EXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH SINGULAR CONDITIONS

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ABSTRACT: *In this paper, we study the existence of generalized solution for a class of singular elliptic equation:* $-\operatorname{diva}\left(x,u(x),\nabla u(x)\right)+f\left(x,u(x),\nabla u(x)\right)=0$.

Using the Galerkin approximation in [2, 10] and test functions introduced by Drabek, Kufner, Nicolosi in [5], we extend some results about elliptic equations in [2, 3, 4, 6, 10].

1.INTRODUCTION

The aim of this paper is to prove the existence of generalized solutions in $W_0^{1,p}(\Omega)$ for the quasilinear elliptic equations:

$$-\operatorname{diva}\left(x,u(x),\nabla u(x)\right)+f\left(x,u(x),\nabla u(x)\right)=0 \qquad (1.1)$$

i.e. proving the existence of $u\in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega a\left(x,u(x),\nabla u(x)\right)\nabla\varphi dx+\int_\Omega f\left(x,u(x),\nabla u(x)\right)\varphi dx=0,\forall\varphi\in C_c^\infty(\Omega)$$

where Ω is a bounded domain in $\mathbb{R}^N, N\geq 2$ with smooth boundary, $p\in(1,N)$ and $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N, f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ satisfy the following conditions:

Each $a_i(x,\eta,\xi)$ is a Caratheodory function, that is, measurable in x for any fixed $\zeta=(\eta,\xi)\in\mathbb{R}^{N+1}$ and continuous in ζ for almost all fixed $x\in\Omega$,

$$\left|a_i(x,\eta,\xi)\right|\leq c_1(x)\left[|\eta|^\alpha+|\xi|^{p-1}+k_1(x)\right],\forall i=\overline{1,N} \qquad (1.2)$$

$$\left[a(x,\eta,\xi)-a(x,\eta,\xi^*)\right]\left[\xi-\xi^*\right]>0 \qquad (1.3)$$

$$a(x,\eta,\xi)\xi\geq\lambda|\xi|^p \qquad (1.4)$$

a.e. $x\in\Omega,\forall\eta\in\mathbb{R},\forall\xi,\xi^*\in\mathbb{R}^N,\xi\neq\xi^*$.

where $c_1\in L^\infty_{loc}(\Omega), c_1\geq 0, k_1\in L^{p'}(\Omega),\alpha\in[0,p-1],\lambda>0$.

and $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Caratheodory function satisfying

$$\left|f(x,\eta,\xi)\right|\leq c_2(x)\left[|\eta|^\beta+|\xi|^\gamma+k_2(x)\right] \qquad (1.5)$$

$$f(x,\eta,\xi)\eta\geq -c_3(x)-b|\eta|^q-d|\xi|^r \qquad (1.6)$$

where c_2 is a positive function in $L^\infty_{loc}(\Omega), c_3$ is a positive function in $L^\infty(\Omega)$, $k_2\in L^{p'}(\Omega)$ and $r,q\in[0,p)$, b, d are positive constants, $\gamma\in[0,p-1],\beta\in[0,p^*-1)$ with $p^*=\dfrac{Np}{N-p}$.

Because $c_1, c_2 \in L^\infty_{loc}(\Omega)$ we cannot define operator on the whole space $W_0^{1,p}(\Omega)$. Therefore, we cannot use the property of (S_+) operator as usual. To overcome this difficulty, in every Ω_n we find solution $u_n \in W_0^{1,p}(\Omega_n)$ of the equation:

$$-\operatorname{div} a(x, u(x), \nabla u(x)) + f(x, u(x), \nabla u(x)) = 0$$

where $\{\Omega_n\}$ is an increasing sequence of open subsets of Ω with smooth boundaries such that $\overline{\Omega_n}$ is contained in Ω_{n+1} and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. In this case, we only have the strong convergence of $\{u_n\}$ to u in $W^{1,p}_{loc}(\Omega)$ by using the same technique of Drabek, Kufner, Nicolosi (in [5], section 2.4). However, it is enough to get the generalized solution.

An example for our conditions:

$$a_i(x, \eta, \xi) = \frac{1}{d^\theta(x)}\left[|\xi_i|^{p-1} + A_1(\eta) + k_1(x)\right] \operatorname{sgn} \xi_i$$

$$f(x, \eta, \xi) = \frac{1}{d^\mu(x)}\left[|\xi|^a + |\eta|^b + k_2(x)\right] \operatorname{sgn} \eta$$

where $d(x) = \operatorname{dist}(x, \partial\Omega); \theta, \mu > 0; A_1, k_1, k_2$ are positive functions

$$k_1, k_2 \in L^{p'}(\Omega); A_1(\eta) \le |\eta|^\alpha; \alpha, a \in [0, p-1]; b \in [0, p^* - 1).$$

The problem is singular because $\dfrac{1}{d^\theta(x)}, \dfrac{1}{d^\mu(x)} \in L^\infty_{loc}(\Omega)$.

Remark:

1) If $c_2 \in L^\infty(\Omega)$ and $\beta, \gamma \in [0, p-1)$ the condition (1.5) implies the condition (1.6).

2) The pseudo-Laplacian $a(x, \eta, \xi) = \left(|\xi_1|^{p-2}\xi_1, ..., |\xi_N|^{p-2}\xi_N\right)$, the p-Laplacian $a(x, \eta, \xi) = \left(|\xi|^{p-2}\xi_1, ..., |\xi|^{p-2}\xi_N\right)$ are some special cases that satisfy our conditions. So our results generalized the corresponding Dirichlet problems in [3, 4]. Our paper also extends the recent result about singular elliptic equations for case p=2 in [6].

2. PREREQUISITES

2.1.Lemma 2.1

(See e.g. [10], Proposition 1.1, page 3) Let G be a measurable set of positive measure in $\mathbb{R}^n$ *and* $h : G \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ *satisfy the following conditions:*

a) *h is a Caratheodory function.*

b) $\left|h(x, u_1, ..., u_m)\right| \le c \sum_{i=1}^{m} |u_i|^{p_i/p'} + g(x), \forall x \in G$

where c is a positive constant, $p_i \in (1, \infty), \forall i = 1, ..., m$, $g \in L^{p'}(G)$.

Then the Nemytskii operator defined by the equality

$H(u_1, ..., u_m)(x) = h(x, u_1(x), ..., u_m(x))$ *acts continuously from* $L^{p_1}(G) \times ... \times L^{p_m}(G)$ *to* $L^{p'}(G)$. *Moreover, it is bounded, i.e. it transforms any set which is* *bounded into another bounded set. (Proof of this fact for the simple case can be found in [8], theorem 2.2, page 26).*

2.2.Lemma 2.2

(See e.g. [10], lemma 4.1, page 14) Let $F:\overline{U}\to\mathbb{R}^m$ *be a continuous mapping of the closure of a bounded domain* $U\subset\mathbb{R}^m$. *Suppose that the origin is an interior point of D and that the condition*

$$\left(F(x),x\right)=\sum_{i=1}^{m}F_i(x)x_i\geq 0,\forall x\in\partial U \qquad (1.7)$$

Then the equation F(x) =0 has at least one solution in $\overline{U}$.

We recall some results about Schauder bases.

Definition: A sequence $\{x_i\}$ in a Banach space X is a Schauder basis if every $x\in X$ can be written uniquely $x=\sum_{i=1}^{\infty}c_ix_i=\lim_{n\to\infty}\sum_{i=1}^{n}c_ix_i$, where $\{c_i\}\subset\mathbb{R}$.

Because every $x\in X$ is written uniquely $x=\sum_{i=1}^{\infty}c_ix_i$ we have $x_i\neq 0$ and c_i is a function from X to $\mathbb{R}$, for all i in R.

2.3.Lemma 2.3: *([9], Theorem 3.1, page 20) For all i in* $\mathbb{N}$, c_i *is a continuous linear function on X, i.e.* $\forall i\in\mathbb{N},\exists M_i>0,\left|c_i(x)\right|\leq M_i\|x\|_X,\forall x\in X$

2.4.Lemma 2.4: *([7], Corollary 3) Let D be a bounded domain in* $\mathbb{R}^N$ *with smooth boundary. Then the space* $W_0^{1,p}(D)$ *has a Schauder basis.*

2.5.Lemma 2.5: *Let D be an open set in* Ω, $\overline{D}\subset\Omega$. *If*

$$u_n\xrightarrow{weak}u \quad \text{in } W^{1,p}(D) \qquad (1.8)$$

$$\text{and } \lim_{n\to\infty}\int_D\left[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\right]\left[\nabla u_n-\nabla u\right]dx=0 \qquad (1.9)$$

Then there exists a subsequence of $\{u_n\}$ *still denoted by* $\{u_n\}$ *such that* $\nabla u_n\to\nabla u$ *in* $L^p(D)$.

Proof: Since $c_1,c_2\in L^\infty_{loc}(\Omega)$ we have $c_1,c_2\in L^\infty(D)$ and the conditions (1.2), (1.5) become:

$$\left|a_i(x,\eta,\xi)\right|\leq C_1\left[|\eta|^\alpha+|\xi|^{p-1}+k_1(x)\right],\forall i=\overline{1,N}$$

$$\left|f(x,\eta,\xi)\right|\leq C_2\left[|\eta|^\beta+|\xi|^\gamma+k_2(x)\right]$$

Using the well-known result in [2], Lemma 3, we obtain our Lemma.

□

Let us recall the definition of class (S+): A mapping $T:X\to X^*$ is called belongs to the class (S+) if for any sequence u_n in X with $u_n\xrightarrow{weak}u$ and $\limsup_{n\to\infty}\langle Tu_n,u_n-u\rangle\leq 0$ it follows that $u_n\to u$.

2.6.Lemma 2.6: *(see [2, 10]) Let D be an open set in* Ω, $\overline{D}\subset\Omega$ *and A be a mapping from* $W_0^{1,p}(D)$ *to* $\left[W_0^{1,p}(D)\right]^*$, *such that* $\langle Au,v\rangle=\int_D\sum_{i=1}^{N}a_i(x,u,\nabla u)\frac{\partial v}{\partial x_i}dx+\int_D f(x,u,\nabla u)vdx$

Then A is a (S_+) operator.

3 . MAIN RESULTS

Let $\{\Omega_n\}$ be an increasing sequence of open subsets of Ω with smooth boundaries such that $\overline{\Omega_n}$ is contained in Ω_{n+1} and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$.

First, in every Ω_n we find solution $u_n \in W_0^{1,p}(\Omega_n)$ of the equation:

$$-\operatorname{div} a(x,u(x),\nabla u(x)) + f(x,u(x),\nabla u(x)) = 0 \tag{3.1}$$

Applying the same technique as in [10], Theorem 4.1, page 14, we can show that (3.1) has a bounded solution in $W_0^{1,p}(\Omega_n)$.

3.1.Lemma 3.1:

For each Ω_n, the equation: $-\operatorname{div} a(x,u(x),\nabla u(x)) + f(x,u(x),\nabla u(x)) = 0$ (3.2)

has a solution $u_n \in W_0^{1,p}(\Omega_n)$. *Furthermore, there exists a positive constant R independent of n satisfying that* $\|u_n\|_{W_0^{1,p}(\Omega_n)} \le R, \forall n \in \mathbb{N}$.

Proof: Fix $n \in \mathbb{N}$. Let $D = \Omega_n, X = W_0^{1,p}(D)$ and A be a mapping from $W_0^{1,p}(D)$ to $\left[W_0^{1,p}(D)\right]^*$, such that

$$\langle Au, v\rangle = \int_D \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial v}{\partial x_i}dx + \int_D f(x,u,\nabla u)v dx, \forall u, v \in W_0^{1,p}(D)$$

By Lemma 2.6, *A* belongs to class (*S*+).

We will prove that *A* is a demicontinuous operator, i.e. if $u_m \to u$ in $W_0^{1,p}(D)$, then

$$\langle Au_m, v\rangle \to \langle Au, v\rangle, \forall v \in W_0^{1,p}(D)$$

By $u_m \to u$ in $W_0^{1,p}(D)$ and (1.2), (1.5), applying Lemma 2.1, we get

$$a_i(.,u_m,\nabla u_m) \to a_i(.,u,\nabla u), \forall i = 1,..,N \text{ In } L^{p'}(D) \text{ as } m \to \infty$$

and $$f(.,u_m,\nabla u_m) \to f(.,u,\nabla u) \text{ in } L^{p'}(D) \text{ as } m \to \infty$$

Hence $\langle Au_m, v\rangle = \int_D \sum_{i=1}^N a_i(x,u_m,\nabla u_m)\frac{\partial v}{\partial x_i}dx + \int_D f(x,u_m,\nabla u_m)v dx \to$

$$\int_D \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial v}{\partial x_i}dx + \int_D f(x,u,\nabla u)v dx = \langle Au, v\rangle, \forall v \in W_0^{1,p}(D)$$

Therefore, *A* is demicontinuous.

Besides, by applying the boundedness of Nemytskii operator for $a(.,u,\nabla u)$ and $f(.,u,\nabla u)$ one deduces that *A* is bounded.

For any arbitrary *u* in $W_0^{1,p}(D)$, due to (1.4), (1.6), we have

$$\begin{aligned}\langle Au, u\rangle &= \int_D a(x,u,\nabla u)\nabla u dx + \int_D f(x,u,\nabla u)u dx \\ &\ge \lambda \int_D |\nabla u(x)|^p dx - \int_D \left[c_3(x) + b.|u(x)|^q + d.|\nabla u(x)|^r\right]dx \\ &\ge \lambda \|u\|_X^p - \|c_3\|_{L^\infty(D)} - b\|u\|_{L^q(D)}^q - d\int_D |\nabla u(x)|^r dx\end{aligned}$$

Let $\widehat{u}(x) = u(x), \forall x \in D$ and $\widehat{u}(x) = 0, \forall x \in \Omega \setminus D$, we have

$$\left\langle Au,u\right\rangle \ge \lambda\left\|u\right\|_X^p - \left\|c_3\right\|_{L^\infty(D)} - b\left\|\hat{u}\right\|_{L^q(\Omega)}^q - d\left(\int_D \left|\nabla u(x)\right|^p dx\right)^{r/p}\left(\int_D dx\right)^{1-r/p}$$

Since $q < p < p^*$, the continuous imbedding $W_0^{1,p}(\Omega) \to L^q(\Omega)$ implies that

$$\begin{aligned}\left\langle Au,u\right\rangle &\ge \lambda\left\|u\right\|_X^p - \left\|c_3\right\|_{L^\infty(\Omega)} - b\left(M.\left\|\hat{u}\right\|_{W_0^{1,p}(\Omega)}\right)^q - d.K\left\|\left|\nabla u\right|\right\|_{L^p(D)}^r \\ &\ge \lambda\left\|u\right\|_X^p - \left\|c_3\right\|_{L^\infty(\Omega)} - bM^q.\left\|u\right\|_{W_0^{1,p}(D)}^q - d.K\left\|u\right\|_{W_0^{1,p}(D)}^r \\ &\ge \left\|u\right\|_X^p\left(\lambda - \frac{\left\|c_3\right\|_{L^\infty(\Omega)}}{\left\|u\right\|_X^p} - \frac{bM^q}{\left\|u\right\|_X^{p-q}} - \frac{d.K}{\left\|u\right\|_X^{p-r}}\right)\end{aligned}$$

Since 1, r, q<p, one can choose a positive constant R independent of n such that

$$\left\langle Au,u\right\rangle \ge 0, \forall u \in \partial B_X(0,R) \tag{3.3}$$

Applying Lemma 2.4 there exists a Schauder basis $\{v_i\}$ in the space X. We consider in $\mathbb{R}^m$ the domain $U_m = \left\{c = (c_1,...,c_m) : \left\|\sum_{i=1}^m c_i v_i\right\|_X < R\right\}$

Applying Lemma 2.3, there exists

$$M_i > 0, \left|c_i\right| \le M_i\left\|\sum_{j=1}^m c_j v_j\right\|_x < M_i R, \forall i = \overline{1,m}, \forall (c_1,...,c_m) \in U_m$$

So U_m is bounded in $\mathbb{R}^m$. We apply Lemma 2.2 to this domain U_m and to the mapping

$$F : \overline{U}_m \to \mathbb{R}^m,\quad F(c) = \left(F_1(c),...,F_m(c)\right),\quad F_i(c) = \left\langle A\left(\sum_{j=1}^m c_j v_j\right), v_i\right\rangle$$

Let $c = (c_1,...,c_m) \in \partial U_m$ and $u = \sum_{j=1}^m c_j v_j$ then $\left\|u\right\|_X = R$. We have

$$\left(F(c),c\right) = \sum_{j=1}^m F_j(c)c_j = \left\langle A\left(\sum_{j=1}^m c_j v_j\right), \sum_{j=1}^m c_j v_j\right\rangle = \left\langle Au,u\right\rangle \ge 0$$

because of (3.3). By Lemma 2.2, the equation F(c) =0 has at least one solution in $\overline{U}_m$, for example c= (c_1,…, c_m). Hence $F_i(c) = \left\langle A\left(\sum_{j=1}^m c_j v_j\right), v_i\right\rangle = 0, \forall i = \overline{1,m}$

Consequently, $u_m = \sum_{j=1}^m c_j v_j$ satisfies the inequality

$$\left\|u_m\right\|_X \le R \tag{3.4}$$

And is a solution of the system

$$\left\langle Au_m, v_i\right\rangle = 0, \forall i = \overline{1,m} \tag{3.5}$$

Let m go through $\mathbb{N}$ we have a sequence $\{u_m\}$ satisfying (3.4) and is a solution of (3.5). By virtue of the reflexivity of the space X, the sequence u_m contains weakly convergent subsequence u_{m_k}. So $u_{m_k} \xrightarrow{weak} u_0$. Since u_0 is in X with the Schauder basis $\{v_i\}$, we have $u_0 = \sum_{j=1}^{\infty} \alpha_j v_j = \lim_{m\to\infty} \sum_{j=1}^{m} \alpha_j v_j$. Let $w_m = \sum_{j=1}^{m} \alpha_j v_j$ then $w_m \to u_0$ so $w_{m_k} \to u_0$. We have

$$\left\langle Au_{m_k}, u_{m_k} - u_0 \right\rangle = \left\langle Au_{m_k}, u_{m_k} - w_{m_k} \right\rangle + \left\langle Au_{m_k}, w_{m_k} - u_0 \right\rangle \tag{3.6}$$

Moreover,

$$\lim_{k\to\infty} \left\langle Au_{m_k}, w_{m_k} - u_0 \right\rangle = 0 \tag{3.7}$$

because of (3.4), the boundedness of the operator *A*, and the strong convergence of w_{m_k} to u_0. Since $u_{m_k} - w_{m_k} = \sum_{j=1}^{m_k} \beta_j v_j$ and (3.5), we get $\left\langle Au_{m_k}, u_{m_k} - w_{m_k} \right\rangle = 0, \forall k$.

Hence

$$\lim_{k\to\infty} \left\langle Au_{m_k}, u_{m_k} - u_0 \right\rangle = 0 \tag{3.8}$$

Because *A* belongs to class (*S*+) and (3.8), we deduce that $u_{m_k} \to u_0$. Since *A* is demicontinuous, passing to limit the equality (3.5) for a fixed i, we have

$$\left\langle Au_0, v_i \right\rangle = 0 \tag{3.9}$$

Let $v \in X$, then $v = \sum_{j=1}^{\infty} \alpha_j v_j = \lim_{m\to\infty} \sum_{j=1}^{m} \alpha_j v_j$. Since i is an arbitrary index, it follows from (3.9) that $\left\langle Au_0, \sum_{i=1}^{m} \alpha_i v_i \right\rangle = 0, \forall m \in \mathbb{N}$. Let m tend to infinity, we get $\left\langle Au_0, v \right\rangle = 0$

Hence, u_0 is a solution of the equation (3.2). Moreover, since $u_{m_k} \xrightarrow{weak} u_0$, we get $\|u_0\|_{W_0^{1,p}(\Omega_n)} = \|u_0\|_X \le \liminf_{k\to\infty} \|u_{m_k}\|_X \le R$, where R does not depend on n. This completes the proof of Lemma 3.1

$\square$

By Lemma 3.1, we have proved that (3.2) has a bounded solution $u_n \in W_0^{1,p}(\Omega_n)$ satisfying $\|u_n\|_{W_0^{1,p}(\Omega_n)} \le R, \forall n \in \mathbb{N}$. Next, we expand u_n to all $\Omega : u_n(x) = 0, \forall x \in \Omega \setminus \Omega_n$. So $u_n \in W_0^{1,p}(\Omega)$ and $\|u_n\|_{W_0^{1,p}(\Omega)} = \|u_n\|_{W_0^{1,p}(\Omega_n)} \le R, \forall n \in \mathbb{N}$. By virtue of the reflexivity of the space $W_0^{1,p}(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \xrightarrow{weak} u$ in $W_0^{1,p}(\Omega)$ for some subsequence. We will prove that u is a generalized solution of the equation (1.1) in $W_0^{1,p}(\Omega)$, i.e. $\int_\Omega a(x,u(x),\nabla u(x))\nabla\varphi dx + \int_\Omega f(x,u(x),\nabla u(x))\varphi dx = 0, \forall \varphi \in C_c^\infty(\Omega)$

In order to do that, we need the following lemma:

3.2.Lemma 3.2.

Let m in $\mathbb{N}$, we have

$$\lim_{n\to\infty}\int_{\Omega_m}\left[a\left(x,u_n,\nabla u_n\right)-a\left(x,u_n,\nabla u\right)\right]\left[\nabla u_n-\nabla u\right]dx=0$$

Proof: We only need to consider n>m+1. Let ϕ_m be a functions in $C_c^\infty(\Omega)$, with $0\le\phi_m\le 1$ in Ω and $\phi_m(x)=\begin{cases}1 & \text{if } x\in\Omega_m\\ 0 & \text{if } x\in\Omega\backslash\Omega_{m+1}\end{cases}$. Then there exists M such that,

$$\left|\phi_m(x)\right|\le M,\ \left|\nabla\phi_m(x)\right|\le M,\ \forall x\in\Omega \tag{3.10}$$

Put $w_n=\phi_m.(u_n-u)$ restricted on Ω_n.

Because $\operatorname{supp}\left[\phi_m.(u_n-u)\right]\subset\Omega_{m+2}\subset\Omega_n,\ \forall n>m+1$, we have $\operatorname{supp}w_n\subset\Omega_n,\ \forall n>m+1$. So $w_n\in W_0^{1,p}(\Omega_n)$. Since u_n is the solution of the equation (3.2), we have

$$\int_{\Omega_n}a\left(x,u_n,\nabla u_n\right)\nabla w_n dx+\int_{\Omega_n}f\left(x,u_n,\nabla u_n\right)w_n dx=0.$$

Hence
$$\int_{\Omega_{m+1}}a\left(x,u_n,\nabla u_n\right)\nabla w_n dx+\int_{\Omega_{m+1}}f\left(x,u_n,\nabla u_n\right)w_n dx=0 \tag{3.11}$$

We shall prove that
$$\lim_{n\to\infty}\int_{\Omega_{m+1}}f\left(x,u_n,\nabla u_n\right)\phi_m\left(u_n-u\right)dx=0 \tag{3.12}$$

by finding a number s such that $f\left(.,u_n,\nabla u_n\right)$ is bounded in $L^{s'}\left(\Omega_{m+1}\right)$ and $u_n\to u$ in $L^s\left(\Omega_{m+1}\right)$. Since $\beta\in[0,p^*-1)$ and p<p*, we can find s satisfying $\beta+1<s<p^*$ and p<s. Hence $\beta<s-1=\frac{s}{s'}$ and $\gamma<p-1<p-\frac{p}{s}=\frac{p}{s'}$. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, the Sobolev imbedding implies that there exists a subsequence still denoted by $\{u_n\}$ such that $u_n\to u$ in $L^s(\Omega)$, so $u_n\to u$ in $L^s\left(\Omega_{m+1}\right)$. From (1.5) and Lemma 2.1, one deduces that $f\left(.,u_n,\nabla u_n\right)$ is bounded in $L^{s'}\left(\Omega_{m+1}\right)$. Combining (3.10), we have (3.12). Hence

$$\lim_{n\to\infty}\int_{\Omega_{m+1}}f\left(x,u_n,\nabla u_n\right)w_n dx=0 \tag{3.13}$$

From (3.11), (3.13), one deduces that $\lim_{n\to\infty}\int_{\Omega_{m+1}}a\left(x,u_n,\nabla u_n\right)\nabla w_n dx=0$ or

$$\lim_{n\to\infty}\int_{\Omega_{m+1}}a\left(x,u_n,\nabla u_n\right)\nabla\left(\phi_m.\left(u_n-u\right)\right)dx=0$$

Hence,

$$\lim_{n\to\infty}\int_{\Omega_{m+1}}a\left(x,u_n,\nabla u_n\right)\left[\phi_m.\nabla\left(u_n-u\right)+\left(u_n-u\right).\nabla\phi_m\right]dx=0 \tag{3.14}$$

Besides, since *p* < *s*, we get
$$u_n\to u \text{ in } L^p\left(\Omega_{m+1}\right) \tag{3.15}$$

Applying Lemma 2.1, we have $a\left(.,u_n,\nabla u_n\right)$ is bounded in $\left[L^{p'}\left(\Omega_{m+1}\right)\right]^N$. Combining with (3.10), (3.15), we obtain
$$\lim_{n\to\infty}\int_{\Omega_{m+1}}a\left(x,u_n,\nabla u_n\right)\left(u_n-u\right).\nabla\phi_m dx=0 \tag{3.16}$$

From (3.14), (3.16) we have
$$\lim_{n\to\infty}\int_{\Omega_{m+1}}a\left(x,u_n,\nabla u_n\right)\phi_m.\nabla\left(u_n-u\right)dx=0 \tag{3.17}$$

On the other hand, (1.2), Lemma 2.1, (3.15) imply

$$a(.,u_n,\nabla u) \to a(.,u,\nabla u) \text{ in } \left[L^{p'}(\Omega_{m+1})\right]^N$$

Combining with (3.10) and the boundedness of *un* in $W_0^{1,p}(\Omega_{m+1})$, one deduces that

$$\lim_{n\to\infty} \int_{\Omega_{m+1}} \left[a(x,u_n,\nabla u) - a(x,u,\nabla u)\right]\phi_m.\nabla(u_n - u)dx = 0 \tag{3.18}$$

and due to the weak convergence of un to u in $W_0^{1,p}(\Omega_{m+1})$ also

$$\lim_{n\to\infty} \int_{\Omega_{m+1}} a(x,u,\nabla u)\phi_m.\nabla(u_n - u)dx = 0 \tag{3.19}$$

It follows from (3.18), (3.19) that

$$\lim_{n\to\infty} \int_{\Omega_{m+1}} a(x,u_n,\nabla u)\phi_m.\nabla(u_n - u)dx = 0 \tag{3.20}$$

Hence (3.17) together with (3.20) yield

$$\lim_{n\to\infty} \int_{\Omega_{m+1}} \left[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\right]\phi_m.\nabla(u_n - u)dx = 0$$

Since $\phi_m.\left[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\right](\nabla u_n - \nabla u) \geq 0$, for all *x* in Ω_{m+1}, we get

$$\lim_{n\to\infty} \int_{\Omega_m} \left[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\right]\phi_m.\nabla(u_n - u)dx = 0$$

The fact that $\phi_m(x) = 1, \forall x \in \Omega_m$ then implies

$$\lim_{n\to\infty} \int_{\Omega_m} \left[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\right]\nabla(u_n - u)dx = 0 \ \square$$

Fix $\varphi \in C_c^\infty(\Omega)$, there exists *m* in $\mathbb{N}$ such that $\operatorname{supp}\varphi \subset \Omega_m$. Applying Lemma 2.5, Lemma 3.2, we have $\nabla u_n \to \nabla u$ in $L^p(\Omega_m)$ for some subsequence. Since $u_n \to u$ in $L^p(\Omega_m)$ also, together with Lemma 2.1, we obtain

$$\int_{\Omega_m} a(x,u_n,\nabla u_n)\nabla\varphi dx \to \int_{\Omega_m} a(x,u,\nabla u)\nabla\varphi dx$$

$$\int_{\Omega_m} f(x,u_n,\nabla u_n)\varphi dx \to \int_{\Omega_m} f(x,u,\nabla u)\varphi dx$$

So

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega f(x,u,\nabla u)\varphi dx = \int_{\Omega_m} a(x,u,\nabla u)\nabla\varphi dx + \int_{\Omega_m} f(x,u,\nabla u)\varphi dx = 0$$

Therefore, we get the main theorem:

Theorem 3.1. Under the conditions (1.2)-(1.6), equation (1.1) has at least a generalized solution u in $W_0^{1,p}(\Omega)$, that is, for any $\varphi \in C_c^\infty(\Omega)$

$$\int_{\Omega} a(x,u,\nabla u)\nabla\varphi dx + \int_{\Omega} f(x,u,\nabla u)\varphi dx = 0$$

SỰ TỒN TẠI NGHIỆM CỦA PHƯƠNG TRÌNH ELLIPTIC QUASILINEAR VỚI ĐIỀU KIỆN KÌ DỊ

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TÓM TẮT: *Trong bài báo này, chúng tôi khảo sát sự tồn tại nghiệm suy rộng của một lớp phương trình elliptic kì dị:*

$$-\mathrm{div}a(x,u(x),\nabla u(x)) + f(x,u(x),\nabla u(x)) = 0$$

Sử dụng phương pháp xấp xỉ Galerkin trong [2,10] và hàm thử được Drabek, Kufner, Nicolosi nêu trong [5], chúng tôi mở rộng một số kết quả về phương trình elliptic trong [2,3,4,6,10].

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