\tau_{vm}}X_{v} \end{bmatrix} \begin{bmatrix} x^{v}(i,j) \\ x^{v}(i,j-\tau_{vm}) \end{bmatrix}$$

$$\Delta V_{6\beta}^{v}(x^{v}(i,j)) \le \tau_{vm}^{2}\beta\delta^{v\mathrm{T}}(i,j)X_{v}\delta^{v}(i,j) - \begin{bmatrix} x^{v}(i,j) \\ x^{v}(i,j-\tau_{vm}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\beta^{1+\tau_{vm}}X_{v} & \beta^{1+\tau_{vm}}X_{v} \\ & -\beta^{1+\tau_{vm}}X_{v} \end{bmatrix} \begin{bmatrix} x^{v}(i,j) \\ x^{v}(i,j-\tau_{vm}) \end{bmatrix}$$

$$\Delta V_{7\beta}^{v}(x^{v}(i,j)) \le r_{\tau v}^{2}\beta\delta^{v\mathrm{T}}(i,j)S_{v}\delta^{v}(i,j) - \begin{bmatrix} x^{v}(i,j-\tau_{v}(j)) \\ x^{v}(i,j-\tau_{hv}) \\ x^{v}(i,j-\tau_{vM}) \end{bmatrix}^{\mathrm{T}} C_{o} \otimes \left(\beta^{1+\tau_{vM}} S_{v}\right) \begin{bmatrix} x^{v}(i,j-\tau_{v}(j)) \\ x^{v}(i,j-\tau_{vm}) \\ x^{v}(i,j-\tau_{vM}) \end{bmatrix}$$

$$\Delta V_{8\beta}^{v}(x^{v}(i,j)) \le r_{dv}\beta x^{v\mathrm{T}}(i,j)Z_{v}x^{v}(i,j) - \Big(\sum_{t=1}^{d_{v}(j)} x^{v}(i,j-t)\Big)^{\mathrm{T}} Z_{v}\Big(\sum_{t=1}^{d_{v}(j)} x^{v}(i,j-t)\Big) \tag{19}$$

Therefore

$$\sum_{n=1}^{8}\Delta V_{n\beta}^{v}(x^{v}(i,j)) \le x^{v\mathrm{T}}(i,j+1)P_{v}x(i,j+1) + \delta^{v\mathrm{T}}(i,j)(X_{v}+Y_{v}+S_{v})\delta^{v}(i,j) + \eta^{v\mathrm{T}}(i,j)\Psi_{v}\eta^{v}(i,j) \tag{20}$$

where $X_{v} = \tau_{vm}^{2}\beta X_{v}$, $Y_{v} = \tau_{vM}^{2}\beta Y_{v}$, $S_{v} = r_{\tau v}^{2}\beta S_{v}$ and

$$\eta^{v}(i,j) = \begin{bmatrix} x^{v\mathrm{T}}(i,j) & x^{v\mathrm{T}}(i,j-\tau_{v}(j)) & x^{v\mathrm{T}}(i,j-\tau_{vm}) & x^{v\mathrm{T}}(i,j-\tau_{vM}) & \sum_{t=1}^{d_{v}(j)} x^{v\mathrm{T}}(i,j-t) \end{bmatrix}^{\mathrm{T}},$$

$$\Psi_{v} = \begin{bmatrix} \Psi_{v11} & 0 & \beta^{1+\tau_{vm}}X_{v} & \beta^{1+\tau_{vM}}Y_{v} & 0 \\ & -\beta^{1+\tau_{vM}}R_{v} - 2\beta^{1+\tau_{vM}}S_{v} & \beta^{1+\tau_{vM}}S_{v} & \beta^{1+\tau_{vM}}S_{v} & 0 \\ & * & -\beta^{1+\tau_{vm}}(Q_{v}+X_{v}) - \beta^{1+\tau_{vM}}S_{v} & 0 & 0 \\ & * & * & -\beta^{1+\tau_{vM}}(X_{v}+S_{v}) & 0 \\ & * & * & * & -Z_{v} \end{bmatrix},$$

$$\Psi_{v11} = \beta Q_{v} + \beta(1+r_{\tau v})R_{v} + \beta r_{dv}Z_{v} - \beta P_{v} - \beta^{1+\tau_{vm}}X_{v} - \beta^{1+\tau_{vM}}Y_{v}.$$

From (18) and (20) we finally obtain

$$\Delta V_{\beta}(x(i,j)) = \sum_{n=1}^{8}(\Delta V_{n\beta}^{h}(x^{h}(i,j)) + \Delta V_{n\beta}^{v}(x^{v}(i,j))) \le \eta^{\mathrm{T}}(i,j)\left[\Psi + A^{\mathrm{T}}PA + D^{\mathrm{T}}\left(X+Y+S\right)D\right]\eta(i,j) \tag{21}$$

By Schur complement lemma $\Psi + A^{\mathrm{T}}PA + D^{\mathrm{T}}\left(X+Y+S\right)D < 0$

if and only if $$\Phi = \begin{bmatrix} \Psi & A^{\mathrm{T}}P & D^{\mathrm{T}}\Upsilon \\ & -P & 0 \\ & * & -\Pi \end{bmatrix} < 0.$$

Hence, if (9) holds then, by (21), we obtain

$$V_h(i+1,j)+V_v(i,j+1) \le \beta(V_h(i,j)+V_v(i,j)) \quad i,j \in Z^{+} \tag{22}$$

For any integer $\Gamma > \kappa_0 = \max(z_1, z_2)$, one has that $V_h(0,\Gamma+1) = V_v(\Gamma+1,0) = 0$. Then summing up both sides of (22) from $\Gamma+1$ to 0 with respect to j and 0 to $\Gamma+1$ with respect to i, one gets

$$\begin{aligned}
&\sum_{i+j=\Gamma+1} V(i,j) = \sum_{i+j=\Gamma+1} \left(V_h(i,j)+V_v(i,j)\right) \\
&= V_h(0,\Gamma+1)+V_v(0,\Gamma+1)+V_h(1,\Gamma)+V_v(1,\Gamma)+\cdots+V_h(\Gamma+1,0)+V_v(\Gamma+1,0) \\
&= \{V_h(1,\Gamma)+V_v(0,\Gamma+1)\}+\{V_h(2,\Gamma-1)+V_v(1,\Gamma)\}+\cdots+\{V_h(\Gamma+1,0)+V_v(\Gamma,1)\} \\
&\le \beta\{V_h(0,\Gamma)+V_v(0,\Gamma)\}+\beta\{V_h(1,\Gamma-1)+V_v(1,\Gamma-1)\}+\cdots+\beta\{V_h(\Gamma,0)+V_v(\Gamma,0)\} \\
&\le \beta \sum_{i+j=\Gamma} V(i,j).
\end{aligned} \tag{23}$$

From (23), using the above relationship iteratively, it follows that

$$\sum_{i+j=\Gamma} V(i,j) \le \beta \sum_{i+j=\Gamma-1} V(i,j) \le \beta^2 \sum_{i+j=\Gamma-2} V(i,j) \le \cdots \le \beta^{\Gamma-\kappa_0} \sum_{i+j=\kappa_0} V(i,j). \tag{24}$$

On the other hand, from (10), we can find two positive scalars ξ_1 and ξ_2, such that:

$$\xi_1 \| x(i,j) \|^2 \le V(i,j) \le \xi_2 \| x(i,j) \|_C^2 \tag{25}$$

Where $\xi_1 = \min\{\lambda_{\min}(P_h)+\lambda_{\min}(P_v)\}$

$$\begin{aligned}
\xi_2 = \max\{&\lambda_{\max}(P_h)+\lambda_{\max}(P_v)+\tau_{hM}\lambda_{\max}(R_h)+\tau_{vM}\lambda_{\max}(R_v)+\tau_{hm}\lambda_{\max}(Q_h)+\tau_{vm}\lambda_{\max}(Q_v) \\
&+r_{dh}\lambda_{\max}(R_h)+r_{dv}\lambda_{\max}(R_v)+\tau_{hm}^2\lambda_{\max}(X_h)+\tau_{vm}^2\lambda_{\max}(X_v)+\tau_{hM}^2\lambda_{\max}(Y_h)+\tau_{vM}^2\lambda_{\max}(Y_v) \\
&+r_{\tau h}^2\lambda_{\max}(S_h)+r_{\tau v}^2\lambda_{\max}(S_v)+r_{dh}^2\lambda_{\max}(Z_h)+r_{dv}^2\lambda_{\max}(Z_v)\}.
\end{aligned}$$

From (24) and (25) it follows that

$$\sum_{i+j=\Gamma} \| x(i,j) \|^2 \le N\beta^{(\Gamma-\kappa_0)} \sum_{i+j=\kappa_0} \| x(i,j) \|_C^2 . \tag{26}$$

where, $N = \dfrac{\xi_2}{\xi_1}$. Thus, the system with mixed (1) is exponentially stable. The proof is completed.

Corollary 1. *For given nonnegative integers* τ_{hm}, τ_{hM}, τ_{vm}, *and* τ_{vM} *if there exist symmetric positive definite matrices* $P = diag(P_h, P_v)$, $Q = diag(Q_h, Q_v)$, $R = diag(R_h, R_v)$,

$X = diag(X_h, X_v)$, $Y = diag(Y_h, Y_v)$, $S = diag(S_h, S_v)$, $Z = diag(Z_h, Z_v)$ *and* $0 < \beta < 1$ *such that the following LMI holds*

$$\Phi = \begin{bmatrix} \Psi & \mathcal{A}^T P & \mathcal{D}^T \Upsilon \\ & -P & 0 \\ & * & -\Pi \end{bmatrix} < 0 \tag{27}$$

where $\Upsilon = \begin{bmatrix} X & Y & S \end{bmatrix}$, $\Pi = diag(X, Y, S)$, and

$$\Psi = \begin{bmatrix} \Psi 11 & 0 & X & \overline{Y} \\ & -(\overline{R} + 2\overline{S}) & \overline{S} & \overline{S} \\ & * & -(Q + X + \overline{S}) & 0 \\ & * & * & -(\overline{X} + \overline{S}) \end{bmatrix} < 0,$$

$$\Psi 11 = \overline{Q} + R - P - X - \overline{Y},$$

$$X = I(\beta\tau_{hm}^2, \beta\tau_{vm}^2)X, \quad Y = I(\beta\tau_{hM}^2, \beta\tau_{vM}^2)Y, \quad S = I(\beta r_{\tau h}^2, \beta r_{\tau v}^2)S,$$

$$R = I(\beta(1 + r_{\tau h}), \beta(1 + r_{\tau v}))R, \quad P = I(\beta, \beta)P, \quad \overline{Q} = I(\beta, \beta)Q,$$

$$X = I(\beta^{1+\tau_{hm}}, \beta^{1+\tau_{vm}})X, \quad \overline{Y} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})Y, \quad \overline{R} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})R,$$

$$\overline{S} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})S, \quad Q = I(\beta^{1+\tau_{hm}}, \beta^{1+\tau_{vm}})Q, \quad \overline{X} = I(\beta^{1+\tau_{hM}}, \beta^{1+\tau_{vM}})X,$$

$$r_{\tau h} = \tau_{hM} - \tau_{hm}, \quad r_{\tau v} = \tau_{vM} - \tau_{vm},$$

$$\mathcal{A} = \begin{bmatrix} A & A_\tau & 0 & 0 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} A - I & A_\tau & 0 & 0 \end{bmatrix},$$

then system (1) with $A_d = 0$ *is exponentially stable.*

4. Conclusion

The exponential stability a class of 2D discrete-time systems with mixed delays has been studied in this paper. Sufficient conditions fordelay-dependent exponential stability of 2D system have been established in terms of a set of LMIs. Future work will be devoted to 2D continuous-time systems.

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