An approximate Hahn-Banach-Lagrange theorem

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(Received on November 5th 2015, accepted on November 21th 2016)

ABSTRACT

In this paper, we proved a new extended version of the Hahn-Banach-Lagrange theorem that is valid in the absence of a qualification condition and is called an approximate Hahn-Banach-Lagrange theorem. This result, in special cases, gives rise to approximate sandwich and approximate Hahn-Banach theorems. These results extend the Hahn-Banach-Lagrange theorem, the sandwich theorem in [18], and the celebrated Hahn-Banach theorem. The mentioned results extend the original ones into two features: Firstly, they extend the original versions to the case with extended sublinear functions (i.e., the sublinear functions that possibly possess extended real values). Secondly, they are topological versions which held without any qualification condition. Next, we showed that our approximate Hahn-Banach-Lagrange theorem was actually equivalent to the asymptotic Farkas-type results that were established recently [10]. This result, together with the results [5, 16], give us a general picture on the equivalence of the Farkas lemma and the Hahn-Banach theorem, from the original version to their corresponding extensions and in either non-asymptotic or asymptotic forms.

INTRODUCTION AND PRELIMINARY

It is well-known that the Farkas lemma for convex systems is equivalent to the celebrated Hahn-Banach theorem [16]. In the last decades, many generalized versions of the Farkas lemma have been developed (see [3, 5, 4, 9, 11, 15, 17], and, in particular, the recent survey [7]). For the generalizations of non-asymptotic Farkas lemma, i.e., the versions of Farkas-type results were hold under some qualification condition. It was shown in [5] that these versions are equivalent to some extended versions of the Hahn-Banach theorem. A natural question arises: Are there any similar results for generalized asymptoic/sequential Farkas lemmas and certain types of extended Hahn-Banach theorems? This paper is aimed to answer this question. Fortunately, the answer is affirmative, and so the result in this paper can be considered as a counter part of [5] concerning versions of sequential Farkas lemmas and the so-called approximate Hahn-Banach-Lagrange theorems (which are extended versions of the Hahn-Banach theorem).

In this paper, we establish a new extended version of Hahn-Banach-Lagrange theorem which extends the original one in [5, 18], and it is valid in the absence of a regularity condition. It is called the approximate Hahn-Banach-Lagrange theorem. The results then gives rise to an approximate sandwich theorem and an approximate Hahn-Banach theorem in the manner as in [5]. The generalization of these reults in comparison with [5, 18] is twofold: firstly, they extend the original version to the case with extended sublinear functions (i.e., the sublinear functions which possibly possess extended real values); secondly, in contrast to [5], they are topological versions which hold without any qualification condition. The paper can be considered as a continuation of the previous ones (of the authors and their co-authors) [5, 10, 12]. Some tools and some ideas of generalizations to Hahn-Banach-Lagrange theorem and to real-extended sublinear functions are modifications of the one in [5] to adapt to the case where no qualification condition is assumed.

Let X and Y be locally convex Hausdorff topological vector spaces (lcHtvs in short), with their topological dual spaces X^* and Y^*, respectively. The only topology we consider on X^*, Y^* is the w^*-topology. For a set $A \subset X^*$, the closure of A w.r.t. the weak*-topology is denoted by $\mathrm{cl}A$. The indicator function of A is denoted by i_A, i.e., $i_A(x) = 0$ if $x \in A$, $i_A(x) = +\infty$ if $x \in X \setminus A$. Let B, C be two subsets of some locally convex Hausdorff topological vector space. We say that B is closed regarding C if $(\mathrm{cl}B) \cap C = B \cap C$ (see [1], [5]).

Let $f : X \to \mathbb{R} \cup \{+\infty\}$. The effective domain of f is the set $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$. The function f is proper if $\mathrm{dom} f \neq \emptyset$. The set of all proper, lower semi-continuous (lsc in short) and convex functions on X will be denoted by $\Gamma(X)$. The epigraph of f is

$$\mathrm{epi}\, f := \{(x,\alpha) \in X \times \mathbb{R} \,:\, f(x) \le \alpha\}.$$

The Legendre-Fenchel conjugate of f is the function $f^* : X^* \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ defined

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}, \ \ \forall x^* \in X^*.$$

Now let K be a closed convex cone in Y and let $\le_K$ be the partial order on Y generated by K, i.e.,

$$y_1 \le_K y_2 \ \text{ if and only if } \ y_2 - y_1 \in K.$$

We add to Y a greatest element with respect to $\le_K$, denoted by ∞_K, which does not belong to Y, and let $Y^\bullet = Y \cup \{\infty_K\}$. Then one has $y \le_K \infty_K$ for every $y \in Y^\bullet$. We assume by convention: $y + \infty_K = \infty_K + y = \infty_K$ for all $y \in Y^\bullet$, and $\alpha\infty_K = \infty_K$ *if* $\alpha \ge 0$. *The dual cone of* K, *denoted by* K^+, *is defined by*

$$K^+ := \{y^* \in Y^* : \langle y^*, y\rangle \ge 0 \ \text{for all } y \in K\}.$$

A mapping $h : X \to Y^\bullet$ *is called* K*-convex if*

$$x_1, x_2 \in X,\ \mu_1, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$$
$$\Rightarrow h(\mu_1 x_1 + \mu_2 x_2) \le_K \mu_1 h(x_1) + \mu_2 h(x_2),$$

where "$\le_K$" *is the binary relation (generated by* K*) extended to* $Y^\bullet$ *by setting* $y \le_K \infty_K$ for all $y \in Y^\bullet$. *The domain of* h, *denoted by* $\mathrm{dom}\, h$, *is defined to be the set* $\mathrm{dom}\, h := \{x \in X :\ h(x) \in Y\}$. *The* K*-epigraph of* h *is the set*

$$\mathrm{epi}_K h := \{(x,y) \in X \times Y \,:\, y \in h(x) + K\}.$$

It is clear that h *is* K*-convex if and only if* $\mathrm{epi}_K h$ *is convex. In addition,* $h : X \to Y^\bullet$ *is said to be* K*-epi closed if* $\mathrm{epi}_K h$ *is a closed set in the product space. Then,* $h^{-1}(-K)$ *is closed (see [5]). It is worth observing that if* h *is* K*-convex, then* $h^{-1}(-K)$ *is convex.*

Moreover, for any $y^* \in Y^*$ and $g : X \to Y^\bullet$, we define the composite function $y^* \circ g : X \to \mathbb{R} \cup \{+\infty\}$ as follows

$$(y^* \circ g)(x) = \begin{cases} \langle y^*, g(x)\rangle, & \text{if } x \in \mathrm{dom}\, g, \\ +\infty, & \text{otherwise}. \end{cases}$$

The function $S : Y \to \mathbb{R} \cup \{+\infty\}$ is called (extended) sublinear if

$$S(y + y') \le S(y) + S(y') \quad \textit{and}$$
$$S(\lambda y) = \lambda S(y),\ \ \forall y, y' \in Y,\ \forall \lambda > 0.$$

By convention, we set $S(0_Y) = 0$. *We extend* S *to* $Y^\bullet$ *by setting* $S(\infty_K) = +\infty$. *An extended sublinear function* $S : Y \to \mathbb{R} \cup \{+\infty\}$ *allows us to introduce in* $Y^\bullet$ *a binary relation which is reflexive and transitive as:*

$y_1 \le_S y_2$ *if* $y_1 \le_K y_2$, *where* $K := \{y \in Y : S(-y) \le 0\}$.

It is worth mentioning that the definition of the relation $\le_S$ can be understood in the extended sense of $S : Y^\bullet \to \mathbb{R} \cup \{+\infty\}$. The relation $\le_S$ can be extended to $Y^\bullet$ in a suitable way.

Given a sublinear function $S : Y \to \mathbb{R} \cup \{+\infty\}$, we adapt the notion S-convex (i.e.,convex with respect to a sublinear function) in [20] and introduce the one corresponding to an extended sublinear function S:

A mapping $h : X \to Y^\bullet$ is said to be S-convex if for all $x_1, x_2 \in X$, $\mu_1, \mu_2 > 0$, $\mu_1 + \mu_2 = 1$ one has

$$h(\mu_1 x_1 + \mu_2 x_2) \le_S \mu_1 h(x_1) + \mu_2 h(x_2).$$

It is worth observing that, as mentioned in [19, Remark 1.10], "S-convex can mean different things under different circumstances" such as, when $Y = \mathbb{R}$, if $S(y) := |y|$, $S(y) := y$, $S(y) := -y$, or $S(y) = 0$, respectively, then "S-convex" means "affine", "convex", "concave" or "arbitrary", respectively.

It can be easily verified that if h is S-convex then h is K-convex with $K := \{y \in Y \ : \ S(-y) \le 0\}$. Conversely, if h is K-convex with some convex cone K then h is S-convex with $S = i_{-K}$ (see [5]).

The organization of the paper is as follows: In the next section, Section 2, we recall two new versions of sequential Farkas lemma for cone-convex systems and sublinear-convex systems established in [10]. In Section 3, we establish the so-called approximate Hahn-Banach-Lagrange theorem, a topological and asymptotic extended version of the original algebraic one in [18, 19, 20]. Versions of approximate sandwich theorem and approximate Hahn-Banach theorem are derived from this approximate Hahn-Banach-Lagrange theorem. The last section, Section 4, we show that our new approximate Hahn-Banach-Lagrange theorem is actually equivalent to the asymptotic Farkas-type results that were established recently in [10]. This equivalence can be considered as the last piece of the whole picture on the equivalence of the Farkas lemma and the Hahn-Banach theorem for which the other pieces are the equivalence of non-asymptotic extended convex Farkas lemmas and extended Hahn-Banach-Lagrange theorem established in [5], and the one between the linear Farkas lemma and the celebrated Hahn-Banach theorem [16].

Sequential Farkas lemma for convex systems

In this section we will recall the sequential Farkas lemmas for convex systems in [10] which hold without any qualification condition: the asymptotic version of the Farkas lemma for systems which is convex w.r.t. a convex cone and the one for systems which is convex w.r.t. an extended sublinear function.

Let X, Y be lcHtvs, K be a closed convex cone in Y, C be a nonempty closed convex subset of X and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc and convex function.

Sequential Farkas lemma for cone-convexsystems

Consider $g : X \to Y^\bullet$ be a K-convex and K-epi closed mapping. Let $A := C \cap g^{-1}(-K)$ and assume that $(\operatorname{dom} f) \cap A \neq \emptyset$. The following sequential Farkas lemmas in [10] will be used in the sequence.

Theorem 2.1 [10] The following statements are equivalent:

(i) $x \in C,\ g(x) \in -K \ \Rightarrow \ f(x) \ge 0$,

(ii) there exist nets $(y_i^*)_{i\in I} \subset K^+$ *and* $(x_{1i}^*, x_{2i}^*, x_{3i}^*, \varepsilon_i)_{i\in I} \subset X^* \times X^* \times X^* \times \mathbb{R}$ *such that*

$\varepsilon_i \ge f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) \ \ \forall i \in I$,

and $(x_{1i}^* + x_{2i}^* + x_{3i}^*, \varepsilon_i) \to (0_{X^*}, 0)$,

(iii) there exist nets $(y_i^*)_{i\in I} \subset K^+$ *and* $(x_i^*, \varepsilon_i)_{i\in I} \subset X^* \times \mathbb{R}$ *such that*

$\varepsilon_i \ge (f + y_i^* \circ g + i_C)^*(x_i^*) \ \ \forall i \in I$,

and $(x_i^*, \varepsilon_i) \to (0_{X^*}, 0)$,

(iv)there exists a net $(y_i^*)_{i\in I} \subset K^+$ *such that*

$f(x) + \liminf_{i\in I} (y_i^* \circ g)(x) \ge 0, \ \ \forall x \in C.$

From the previous theorem, it is easy to see that under some closedness conditions, one gets back stable Farkas lemma established recently in [5] (see [10]).

Sequential Farkas lemma for sublinear-convex systems

Let X, Y be lcHtvs, C be a nonempty closed convex subset of X, $S: Y \to \mathbb{R} \cup \{+\infty\}$ be an lsc sublinear function and $g: X \to Y^{\bullet}$ be an S-convex mapping such that the set

$\{(x, y, \lambda) \in X \times Y \times \mathbb{R} \ : S(g(x) - y) \le \lambda\}$ *(2.1)*

is closed in the product space $X \times Y \times \mathbb{R}$. *Let us consider* $f: X \to \mathbb{R} \cup \{+\infty\}$ *and* $\psi: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be proper convex lsc functions.*

We now recall two versions of asymptotic Farkas lemma for systems that are convex w.r.t. the sublinear function $S: Y \to \mathbb{R} \cup \{+\infty\}$ in [10]

Theorem 2.2 *[10] Assume that the following condition holds:*

$$(\operatorname{dom} f) \cap \{x \in C \ : \exists \alpha \in \operatorname{dom}\psi, \ (S \circ g)(x) \le \alpha\} \ne \emptyset. \quad (2.2)$$

Then the following statements are equivalent:

(a) $x \in C, \ \alpha \in \mathbb{R}, \ (S \circ g)(x) \le \alpha \ \Rightarrow \ f(x) + \psi(\alpha) \ge 0$,

(b) there exist nets $(y_i^*, \gamma_i)_{i \in I} \subset Y^* \times \mathbb{R}_+$ and $(x_{1i}^*, x_{2i}^*, x_{3i}^*, \eta_i, \varepsilon_i)_{i \in I} \subset X^* \times X^* \times X^* \times \mathbb{R} \times \mathbb{R}$ with $y_i^* \le \gamma_i S$ on Y for all $i \in I$ such that

$$\varepsilon_i \ge f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) + \psi^*(\eta_i + \gamma_i) \quad \forall i \in I,$$

and $(x_{1i}^* + x_{2i}^* + x_{3i}^*, \eta_i, \varepsilon_i) \to (0_{X^*}, 0, 0)$,

(c) there exist nets $(y_i^*, \gamma_i)_{i \in I} \subset Y^* \times \mathbb{R}_+$ and $(x_i^*, \eta_i, \varepsilon_i)_{i \in I} \subset X^* \times \mathbb{R} \times \mathbb{R}$ with $y_i^* \le \gamma_i S$ on Y for all $i \in I$ such that

$$\varepsilon_i \ge (f + y_i^* \circ g + i_C)^*(x_i^*) + \psi^*(\eta_i + \gamma_i) \ \ \forall i \in I, \quad (2.3)$$

$$\text{and } (x_i^*, \eta_i, \varepsilon_i) \to (0_{X^*}, 0, 0). \quad (2.4)$$

Theorem 2.3 [10] Assume that (2.2) holds. Then the following statements are equivalent:

(a) $x \in C, \ \alpha \in \mathbb{R}, \ (S \circ g)(x) \le \alpha \Rightarrow f(x) + \psi(\alpha) \ge 0$,

(b) there exists a net $(y_i^*, \gamma_i, \eta_i)_{i \in I} \subset Y^* \times \mathbb{R}_+ \times \mathbb{R}$ with $y_i^* \le \gamma_i S$ on Y for all $i \in I$ such that $\eta_i \to 0$, $(\eta_i + \gamma_i)_{i \in I} \subset \operatorname{dom}\psi^*$ and

$$f(x) + \liminf_{i \in I} \big((y_i^* \circ g)(x) - \psi^*(\eta_i + \gamma_i)\big) \ge 0 \ \ \forall x \in C. \quad (2.5)$$

Approximate Hahn-Banach-Lagrange theorem

In this section we establish the so-called approximate Hahn-Banach-Lagrange theorem, a topological and asymptotic extended version of the original algebraic version in [18], [19], and [20]. An approximate sandwich theorem and an approximate Hahn-Banach theorem are derived from this approximate Hahn-Banach-Lagrange theorem.

It is worth mentioning that these extended versions of Hahn-Banach-Lagrange theorem, sandwich theorem, and Hahn-Banach theorem extended the original ones in two features: they extend the original version to the case with extended sublinear functions and, in contrast to [5], they are topological versions which hold without any qualification condition.

We will maintain the notations used in Section 2.

Theorem 3.1 [Approximate Hahn-Banach-Lagrange theorem] Let X, Y be lcHtvs, C be a nonempty closed convex subset of X, $S: Y \to \mathbb{R} \cup \{+\infty\}$ be an lsc extended sublinear function, and $g: X \to Y^{\bullet}$ be an S-convex mapping such that the set in (2.1) is closed in the product space $X \times Y \times \mathbb{R}$. Let further $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function.

Assume that

$$(\operatorname{dom} f) \cap \{x \in C : \exists \alpha \in \mathbb{R} \ \textit{such that} \ (S \circ g)(x) \le \alpha\} \ne \emptyset. \quad (3.1)$$

Then the *following statements are equivalent:*

(i) $\inf_{x \in C} \big[f(x) + (S \circ g)(x)\big] \in \mathbb{R}$,

(ii) there exist nets $(y_i^*, \gamma_i)_{i \in I} \subset Y^* \times \mathbb{R}_+$ *and* $(x_{1i}^*, x_{2i}^*, x_{3i}^*)_{i \in I} \subset X^* \times X^* \times X^*$ *with* $y_i^* \le \gamma_i S$ *on* Y for all $i \in I$ such that $\gamma_i \to 1$, $x_{1i}^* + x_{2i}^* + x_{3i}^* \to 0_{X^*}$, and

$$\liminf_{i \in I} \big\{-f^*(x_{1i}^*) - (y_i^* \circ g)^*(x_{2i}^*) - i_C^*(x_{3i}^*)\big\}$$

$$= \inf_{x \in C} \Big[f(x) + \liminf_{i \in I} (y_i^* \circ g)(x)\Big]$$

$$= \inf_{x \in C} \big[f(x) + (S \circ g)(x)\big] \in \mathbb{R}.$$

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\psi(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$. It is clear that ψ is proper convex continuous function and

$$\psi^*(\gamma) = \begin{cases} 0 & \text{if } \gamma = 1, \\ +\infty & \text{else}. \end{cases} \qquad (3.2)$$

The conclusion follows from Theorem 2.2. Firstly, (2.2) follows from the assumption (3.1).

[(i) $\Rightarrow$ (ii)]

Assume that $\beta := \inf_{x\in C} [f(x) + (S \circ g)(x)] \in \mathbb{R}$.

Then $f(x) + (S \circ g)(x) \ge \beta$ for all $x \in C$. Note that $x \in C,\ \alpha \in \mathbb{R}$, $(S \circ g)(x) \le \alpha$ then

$$f(x) + \psi(\alpha) = f(x) + \alpha \ge f(x) + (S \circ g)(x) \ge \beta.$$

Thus, with $\tilde f := f - \beta$ then

$$x \in C,\ \alpha \in \mathbb{R},\ (S \circ g)(x) \le \alpha \Rightarrow \tilde f(x) + \psi(\alpha) \ge 0,$$

i.e., (a) inTheorem 2.2 holds, where $\tilde f$ plays the role of f. By this theorem, (a) is equivalent to (b), namely, there exist nets $(y_i^*, \gamma_i)_{i\in I} \subset Y^* \times \mathbb{R}_+$ and $(x_{1i}^*, x_{2i}^*, x_{3i}^*, \eta_i, \varepsilon_i)_{i\in I} \subset X^* \times X^* \times X^* \times \mathbb{R} \times \mathbb{R}$ with $y_i^* \le \gamma_i S$ on Y for all $i \in I$ such that

$$\varepsilon_i \ge \tilde f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) + \psi^*(\eta_i + \gamma_i) \quad \forall i \in I. \qquad (3.3)$$

and

$$(x_{1i}^*, x_{2i}^*, x_{3i}^*, \eta_i, \varepsilon_i) \to (0_{X^*}, 0, 0). \qquad (3.4)$$

It follows from (3.3) that $(\eta_i + \gamma_i)_{i\in I} \subset \operatorname{dom}\psi^*$. Hence, by (3.2), one has

$$\psi^*(\eta_i + \gamma_i) = 0 \text{ and } \eta_i + \gamma_i = 1 \text{ for all } i \in I. \qquad (3.5)$$

As $\eta_i \to 0$ we have $\gamma_i \to 1$. This and the fact that $y_i^* \le \gamma_i S$ on Y for all $i \in I$ imply that $\liminf_{i\in I} y_i^*(y) \le S(y)$ for all $y \in Y$. Hence, one gets $\liminf_{i\in I} (y_i^* \circ g)(x) \le (S \circ g)(x)$ for all $x \in C$ (note that this inequality still holds in the case where $x \notin \operatorname{dom} g$).

Moreover, by (3.5), (3.3) can be rewritten as

$$\varepsilon_i - f^*(x_{1i}^*) - (y_i^* \circ g)^*(x_{2i}^*) - i_C^*(x_{3i}^*) \ge \beta, \quad \forall i \in I \qquad (3.6)$$

(note that $x_{1i}^* \in \operatorname{dom} f^*$, $x_{2i}^* \in \operatorname{dom}(y_i^* \circ g)^*$ and $x_{3i}^* \in \operatorname{dom} i_C^*$ for all $i \in I$ as (3.3) holds). On the other hand, by the definition ofthe conjugate function, one has

$$\varepsilon_i + f(x) + (y_i^* \circ g)(x) - \langle x_{1i}^* + x_{2i}^* + x_{3i}^*, x\rangle$$
$$\ge \varepsilon_i - f^*(x_{1i}^*) - (y_i^* \circ g)^*(x_{2i}^*) - i_C^*(x_{3i}^*), \quad \forall x \in C,\ \forall i \in I.$$

Combing this inequality and (3.6), we get

$$\varepsilon_i + f(x) + (y_i^* \circ g)(x) - \langle x_{1i}^* + x_{2i}^* + x_{3i}^*, x\rangle$$
$$\ge \varepsilon_i - f^*(x_{1i}^*) - (y_i^* \circ g)^*(x_{2i}^*) - i_C^*(x_{3i}^*) \ge \beta, \quad \forall x \in C, \forall i \in I.$$

Taking liminf in the last inequalities and taking the fact that $\liminf_{i\in I} (y_i^* \circ g)(x) \le (S \circ g)(x)$ for all $x \in C$ into account, one gets

$$f(x) + (S \circ g)(x) \ge f(x) + \liminf_{i\in I} (y_i^* \circ g)(x)$$
$$\ge -\limsup_{i\in I} \left\{ f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) \right\}$$
$$\ge \beta = \inf_{x\in C} [f(x) + (S \circ g)(x)], \quad \forall x \in C,$$

and (ii) follows.

[(ii) $\Rightarrow$ (i)] The converse implication is trivial. The proof is complete.

As a consequence of Theorem 3.1, approximate Hahn-Banach theorem is derived, namely, Corollary 3.1. This result can be considered as a convex version of the approximate Hahn-Banach theorem for positive homogeneous functions established recently in [2].

Corollary 3.1 [Approximate Hahn-Banach theorem] Let X be an lcHtvs, $S : X \to \mathbb{R} \cup \{+\infty\}$ be an lsc extended sublinear function, F be a closed subspace of X, and $\phi : F \to \mathbb{R}$ be a continuous linear functional on F and such that $\phi \le S$ on F. Assume that $F \cap (\operatorname{dom} S) \ne \varnothing$. Then there exists a net $(z_i^*)_{i\in I} \subset X^*$ such that $z_i^* \le S$ on X for all $i \in I$ and $z_i^* \to \phi$ on F.

Proof. Let $Y := X$, $C := F$, $g : X \to X$ with $g(x) := x$ for all $x \in X$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ with

$$f(x) := \begin{cases} -\phi(x) & \text{if } x \in F, \\ +\infty & \text{else}. \end{cases}$$

Then g is S-convex and f is a proper lsc convex function. We first observe that the conditions (2.1) and (3.1) in Theorem 3.1 hold. Indeed, since $F \cap (\operatorname{dom} S) \ne \varnothing$, (3.1) holds. We now set $h : X \times X \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$h(x,y,\lambda) := S(x-y)-\lambda$ for all $(x,y,\lambda)\in X\times X\times\mathbb{R}$. Then h is lsc. So one has

$$\{(x,y,\lambda)\in X\times Y\times\mathbb{R} : S(g(x)-y)\le\lambda\} =$$
$$\{(x,y,\lambda)\in X\times X\times\mathbb{R} : S(x-y)-\lambda\le 0\} = h^{-1}((-\infty,0])$$

is closed in the product space $X\times X\times\mathbb{R}$, i.e., (2.1) holds. Since $F\cap(\operatorname{dom}S)\neq\varnothing$ and $\phi\le S$ on F, one has $\inf_{x\in F}[f(x)+(S\circ g)(x)] = \inf_{x\in F}[-\phi(x)+S(x)]\in\mathbb{R}$. By Theorem 3.1, there exists a net $(y_i^*,\gamma_i)_{i\in I}\subset X^*\times\mathbb{R}_+$ with $y_i^*\le\gamma_i S$ on X for all $i\in I$ such that $\gamma_i\to 1$ and

$$\inf_{x\in F}[-\phi(x)+\liminf_{i\in I} y_i^*(x)] = \inf_{x\in F}[-\phi(x)+S(x)]\ge 0$$

(as $\phi\le S$ on F) which gives rise to

$$\liminf_{i\in I} y_i^*(x)\ge\phi(x),\quad \forall x\in F. \tag{3.7}$$

On the other hand, since F is subspace of X, it follows that $\liminf_{i\in I} y_i^*(-x)\ge\phi(-x),\ \forall x\in F$ ($-x$ also belongs to F), which is equivalent to

$$\limsup_{i\in I} y_i^*(x)\le\phi(x),\quad \forall x\in F. \tag{3.8}$$

From (3.7) and (3.8), we get $y_i^*\to\phi$ on F. Since $(\gamma_i)_{i\in I}\subset\mathbb{R}_+$ and $\gamma_i\to 1$, we can assume that $\gamma_i>0$ for all $i\in I$. Thus, by setting $z_i^* := \frac{1}{\gamma_i}y_i^*$ for all $i\in I$, we obtain $z_i^*\le S$ on X for all $i\in I$ and $z_i^*\to\phi$ on F. The proof is complete.

Corollary 3.2 [Approximate sandwich theorem] Let X be an lcHtvs, $S: X\to\mathbb{R}\cup\{+\infty\}$ be an lsc sublinear function and $f: X\to\mathbb{R}\cup\{+\infty\}$ be a proper lsc convex function satisfying $-f\le S$ on X. Assume that $(\operatorname{dom} f)\cap(\operatorname{dom} S)\neq\varnothing$. Then there exists a net $(y_i^*)_{i\in I}\subset X^*$ such that

$$-f(x)\le\liminf_{i\in I} y_i^*(x)\le\limsup_{i\in I} y_i^*(x)\le S(x),\quad\forall x\in X.$$

Proof. The conclusion follows from Theorem 3.1 by taking $Y\equiv X$, $C:=X$ and $g(x):=x$ for all $x\in X$. Indeed, similar to the proof of Corollary 3.1, the set in (2.1) is closed in the product space $X\times X\times\mathbb{R}$ and (3.1) in Theorem 3.1 also holds as $(\operatorname{dom} f)\cap(\operatorname{dom} S)\neq\varnothing$. Moreover, observe that $(\operatorname{dom} f)\cap(\operatorname{dom} S)\neq\varnothing$ and $-f\le S$ on X entail $\inf_{x\in X}[f(x)+S(x)]\in\mathbb{R}$. Theorem 3.1 ensures the existence of a net $(y_i^*,\gamma_i)_{i\in I}\subset X^*\times\mathbb{R}_+$ with $y_i^*\le\gamma_i S$ on X for all $i\in I$ such that $\gamma_i\to 1$ and $\inf_{x\in X}[f(x)+\liminf_{i\in I} y_i^*(x)] = \inf_{x\in X}[f(x)+S(x)]\ge 0$ (as $-f\le S$ on X) which implies that

$$\limsup_{i\in I} y_i^*(x)\ge\liminf_{i\in I} y_i^*(x)\ge -f(x)\quad\text{for all } x\in X.$$

Since $y_i^*\le\gamma_i S$ on X for all $i\in I$ and $\gamma_i\to 1$, the conclusion of the corollary follows.

The equivalence of sequential Farkas lemmas and approximate Hahn-Banach-Lagrange theorem

It is well-known that the original (linear) Farkas lemma for convex systems is equivalent to the celebrated Hahn-Banach theorem [16]. For the generalizations of non-asymptotic Farkas lemma, i.e., the versions of Farkas-type results that hold under some qualification conditions, it was shown in [5] that these versions are equivalent to some extended versions of the Hahn-Banach theorem. In this section, we establish the counter part of [5] concerning versions of sequential Farkas lemmas and the so-called approximate Hahn-Banach-Lagrange theorem just obtained in Section 3. Concretely, we show that two versions of sequential Farkas lemma for cone-convex systems and for sublinear-convex systems in [10] and the approximate Hahn-Banach-Lagrange established in this paper are equivalent.

Claim: Theorem 2.1, Theorem 2.2 and Theorem 3.1 are equivalent.

Proof of the Claim:

- [Theorem 2.1 $\Rightarrow$ Theorem 2.2] This was proved in [10].
- [Theorem 2.2 $\Rightarrow$ Theorem 3.1] It was proved in Section 3.

• It is sufficient to prove the implication [Theorem 3.1 $\Rightarrow$ Theorem 2.1].

Let $X, Y, C, K, f,$ and g be as in Theorem 2.1. Let $S := i_{-K}$.

We firstly observe that S is an lsc sublinear function (as K is a closed convex cone), and g is S-convex as g is K-convex.

Secondly, since S is the indicator function of $-K$, we get

$\{(x, y, \lambda) \in X \times Y \times \mathbb{R} : S(g(x) - y) \le \lambda\} = \mathrm{epi}_K g \times [0, +\infty[.$
This set is closed in $X \times Y \times \mathbb{R}$ by the K-epi closedness of the mapping g, and hence the set in (2.1) is closed in $X \times Y \times \mathbb{R}$.

Thirdly, note that we also have

$C \cap (\mathrm{dom}\, f) \cap (\mathrm{dom}\,(S \circ g))$

$= (\mathrm{dom}\, f) \cap \{x \in C : g(x) \in -K\} \neq \emptyset$

(If $g(x) \in -K$, $(S \circ g)(x) = 0$; otherwise, i.e., if $g(x) \notin -K$, then $(S \circ g)(x) = i_{-K}(g(x)) = +\infty$). This means that (3.1) holds.

On the other hand, if (i) in Theorem 2.1 holds, i.e., $x \in C,\ g(x) \in -K \ \Rightarrow\ f(x) \ge 0$, then

$$f(x) + (S \circ g)(x) = f(x) + i_{-K}(g(x)) = f(x) \ge 0,$$
$$\forall x \in C \cap g^{-1}(-K). \quad (4.1)$$

Observe that the above inequality, (4.1), still holds for $x \in C$ that does not belong to $g^{-1}(-K)$. Consequently, one gets $f(x) + (S \circ g)(x) \ge 0, \forall x \in C,$

or equivalently, $\inf_{x \in C} [\, f(x) + (S \circ g)(x)] \ \ge\ 0.$

As $(\mathrm{dom}\, f) \cap C \cap \left(g^{-1}(-K)\right) \neq \emptyset$ we have $\inf_{x \in C} [\, f(x) + (S \circ g)(x)] < +\infty.$ So, $\inf_{x \in C} [\, f(x) + (S \circ g)(x)] \in \mathbb{R}$. Now, Theorem 3.1 yields the existence of nets $\left(y_i^*, \gamma_i\right)_{i \in I} \subset Y^* \times \mathbb{R}_+$ and $\left(x_{1i}^*, x_{2i}^*, x_{3i}^*\right)_{i \in I} \subset X^* \times X^* \times X^*$ with $y_i^* \le \gamma_i S$ on Y for all $i \in I$ such that $\gamma_i \to 1$, $x_{1i}^* + x_{2i}^* + x_{3i}^* \to 0_{X^*}$ and $-\limsup_{i \in I} \left\{ f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) \right\}$

$= \inf_{x \in C} [\, f(x) + (S \circ g)(x)] \ge 0$

which implies that

$$\limsup_{i \in I} \left\{ f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) \right\} \le 0.$$

(4.2)

By the definition of limit superior, for any $\varepsilon > 0$, there exists $i_0 \in I$ such that

$f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) \le \varepsilon, \quad \forall i \succeq i_0.$

Therefore, there exists $(\varepsilon_i)_{i \in I} \subset \mathbb{R}$ satisfying $\varepsilon_i \to 0_+$ such that

$f^*(x_{1i}^*) + (y_i^* \circ g)^*(x_{2i}^*) + i_C^*(x_{3i}^*) \le \varepsilon_i, \quad \forall i \in I.$

Note that $\left(y_i^*\right)_{i \in I} \subset K^+$. Indeed, as for any $i \in I$ we have $y_i^*(y) \le \gamma_i S(y) = \gamma_i\, i_{-K}(y) = 0, \quad \forall y \in -K$.

The implication [(i) $\Rightarrow$ (ii)] in Theorem 2.1 follows.

The proof of the implications [(ii) $\Rightarrow$ (iii)], [(iii) $\Rightarrow$ (iv)], and [(iv) $\Rightarrow$ (i)] are similar to that of Theorem 2.1.

Acknowledgement*: The authors would like to thank the anonymous referees for their valuable comments that significantly improved the quality of the paper.*

Định lý Hahn-Banach-Lagrange xấp xỉ

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TÓM TẮT

Trong bài báo này chúng tôi thiết lập một định lý Hahn-Banach-Lagrange mở rộng mà không có điều kiện chính quy, gọi là định lý xấp xỉ Hahn-Banach-Lagrange. Định lý này trong các trường hợp đặc biệt cho các phiên bản của các định lý Hahn-Banach xấp xỉ và định lý sandwich xấp xỉ. Các dạng định lý xấp xỉ này mở rộng các định lý dạng kinh điển theo *hai khía cạnh: Thứ nhất, các "bản gốc" được mở rộng ra cho hàm dưới tuyến tính (xuất hiện trong các định lý này) có thể nhận giá trị vô cùng; thứ hai, khác với các kết quả trong [5], đây là các phiên bản tôpô của các định lý này nhưng không đòi hỏi bất cứ điều kiện chính quy nào. Chúng tôi cũng chứng minh được rằng các định lý Farkas dạng tiệm cận được thiết lập trong [10] và định lý Hahn-Banach-Lagrange xấp xỉ do chúng tôi thiết lập ở đây là tương đương với nhau. Điều này cùng với các kết qủa trong [5, 10] cho một bức tranh toàn cảnh về sự tương đồng giữa định lý Hahn-Banach và bổ đề Farkas từ các phiên bản "gốc" đến các phiên bản mở rộng của chúng, dù là các mở rộng tiệm cận hay không tiệm cận.*

Từ khóa*: Bổ đề Farkas, Bổ đề Farkas theo dãy,* định lý Hahn-Banach-Lagrange, định lý Hahn-Banach, định lý *sandwich, giới hạn trên, giới hạn dưới.*

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