# ON THE EXISTENCE OF SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

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***ABSTRACT*** : *Using the topological degree of class* $(S)_+$ *introduced by F. E. Browder in* $[1]$ *and* $[2]$*, we extend some results of the papers* $[3]$ *and* $[4]$ *to the case of Banach spaces with locally bounded conditions.*

## 1. INTRODUCTION

Let $N$ be an integer $\geq 2$ and $D$ be a bounded open subset in $R^N$. In this paper we study the following equation:

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u) - \left[ \sum_{i=1}^{N} g_i(x,u) \frac{\partial u}{\partial x_i} + g_0(x,u) + a(x) \right] = 0 \qquad \forall x \in D, \qquad (1.1)$$

The $p-$Laplace equation $-\Delta_p u + f(x,u) = 0$ is a special case of $(1.1)$. If $p = 2$ and $a_i(x, \nabla u) = \dfrac{\partial u}{\partial x_i}$ then $(1.1)$ has the form:

$$-\Delta u + \left[ \sum_{i=1}^{N} g_i(x,u) \frac{\partial u}{\partial x_i} + g_0(x,u) + a(x) \right] = 0 \, . \qquad (1.2)$$

The problem $(1.2)$ has been solved in $[4]$ (Theorem 3.1, p.514) by using the topological degree for operators of class $(B)_+$. However, that method doesn't work when $p \neq 2$ and $a_i(x, \nabla u) = |\nabla u|^{p-2} \dfrac{\partial u}{\partial x_i}$. The one we use here can solve the problem $(1.2)$ for all $p > 1$.

Moreover, our result is also stronger than Theorem 11 in $[3]$ (p.357) where the authors prove the existence result for the Dirichlet problem:

$$\begin{cases} -\Delta_p u = f(x,u) \\ u|_{\partial D} = 0 \end{cases} \quad \text{in } D.$$

with the condition $(10)$ that the function $b$ is in $L^p(D)$ but not in $L^p_{loc}(D)$.

## 2. TOPOLOGICAL DEGREE OF CLASS $(S)_+$

In this section, we recall the class $(S)_+$ introduced by Browder (see $[1]$, $[2]$).

**Definition 2.1**. *Let* $D$ *be a bounded open set of a reflexive Banach space* $X$ *and* $f$ *be a mapping from* $\overline{D}$ *into the dual space* $X^*$ *of* $X$. *We say* $f$ *is of class* $(S)_+$ *if* $f$ *has the following properties:*

$(\mathrm{i})$ $\{f(x_n)\}_n$ *converges weakly to* $f(x)$ *if* $\{x_n\}_n$ *converges strongly to* $x$ *in* $\overline{D}$*, i.e.* $f$ *is a demicontinuous mapping on* $\overline{D}$.

$(\mathrm{ii})$ $\{x_n\}_n$ *converges strongly to* $x$ *if* $\{x_n\}_n$ *converges weakly to* $x$ *in* $\overline{D}$ *and*

$$\limsup_{n\to\infty}\langle f(x_n), x_n - x\rangle \le 0.$$

**Definition 2.1**. *Let* $\{g_t : 0 \le t \le 1\}$ *be a one-parameter family of maps of* $\overline{D}$ *into* $X^*$. *We say* $\{g_t : 0 \le t \le 1\}$ *is a homotopy of class* $(S)_+$, *if the sequences* $\{x_n\}_n$ *and* $\{g_{t_n}(x_n)\}_n$ *converge strongly to* $x$ *and* $g_t(x)$ *respectively for any sequence* $\{x_n\}_n$ *in* $\overline{D}$ *converging weakly to some* $x$ *in* $X$ *and for any sequence* $\{t_n\}_n$ *in* $[0,1]$ *converging to* $t$ *such that* $\limsup_{n\to\infty}\langle g_{t_n}(x_n), x_n - x\rangle \le 0$.

Let $f$ be a mapping of class $(S)_+$ on $\overline{D}$ and let $p$ be in $X^* \setminus f(\partial D)$. By Theorems 4 and 5 in $[2]$, the topological degree of $f$ on $D$ at $p$ is defined as a family of integers and is denoted by $\deg(f, D, p)$. In $[6]$ Skrypnik showed that this topological degree is single-valued (see also $[2]$). The following result was proved in $[2]$.

**Proposition 2.1**. *Let* $f$ *be a mapping of class* $(S)_+$ *from* $\overline{D}$ *into* $X^*$, *and let* $y$ *be in* $X^* \setminus f(\partial D)$. *Then we can define the degree* $\deg(f, D, y)$ *as an integer satisfying the following properties:*

$(a)$ *If* $\deg(f, D, y) \ne 0$ *then there exists* $x \in D$ *such that* $f(x) = y$.

$(b)$ *If* $\{g_t : 0 \le t \le 1\}$ *is a homotopy of class* $(S)_+$ *and* $\{y_t : 0 \le t \le 1\}$ *is a continuous curve in* $X^*$ *such that* $y_t \notin g_t(\partial D)$ *for all* $t \in [0,1]$, *then* $\deg(g_t, D, y_t)$ *is constant in* $t$ *on* $[0,1]$.

**Proposition 2.2**. *Let* $A : \overline{D} \to X^*$ *be a mapping of class* $(S)_+$. *Suppose that* $0 \in \overline{D} \setminus \partial D$ *and*

$$Au \ne 0, \quad \langle Au, u\rangle \ge 0 \quad \textit{for} \quad u \in \partial D.$$

*Then* $\deg(A, D, 0) = 1$.

**Proposition 2.3**. Let $A_t : \overline{D} \to X^*$, $t \in [0,1]$ be the homotopy family of operators of class $(S)_+$. Suppose that $A_t u \ne 0$ for $u \in \partial D$, $t \in [0,1]$. Then $\deg(A_0, D, 0) = \deg(A_1, D, 0)$.

## 3. NONLINEAR ELLIPTIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

Let $p$ be a real number $\ge 2$, $N$ be an integer $\ge 2$, $\Omega$ and $D$ be bounded open subsets in $R^N$. We denote by $W_0^{1,p}(D)$ the completion of $C_c^\infty(D,\mathbb{R})$ in the norm:

$$\|u\|_D = \left(\int_D |\nabla u|^p\, dx\right)^{1/p} \qquad \forall u \in C_c^\infty(D,\mathbb{R}).$$

Let $\Omega_k$ be an increasing sequence of open subsets of $\Omega$ such that $\overline{\Omega_k}$ is contained in $\Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty}\Omega_k$. Put $X = W_0^{1,p}(\Omega)$, $X_k = W_0^{1,p}(\Omega_k)$.

We denote by $p'$ and $p^*$ the conjugate exponent and the Sobolev conjugate exponent of $p$, i.e.,

$$p' = \left(1 - \frac{1}{p}\right)^{-1} \qquad \text{and} \qquad p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } N \geq p \\ \infty & \text{if } N \leq p \end{cases}.$$

Let $g_0, g_1, ..., g_N$ be real functions on $\Omega \times \mathbb{R}$ satisfying the following conditions:

$(C1)$ The function $g_i(x,t)$ is measurable in $x$ for fixed $t$ in $\mathbb{R}$ and continuous in $t$ for fixed $x$ in $\Omega$ for any $i = 0, ..., N$.

$(C2)$ $g_0(x,0) = 0 \quad \forall x \in \Omega$.

$(C3)$ $|g_i(x,t)| \leq b_i(x) + k_i |t|^{s_i} \quad \forall (x,t) \in \Omega \times \mathbb{R}$, $i = 0,...,N$ and

$(C4)$

$$-\alpha(x)|z||t|^q - \beta(x)|t|^r - c(x) \leq \left[\sum_{i=1}^{N} g_i(x,t) z_i + g_0(x,t) + a(x)\right] \quad \forall (x,t,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$$

where $s_0, ..., s_N$, $k_0, ..., k_N$, $r_0, ..., r_N$ and $r, q$ are non-negative real numbers and $b_0, ..., b_N$ and $c, \alpha, \beta$ are measurable functions such that $\alpha \in L^b(\Omega)$, $b \in \left(\dfrac{Np}{N(p-q-1)+pq}, \infty\right)$, $\beta \in L^d(\Omega)$, $d \in \left(\dfrac{Np}{N(p-r)+pr}, \infty\right)$, $c \in L^1(\Omega)$, $r \in (1,p)$, $q \in (1, p-1)$, $r_0 \in \left(\dfrac{Np}{N(p-1)+p}, \infty\right)$, $s_0^{-1} \in \left(\dfrac{N-p}{Np} r_0, \infty\right)$, $a \in L^{r_0}(\Omega)$, $r_i \in \left(\dfrac{Np}{N(p-2)+p}, \infty\right)$, $s_i^{-1} \in \left(\dfrac{N-p}{Np} r_i, \infty\right)$ and $b_i \in L^{r_i}_{loc}(\Omega)$ for any $i = 0, ..., N$.

We assume that the functions $a_i(x,s)$, $i = 1, ..., N$, $s = (s_1, ..., s_N) \in \mathbb{R}^N$ satisfy:

$(C5)$ $a_i(x,s)$ is defined and differentiable w.r.t all of its arguments for $x \in \overline{\Omega}$, $s = (s_1, ..., s_N) \in \mathbb{R}^N$. Moreover, $a_i(x,0) = 0$ for all $i = 1, ..., N$, $x \in \overline{\Omega}$.

$(C6)$ There exist positive constants $M_1, M_2$ such that the inequalities :

$$\sum_{i,j=1}^{N} \frac{\partial a_i(x,s)}{\partial s_j} \xi_i \xi_j \geq M_1 (1 + |s|)^{p-2} \sum_{i=1}^{N} \xi_i^2,$$

$$\left|\frac{\partial a_i(x,s)}{\partial s_j}\right| \leq d(x)(1+|s|)^{p-2} \quad \text{and} \quad \left|\frac{\partial a_i(x,s)}{\partial x_k}\right| \leq M_2 (1+|s|)^{p-1}$$

are satisfied, where $d \in L^\infty_{loc}(\Omega)$.

**Theorem 3.1**. *Under conditions* $(C1)-(C6)$*, there exists* $u$ *in* $X$ *such that for any* $v \in Y$*,*

$$\int_\Omega \sum_{i=1}^{N} a_i(x, \nabla u) \frac{\partial v}{\partial x_i} dx + \int_\Omega \left[\sum_{i=1}^{N} g_i(x,u) \frac{\partial u}{\partial x_i} + g_0(x,u) + a(x)\right] v dx = 0. \tag{3.1}$$

To prove the theorem we need the following lemma.

**Lemma 3.1**. *Let* $X_k = W_0^{1,p}(\Omega_k)$. *Under conditions* $(C1)-(C6)$ *there exists* $u_k$ *in* $X_k$ *such that for any* $v \in X_k$,

$$\int_{\Omega}\sum_{i=1}^{N} a_i(x,\nabla u_k)\frac{\partial v}{\partial x_i}dx + \int_{\Omega}\left[\sum_{i=1}^{N} g_i(x,u_k)\frac{\partial u_k}{\partial x_i} + g_0(x,u_k) + a(x)\right]vdx = 0.$$

**Proof**. Fix a $u$ in $X_k$. We will show that there exists a unique $T_k(u)$ in $X_k^*$ satisfying

$$\langle T_k(u), v\rangle = \int_{\Omega_k}\sum_{i=1}^{N} a_i(x,\nabla u)\frac{\partial v}{\partial x_i}dx + \int_{\Omega_k}\left[\sum_{i=1}^{N} g_i(x,u)\frac{\partial u_k}{\partial x_i} + g_0(x,u_k) + a(x)\right]vdx = 0 \tag{3.2}$$

for all $v \in X_k$.

Since $a_i(x,0) = 0$ for $x \in \overline{\Omega}$ and condition $(C6)$,

$$\left|\int_{\Omega_k}\sum_{i=1}^{N} a_i(x,\nabla u)\frac{\partial v}{\partial x_i}dx\right| = \left|\int_{\Omega_k}\sum_{i=1}^{N}\left[\int_0^1\sum_{j=1}^{N}\frac{\partial a_i(x,t\nabla u)}{\partial s_j}.\frac{\partial u}{\partial x_j}dt\right]\frac{\partial v}{\partial x_i}dx\right|$$

$$\le \int_{\Omega_k}\left[\int_0^1\sum_{j=1}^{N}\left|\frac{\partial a_i(x,t\nabla u)}{\partial s_j}\right|dt\right]|\nabla u||\nabla v|dx$$

$$\le N\|d\|_{\infty,\Omega_k}\int_{\Omega_k}\left[\int_0^1(1+|t\nabla u|)^{p-2}dt\right]|\nabla u||\nabla v|dx \le c\|v\|_{\Omega_k}, \tag{3.3}$$

where $c$ is a positive number depending on $k, N, u$ and $d$.

Put $G_{k,i}(u)(x) = g_i(x,u(x))$ $\forall x \in \Omega_k$, $i = 0, ..., N$. Then $G_{k,i}$ is a bounded, continuous mapping from $L^{r_i s_i}(\Omega_k)$ into $L^{r_i}(\Omega_k)$ by conditions $(C2),(C3)$ and by a result in [5], p.30. Moreover, by Sobolev embedding theorem there exists a positive $C$ such that:

$$\left|\int_{\Omega_k}\left[\sum_{i=1}^{N} g_i(x,u)\frac{\partial u_k}{\partial x_i} + g_0(x,u_k) + a(x)\right]vdx\right| \le$$

$$\le C\left[\sum_{i=1}^{N}\|G_{k,i}(u)\|_{r_i,k}\|u\|_{\Omega} + \|G_{k,0}(u)\|_{r_0,k} + \|a\|_{r_0}\right]\|v\|_{\Omega_k} \qquad \forall v \in X_k.$$

From this and $(3.3)$ we get $(3.2)$. Next, we show that $T_k$ is of class $(S)_+$. First, we check that $T_k$ is demicontinuous in $X_k$. Let $\{w_n\}_n$ be a sequence converging strongly to $w$ in $X_k$. Then for every $v$ in in $X_k$ we have:

$$|\langle T_k(w_n) - T_k(w), v\rangle| = \int_{\Omega_k}\sum_{i=1}^{N}\left(a_i(x,\nabla w_n) - a_i(x,\nabla w)\right)\frac{\partial v}{\partial x_i}dx +$$

$$+\int_{\Omega_k}\left[\sum_{i=1}^{N}\left(g_i(x,w_n)\frac{\partial w_n}{\partial x_i} - g_i(x,w)\frac{\partial w}{\partial x_i}\right) + \left(g_0(x,w_n) - g_0(x,w)\right) + a(x)\right]vdx. \tag{3.4}$$

On the other hand:

$$\int_{\Omega_k}\sum_{i=1}^{N}\left(a_i(x,\nabla w_n) - a_i(x,\nabla w)\right)\frac{\partial v}{\partial x_i}dx =$$

$$= \int_{\Omega_k} \sum_{i=1}^{N} \left[ \int_0^1 \sum_{j=1}^{N} \frac{\partial a_i\left(x, \nabla w_n + t\nabla\left(w_n - w\right)\right)}{\partial s_j} . \frac{\partial\left(w_n - w\right)}{\partial x_j} dt \right] \frac{\partial v}{\partial x_i} dx$$

$$\leq N \|d\|_{\infty,\Omega_k} \int_{\Omega_k} \left[ \int_0^1 \left(1 + \left|\nabla w_n + t\nabla\left(w_n - w\right)\right|\right)^{p-2} dt \right] \left|\nabla\left(w_n - w\right)\right| \left|\nabla v\right| dx \leq M_3 \|w_n - w\|_{\Omega_k} . \quad (3.5)$$

where $M_3$ is a positive number depending on $k, N, v$ and $d$.

And:

$$\int_{\Omega_k} \left[ \sum_{i=1}^{N} \left( g_i\left(x, w_n\right) \frac{\partial w_n}{\partial x_i} - g_i\left(x, w\right) \frac{\partial w}{\partial x_i} \right) + \left( g_0\left(x, w_n\right) - g_0\left(x, w\right) \right) + a\left(x\right) \right] v dx =$$

$$= \int_{\Omega_k} \left[ \sum_{i=1}^{N} \left( G_{k,i}\left(w_n\right) \frac{\partial w_n}{\partial x_i} - G_{k,i}\left(w\right) \frac{\partial w}{\partial x_i} \right) + \left( G_{k,0}\left(w_n\right) - G_{k,0}\left(w\right) \right) \right] v dx$$

$$\leq M_4 \left[ \sum_{i=1}^{N} \left\|G_{k,i}\left(w_n\right) - G_{k,i}\left(w\right)\right\|_{r_i,k} \|w_n\|_{\Omega} + \left\|G_{k,i}\left(w\right)\right\|_{r_i,k} \|w_n - w\|_{\Omega} \right] \|v\|_{\Omega} +$$

$$+ M_4 \left\|G_{k,0}\left(w_n\right) - G_{k,0}\left(w\right)\right\|_{r_0,k} \|v\|_{\Omega} \bullet \quad (3.6)$$

Since $G_{k,i}$ is a bounded, continuous mapping from $L^{r_i s_i}\left(\Omega_k\right)$ into $L^{r_i}\left(\Omega_k\right)$ and $\left\{w_n\right\}_n$ converges strongly to $w$ in $X_k$, from $(3.4)$ and $(3.6)$, we have $T_k$ is demicontinuous in $X_k$.

Now let $\left\{u_m\right\}_m$ be a sequence converging weakly to $u$ in $X_k$ and

$$\limsup_{m\to\infty} \left\langle T_k\left(u_m\right), u_m - u\right\rangle \leq 0 \text{ or}$$

$$\limsup_{m\to\infty} \int_{\Omega_k} \sum_{i=1}^{n} a_i\left(x, \nabla u_m\right) \frac{\partial\left(u_m - u\right)}{\partial x_i} dx +$$

$$+ \int_{\Omega_k} \left[ \sum_{i=1}^{N} g_i\left(x, u_m\right) \frac{\partial u_m}{\partial x_i} + g_0\left(x, u_m\right) + a\left(x\right) \right] \left(u_m - u\right) dx \leq 0 \bullet \quad (3.7)$$

Since $r_i^{-1} s_i^{-1} > \dfrac{N-p}{pN}$ for all $i = 0, ..., N$, the theorem of Rellich-Konkrachov gives us that the sequence $\left\{G_{k,i}\left(u_m\right)\right\}_m$ converges to $G_{k,i}\left(u\right)$ in $L^{r_i}\left(\Omega_k\right)$. Thus, $\left\{G_{k,0}\left(u_m\right)\right\}_m$ converges to $G_{k,0}\left(u\right)$ in $L^{r_i}\left(\Omega_k\right)$. This implies : $\int_{\Omega_k} \left[g_0\left(x, u_m\right) + a\left(x\right)\right]\left(u_m - u\right) dx \to 0.$

On the other hand, since $\left\{u_m\right\}_m$ converges to $u$ in $L^p$, $\partial u_m$ converges weakly to $\partial u$ and $\left\{G_{k,i}\left(u_m\right)\right\}_m$ converges to $G_{k,i}\left(u\right)$ in $L^{r_i}\left(\Omega_k\right)$, we get $\int_{\Omega_k} \sum_{i=1}^{N} g\left(x, u_m\right) \frac{\partial u_m}{\partial x_i} \left(u_m - u\right) dx \to 0.$

Hence

$$\lim_{m\to\infty} \int_{\Omega_k} \left[ \sum_{i=1}^{N} g_i\left(x, u_m\right) \frac{\partial u_m}{\partial x_i} + g_0\left(x, u_m\right) + a\left(x\right) \right] \left(u_m - u\right) dx = 0. \quad (3.8)$$

So, it follows from $(3.7)$ and $(3.8)$ that $\limsup_{m\to\infty}\int_{\Omega_k}\sum_{i=1}^{n} a_i(x,\nabla u_m)\frac{\partial(u_m-u)}{\partial x_i}dx \le 0$ or

$$\limsup_{m\to\infty}\int_{\Omega_k}\sum_{i=1}^{n}\left[a_i(x,\nabla u_m)-a_i(x,\nabla u)\right]\frac{\partial(u_m-u)}{\partial x_i}dx \le 0 \,. \tag{3.9}$$

By condition $(C6)$

$$\int_{\Omega_k}\sum_{i=1}^{N}\left(a_i(x,\nabla v)-a_i(x,\nabla u)\right)\frac{\partial(v-u)}{\partial x_i}dx$$

$$=\int_{\Omega_k}\sum_{i=1}^{N}\left[\int_0^1\sum_{j=1}^{N}\frac{\partial a_i\left(x,\nabla u+t\nabla(v-u)\right)}{\partial s_j}.\frac{\partial(v-u)}{\partial x_j}dt\right]\frac{\partial(v-u)}{\partial x_i}dx$$

$$\ge M_1\int_{\Omega_k}\left[\int_0^1\left(1+\left|\nabla u+t\nabla(v-u)\right|\right)^{p-2}dt\right]\left|\nabla(v-u)\right|^2 \ge M_5\left|\nabla(v-u)\right|^p. \tag{3.10}$$

Combining $(3.9)$ and $(3.10)$, we have the conclusion that the sequence $\{u_m\}_m$ converges to $u$ in $X_k$. Thus, $T_k$ is of class $(S)_+$ in $X_k$. Next we calculate the topological degree of the operator $T_k$.

By condition (C4), the Holder enequality and $(3.10)$, we have:

$$\left\langle T_k(u),u\right\rangle_\Omega = \int_{\Omega_k}\sum_{i=1}^{n}a_i(x,\nabla u)\frac{\partial(u)}{\partial x_i}+\int_{\Omega_k}\left[\sum_{i=1}^{N}g_i(x,u)\frac{\partial u}{\partial x_i}+g_0(x,u)+a(x)\right]u(x)dx$$

$$\ge M_5\|u\|_\Omega^p-\|\alpha\|_b\|\nabla u\|_p\|u\|_{qb_1}^q-\|\beta\|_b\|u\|_{d_1r}^r-\|c\|_1.$$

where $b_1,d_1$ are positive numbers such that $b^{-1}+p^{-1}+b_1^{-1}=1$, $d^{-1}+d_1^{-1}=1$. From conditions of $b,d$ we have: $1<qb_1<p^*$, $1<rd_1<p^*$. By Poincare inequalities, the Sobolev embedding theorem there exists $C>0$ such that:

$$\left\langle T_k(u),u\right\rangle_\Omega \ge M_5\|u\|_\Omega^p-C\|\alpha\|_b\|u\|_{qb_1}^{q+1}-C\|\beta\|_d\|u\|_\Omega^r-\|c\|_1.$$

Since $r,q+1\in(1,p)$, we can choose $s>0$ such that :

$$C\|\alpha\|_b s^{q+1-p}-C\|\beta\|_d s^{r-p}-\|c\|_1 s^{-p}<\frac{M_5}{2}.$$

Let $G=\{w\in X:\|w\|_\Omega<s\}$ and $G_k=G\cap X_k$. Then $G_k$ is an open bounded set in $X_k$ and $\left\langle T_k(u),u\right\rangle_\Omega \ge \frac{M_5}{2}s^p, \forall u\in\partial_k G_k$ .

Since $T_k$ satisfies condition $(S)_+$ on $X_k$, by Proposition 2.2 we conclude that

$$\deg\left(T_k,\overline{G_k}^{X_k},0\right)=1.$$

Then there exists $u_k\in\partial_{X_k}G_k$ such that $T_k(u_k)=0$, i.e.

$$\int_\Omega\sum_{i=1}^{N}a_i(x,\nabla u_k)\frac{\partial v}{\partial x_i}dx+\int_\Omega\left[\sum_{i=1}^{N}g_i(x,u_k)\frac{\partial u_k}{\partial x_i}+g_0(x,u_k)+a(x)\right]vdx=0, \forall v\in X_k$$

which completes the proof of the lemma .

**Proof of Theorem 3.1**. By Lemma 3.1, there exists a sequence $\{u_k\}_k \subset \partial_{X_k} G_k$ such that:

$$T_k(u_k) = 0 \bullet \tag{3.11}$$

Since $\{u_k\}_k \subset \partial_X G$, it is bounded in $X$. Let $u$ be the weak limit of $\{u_k\}_k$ in $W_0^{1,p}(\Omega)$.

By $(3.11)$ we have

$$\langle T_k(u_k), v\rangle = 0, \quad \forall v \in X_k \bullet \tag{3.12}$$

Fix $l \in \mathbb{N}^+$. We consider the function $\rho_l \in C_c^\infty(\Omega)$ which satisfies $0 \le \rho_l \le 1$ and

$$\rho_l(x) = \begin{cases} 1 & \text{if } x \in \Omega_{l-1} \\ 0 & \text{if } x \notin \Omega_l \end{cases}.$$

For all $k \ge 1$ we have $\rho_l u_k - \rho_l u \in X_k$. Then, $(3.12)$ implies

$$\langle T_k(u_k), \rho_l u_k - \rho_l u\rangle = 0 \bullet \tag{3.13}$$

This yields $\lim\limits_{k\to\infty} \langle T_k(u_k), \rho_l u_k - \rho_l u\rangle = 0$, that is

$$\lim_{k\to\infty} \int_{\Omega_l} \sum_{i=1}^N a_i(x, \nabla u_k) \frac{\partial(\rho_l u_k - \rho_l u)}{\partial x_i} dx +$$

$$+ \int_{\Omega_l} \left[\sum_{i=1}^N g_i(x, u_k) \frac{\partial u_k}{\partial x_i} + g_0(x, u_k) + a(x)\right](\rho_l u_k - \rho_l u) dx = 0. \tag{3.14}$$

Since $\{\rho_l u_k\}_k$ converges weakly to $\rho_l u$ in $X$, arguing as in the Lemma 3.1 (the proof of $T_k$ satisfying condition $(S)_+$, we have

$$\lim_{k\to\infty} \int_{\Omega_l} \left[\sum_{i=1}^N g_i(x, u_k) \frac{\partial u_k}{\partial x_i} + g_0(x, u_k) + a(x)\right](\rho_l u_k - \rho_l u) dx = 0. \tag{3.15}$$

Therefore, $(3.14)$ and $(3.15)$ imply $\lim\limits_{k\to\infty} \int_{\Omega_l} \sum_{i=1}^N a_i(x, \nabla u_k) \frac{\partial(\rho_l u_k - \rho_l u)}{\partial x_i} dx = 0$, or

$$\lim_{k\to\infty} \int_{\Omega_l} \sum_{i=1}^N a_i(x, \nabla u_k) \left[(u_k - u)\frac{\partial \rho_l}{\partial x_i} + \rho_l \frac{\partial(u_k - u)}{\partial x_i}\right] dx = 0. \tag{3.16}$$

Since $\{u_k\}_k$ converges to $u$ in $L^p(\Omega)$, it is easily seen that

$$\lim_{k\to\infty} \int_{\Omega_l} \sum_{i=1}^N a_i(x, \nabla u_k)(u_k - u) \frac{\partial \rho_l}{\partial x_i} dx = 0.$$

Combining this and $(3.16)$ we obtain $\lim\limits_{k\to\infty} \int_{\Omega_l} \rho_l \sum_{i=1}^N a_i(x, \nabla u_k) \frac{\partial(u_k - u)}{\partial x_i} dx = 0$, or

$$\lim_{k\to\infty}\int_{\Omega_l}\rho_l\sum_{i=1}^{N}\left[a_i\left(x,\nabla u_k\right)-a_i\left(x,\nabla u\right)\right]\frac{\partial\left(u_k-u\right)}{\partial x_i}dx=0. \qquad (3.17)$$

On the other hand, by $(3.10)$: $\sum_{i=1}^{N}\left[a_i\left(x,\nabla u_k\right)-a_i\left(x,\nabla u\right)\right]\frac{\partial\left(u_k-u\right)}{\partial x_i}\geq M_8\left|\nabla\left(u_k-u\right)\right|^p$.

Hence $\lim_{k\to\infty}\int_{\Omega_l}\rho_l\left|\nabla\left(u_k-u\right)\right|^p dx=0$.

This means that $\{u_k\}_k$ strongly converges to $u$ on $\Omega_l$ for all $l\in\mathbb{N}^+$. Now fix $v\in Y$. Our goal is to show that

$$\int_{\Omega}\sum_{i=1}^{N}a_i\left(x,\nabla u\right)\frac{\partial v}{\partial x_i}dx+\int_{\Omega}\left[\sum_{i=1}^{N}g_i\left(x,u\right)\frac{\partial u_k}{\partial x_i}+g_0\left(x,u\right)+a\left(x\right)\right]vdx=0. \qquad (3.18)$$

Indeed, since $v\in Y$, there exists a positive integer $m$ such that $\operatorname{supp}(v)\subset\Omega_m$. Then $v\in X_k$ for all $k\geq m$. By Lemma 3.1:

$$\int_{\Omega_m}\sum_{i=1}^{N}a_i\left(x,\nabla u_k\right)\frac{\partial v}{\partial x_i}dx+\int_{\Omega_m}\left[\sum_{i=1}^{N}g_i\left(x,u_k\right)\frac{\partial u_k}{\partial x_i}+g_0\left(x,u_k\right)+a\left(x\right)\right]vdx=0.$$

Since $\{u_k\}_k$ strongly converges to $u$ on $\Omega_m$, it follows from the above equality that $(3.18)$ holds. We now **N** complete the proof of the theorem.

## VỀ SỰ TỒN TẠI NGHIỆM CỦA PHƯƠNG TRÌNH ELLIPTIC PHI TUYẾN VỚI CÁC HỆ SỐ KHÔNG BỊ CHẶN

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***TÓM TẮT***: *Sử dụng bậc tôpô của lớp* $(S)_+$ *được giới thiệu bởi F. E. Browder trong các bài báo* [1] *và* [2], *chúng tôi mở rộng một số kết quả của các bài báo* [3] *và* [4] *sang trường hợp không gian Banach với các điều kiện bị chặn địa phương.*

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