A method of sliding mode control of cart and pole system

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(Manuscript Received on July 15, 2015, Manuscript Revised August 30, 2015)

ABSTRACT

This paper presents a method of using Sliding Mode Control (SMC) for Cart and Pole system. The stability of controller is proved through using Lyapunov function and simulations. A genetic algorithm (GA) program is used to optimize controlling parameters. The GA-based parameters prove good-quality of control through Matlab/Simulink Simulation.

Keywords: *Sliding Mode Control, Cart and Pole, Inverted Pendulum, Genetic Algorithm, Matlab/Simulink.*

1. INTRODUCTION

Cart and Pole system is a popular classical non-linear model used in most laboratories in universities for testing controlling algorithm. Morever, it is a SIMO system in which just one input control must stabilize two outputs: position of cart and angle of pendulum. Many control algorithms were proved to work well on this model [1].

Beside other kinds of control, the nonlinear control, especially Sliding Mode Control (SMC), depends on nonlinear structure of system. So, the stability of system is ensured. Cesar Aguilar [2] set new variable including both Cart's position and Pendulum's angle, neglecting some components in calculating and trying to transform dynamic equation to appropriate form. But it just operated well when the neglected component was not remarkable. Reference [3] introduced other way to set sliding mode for a similar model, the Rotary Inverted Pendulum but did not prove the stability by mathematical methods. Reference [4] and [5] respectively introduced integral SMC and hierarchial SMC applied for Cart and Pole system. But [4] did not prove stability by mathematics or examples in Matlab/Simulink.

This paper presents a new and simple SMC for Cart and Pole system. First, different sliding surfaces are presented. Then, a positive Lyapunov function is set to include both sliding surfaces. A nonlinear way is set to make this function to zero when operating system. After proving stability of controller, GA program is used to optimize controlling parameters.

2. CART AND POLE SYSTEM

The studied system in Fig. 1 is a cart of which a rigid pole is hinged. The cart is free to move within the bounds of a one-dimensional track. The pole can move in the vertical plane parallel to the track. The controller can apply a force to the cart parallel to the track.

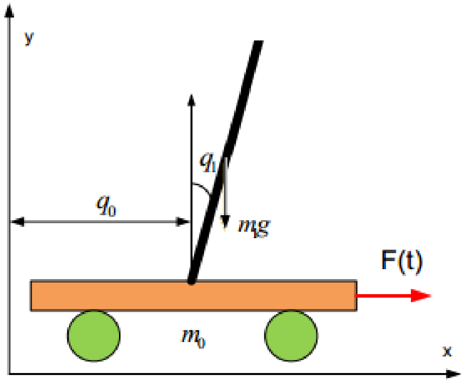

Figure 1: Cart and Pole system

Lagragian equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_p \tag{1}$$

with vector of state variables $q = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$

Kinetic energy of system:

$$T = T_0 + T_1 = \frac{1}{2}m_0\dot{q}_0^2 + \frac{1}{2}J_1\dot{q}_1^2 + \frac{1}{2}m_1(\dot{q}_0 + \dot{q}_1 l_1\cos q_1)^2 \tag{2}$$

Potential energy of system:

$$P = P_0 + P_1 = m_1 g l_1 \cos q_1 \tag{3}$$

Lagrangian operator:

$$L = T - P = \frac{1}{2}m_0\dot{q}_0^2 + \frac{1}{2}J_1\dot{q}_1^2 + \frac{1}{2}m_1(\dot{q}_0 + \dot{q}_1 l_1\cos q_1)^2 - m_1 g l_1 \cos q_1 \tag{4}$$

Lagrangian for motion of cart:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_0}\right) - \frac{\partial L}{\partial q_0} = F - b_0\dot{q}_0 \tag{5}$$

Lagrangian for rotating motion of pendulum:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = -b_1\dot{q}_1 \tag{6}$$

Solve (5) and (6), system dynamic equations are:

$$\begin{cases} (m_0 + m_1)\ddot{q}_0 + m_1 l_1(\ddot{q}_1\cos q_1 - \dot{q}_1\sin q_1) = F - b_0\dot{q}_0 \\ J_1\ddot{q}_1 + m_1 l_1^2(\ddot{q}_1\cos^2 q_1 - 2\dot{q}_1^2\sin q_1\cos q_1) + m_1 l_1(\ddot{q}_0\cos q_1 - \dot{q}_0\dot{q}_1\sin q_1) + m_1\dot{q}_1^2 l_1^2\cos q_1\sin q_1 + \\ + m_1\dot{q}_0\dot{q}_1 l_1\sin q_1 - m_1 g l_1\sin q_1 = -b_1\dot{q}_1 \end{cases} \tag{7}$$

We can transfrom (7) to the form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x) + g_1(x)u \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(x) + g_2(x)u \end{cases} \quad \text{with } x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [q_0 \quad \dot{q}_0 \quad q_1 \quad \dot{q}_1]^T \tag{8}$$

And $f_1(x)$, $f_2(x)$, $g_1(x)$, $g_2(x)$ defined as below:

$$f_1(x) = \frac{\begin{bmatrix} -J_1 b_0 x_2 - g l_1^2 m_1^2\cos x_1\sin x_1 + l_1^3 m_1^2 x_4\cos^2 x_3\sin x_3 - b_0 l_1^2 m_1 x_2\cos^2 x_3 + \\ + J_1 l_1 m_1 x_4\sin x_3 + b_1 l_1 m_1\dot{q}_1\cos(x_3) - l_1^3 m_1^2 x_4^2\cos(x_3)\sin(x_3) \end{bmatrix}}{m_0 m_1 l_1^2\cos^2(x_3) + J_1 m_0 + J_1 m_1} \tag{9}$$

$$g_1(x) = \frac{J_1 + l_1^2 m_1 \cos^2 x_3}{m_0 m_1 l_1^2 \cos^2(x_3) + J_1 m_0 + J_1 m_1} \tag{10}$$

$$f_2(x) = \frac{\begin{bmatrix} g l_1 m_1^2 \sin x_3 - b_1 m_1 x_4 - b_1 m_0 x_4 - l_1^2 m_1^2 x_4 \cos x_3 \sin x_3 + l_1^2 m_1^2 x_4^2 \cos x_3 \sin x_3 + \\ + b_0 l_1 m_1 x_2 \cos x_3 + g l_1 m_0 m_1 \sin x_3 + l_1^2 m_0 m_1 x_4^2 \cos x_3 \sin x_3 \end{bmatrix}}{m_0 m_1 l_1^2 \cos^2(x_3) + J_1 m_0 + J_1 m_1} \tag{11}$$

$$g_2(x) = \frac{l_1 m_1 \cos x_3}{m_0 m_1 l_1^2 \cos^2(x_3) + J_1 m_0 + J_1 m_1} \tag{12}$$

Parameters of system is used from the real system in [6], but taking away the second link of the double-linked Inverted Pendulum to have a Single-linked Inverted Pendulum on Cart (Cart and Pole system). Values of parameters are listed in Table 1.

Table 1: Real System parameters

Parameter	Unit	Definition	Value
m_0	Kg	Mass of cart	0.033
m_1	Kg	Mass of first pendulum	1.999
L_1	M	Length of first pendulum	0.2
l_1	M	Distance between center and rotating axis of first pendulum	0.115
J_1	kgm^2	Inertial moment of first pendulum	0.023
g	m/s^2	Gravitation acceleration	9.81
F	N	Force controlling cart	
b_0	kg/s	Viscous Coefficient of Cart	0.0001
b_1	Nms	Viscous Coefficient of Rotating Axis of first inverted pendulum	0.0001

3. SLIDING MODE CONTROL

Sliding surfaces are chosen as:

$$\begin{cases} s_1 = x_1 + \lambda_1 x_2 \\ s_2 = x_3 + \lambda_2 x_4 \end{cases} \text{ with } \lambda_1 = const > 0 \text{ and } \lambda_2 = const > 0 \tag{13}$$

Choosing Lyapunov function:

$$V = |s_1| + \lambda_3 |s_2| > 0 \tag{14}$$

$$\dot{V} = \dot{s}_1 \operatorname{sgn}(s_1) + \lambda_3 \dot{s}_2 \operatorname{sgn}(s_2) = (x_2 + \lambda_1 \dot{x}_2) \operatorname{sgn}(s_1) + \lambda_3 (x_4 + \lambda_2 \dot{x}_4) \operatorname{sgn}(s_2)$$

$$= \{x_1 + \lambda_1 [f_1(x) + g_1(x)u]\} \operatorname{sgn}(s_1) + \{x_3 + \lambda_2 [f_2(x) + g_2(x)u]\} \operatorname{sgn}(s_2)$$

$$= \alpha(x) + \beta(x)u \tag{15}$$

With $\alpha(x) = \lambda_1 \operatorname{sgn}(s_1) f_1(x) + \lambda_3 \operatorname{sgn}(s_2) f_2(x) + x_2 \operatorname{sgn}(s_1) + \lambda_3 x_4 \operatorname{sgn}(s_2)$ (16)

And $\beta(x) = \lambda_1 \operatorname{sgn}(s_1) g_1(x) + \lambda_3 \operatorname{sgn}(s_2) g_2(x)$ (17)

Choosing u that makes:

$$\dot{V} = -\lambda_4 < 0 \tag{18}$$

So, we choose $\lambda_4 = const > 0$

From (15) and (18), we have:

$$u = \left[\frac{-\lambda_4 - \alpha(x)}{\beta(x)} \right] \tag{19}$$

In (13), two sliding surfaces are presented with S_1 includes elements of Cart and S_2 includes elements of Pendulum. When model is balanced, S_1 and S_2 will move to zero. In this case, we try to reduce S_1 and S_2 by setting positive function V in (14).

After generating $\dot{V}$ in (12), we choose control signal u that makes $\dot{V} < 0$ in (18). Finally, (19) shows the appropriate control signal u. From (14), (18), we have: $V > 0$ and $V\dot{V} < 0$. So, $V \xrightarrow{t \to \infty} 0$. From (14), we have: $S_1 \xrightarrow{t \to \infty} 0$ and $S_2 \xrightarrow{t \to \infty} 0$.

4. GENETIC ALGORITHM

Stability characteristic of the system is proved in Section 3. With a random parameters of controller like chosen in three examples in Section 5, we have the simulation results are shown in Fig. 4, Fig. 5, Fig. 6.

As in these figures, the cart's position is stable eventhough quality of control is not so good and the Pendulum's angle is not completely stable but it is not unstable. The force on Cart chatters because of using function sign() in controller. So, genetic algorithm (GA) is used here to optimize control parameters.

In this case, GA used is off-line. Parameters for GA program are listed as below:

- Size of population: N=20
- Linear Ranking Selection: $\eta = 0.2$
- Decimal coding
- Two-point crossover
- Crossover parameter: 0.8
- Mutation parameter: 0.2

Choose fitness function:

$$J = \sum_{i=1}^{n} [e_1(i)]^2 + \sum_{i=1}^{n} [e_2(i)]^2 \tag{20}$$

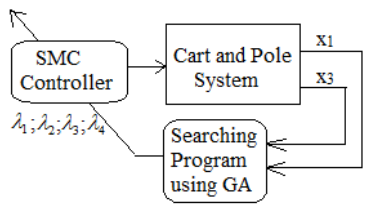

Figure 2: Block diagram of GA program

With $e_1 = q_0$, $e_2 = q_1$ and n is number of samples in one time of simulation. If the controller can stabilize system well, function J will be very small.

In this case, we operate Simulink program of simulating system in 10s, with sample-time is 0.01s. So, we have n = 1001 sample.

After 94 generation, the result is $\lambda_1 = 5.84$; $\lambda_2 = 0.06$; $\lambda_3 = 7.42$; $\lambda_4 = 9.84$ and the fitness function is $J = 0.8677$.

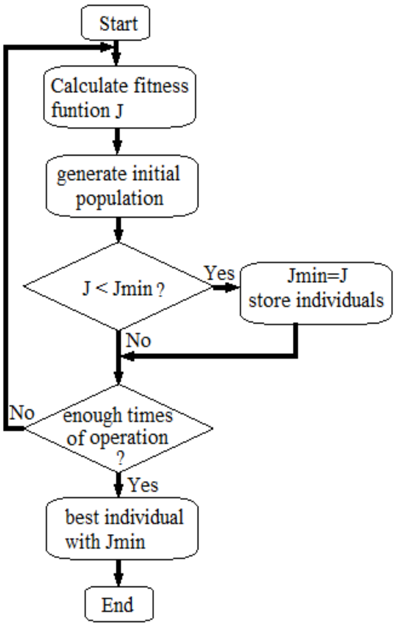

Figure 3: Flow chart of GA Searching process

5. SIMULATION

5.1 Using random controlling parameters

In order to test the stability of system, we can choose some values of λ_1; λ_2; λ_3; λ_4. Three samples are randomly chosen as:

- Example 1:

$$\lambda_1 = 1;\ \lambda_2 = 1;\ \lambda_3 = 1;\ \lambda_4 = 1$$

- Example 2:

$$\lambda_1 = 10;\ \lambda_2 = 10;\ \lambda_3 = 10;\ \lambda_4 = 10$$

- Example 3:

$$\lambda_1 = 1;\ \lambda_2 = 2;\ \lambda_3 = 3;\ \lambda_4 = 4$$

Choosing initial values of variables are chosen as: $q_{0\_init} = 0.1$ (m), $\dot{q}_{0\_init} = -0.1$ (m/s), $q_{1\_init} = -0.1$ (rad), $\dot{q}_{1\_init} = -0.1$ (rad/s), the simulation results are shown in Fig. 4, Fig. 5, Fig. 6.

In Fig. 4, the cart's position is stable eventhough quality of control is not so good. In Fig. 5, the pendulum's angle is not completely stable but it is not unstable. In Fig. 6, the force on cart chatters because of using function sign() in controller. The SMC algorithm ensures the stability of system but quality is not so good.

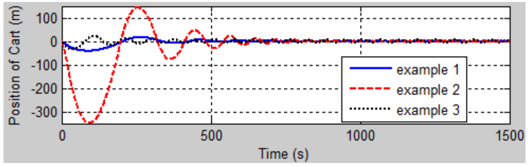

Figure 4: Position of Cart (m) when control parameters are random

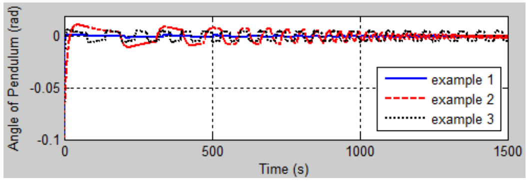

Figure 5: Angle of Pendulum (rad) when control parameters are random

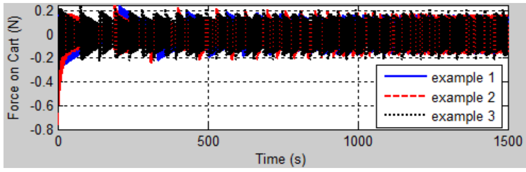

Figure 6: Force on Cart (N) when control parameters are random

5.2 Using controlling parameters from GA program

By using GA program in Chapter 4, we have: $\lambda_1 = 5.84$; $\lambda_2 = 0.06$; $\lambda_3 = 7.42$; $\lambda_4 = 9.84$.

Choosing initial values of variables are $q_{0\_init} = 0.1$ (m); $\dot{q}_{0\_init} = -0.1$ (m/s); $q_{1\_init} = -0.1$ (rad); $\dot{q}_{1\_init} = -0.1$ (rad/s), and the results of simulation are shown from Fig. 7 to Fig. 13. The cart's position and pendulum's angle move to balancing point after 10s and 2.2s, respectively. In Fig. 9, control signal still chatters but with smaller amplitude than in Fig. 6. Through Fig. 7 to Fig. 8, the variables are proved to stabilized quickly. Fig. 10 and Fig. 11 show the robust characteristics of SMC. Fig. 9 proves the chattering of signal control descreases but not be exterminated. Morever, two sliding surfaces s_1 and s_2 are proved to be stabilized quickly in just 3s in Fig. 12 and Fig. 13.

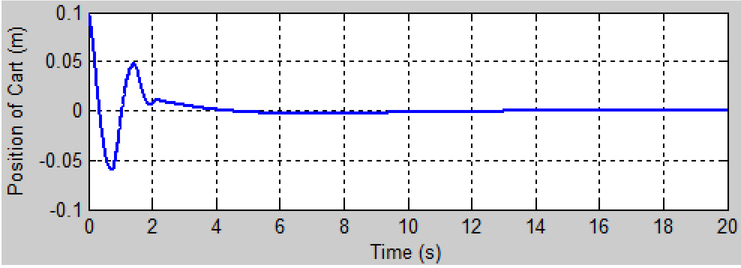

Figure 7: Position of Cart (m) with parameters chosen by GA

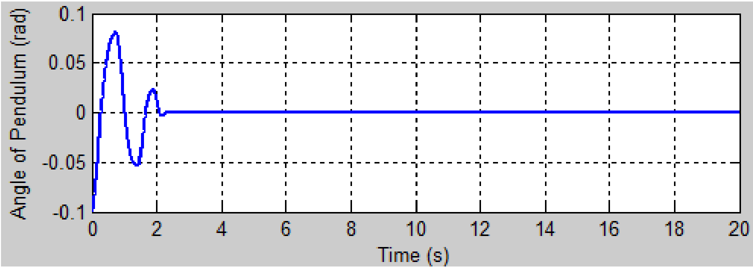

Figure 8: Angle of Pendulum (rad) with parameters chosen by GA

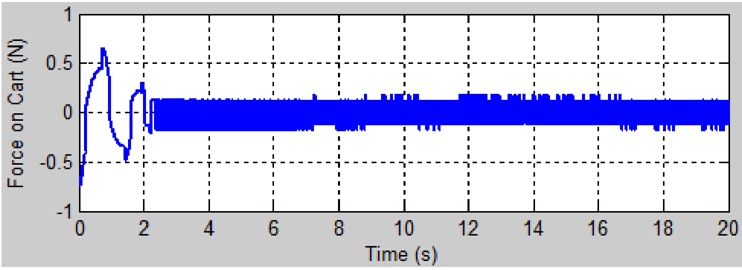

Figure 9: Force on Cart (N) with parameters chosen by GA

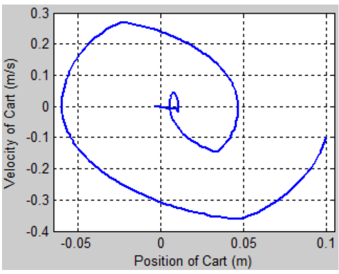

Figure 10: Position and Velocity of Cart in 20s

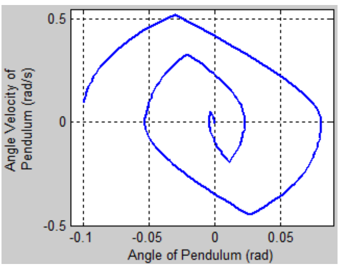

Figure 11: Angle and Angle Velocity of Pendulum in 20s

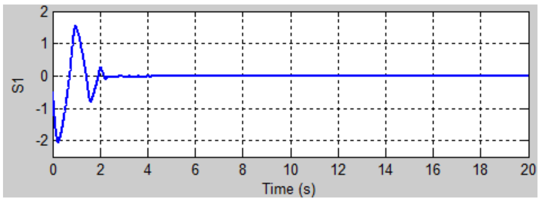

Figure 12: Sliding surface S_1

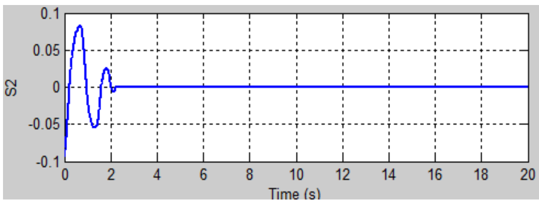

Figure 13: Sliding Surface S_2

6. CONCLUSION

This paper presented a new way of SMC to control Cart and Pole system. The stability of controller was proved through Lyapunov setting and random examples. Anyway, the stability of system was ensured but quality of controller was not ensured. To overcome the difference in choosing controlling parameters, one GA program was used to search the optimized controlling parameters. The controller with these parameters worked well in Simulation.

Một phương pháp điều khiển trượt cho hệ con lắc ngược trên xe

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TÓM TẮT

Bài báo trình bày một phương pháp sử dụng giải thuật điều khiển trượt (SMC) cho hệ con lắc ngược trên xe. Độ ổn định hệ thống của bộ điều khiển được chứng minh thông qua hàm Lyapunov và các kết quả mô phỏng. Một chương trình tính toán áp dụng giải thuật di truyền (GA) được sử dụng để tối ưu hóa các thông số điều khiển.

Từ khóa: *Điều khiển trượt, Con lắc ngược trên xe, Giải thuật di truyền, Matlab/Simulink.*

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