Extended finite element method for simulating the mechanical behavior of multiple random holes and inclusions in functionally graded material

Kim Bang Tran, The Huy Tran, Quoc Tinh Bui and Tich Thien Truong

Abstract— **Analysis of mechanical behavior of a structure containing defects such as holes and inclusions is essential in many engineering applications. In many structures, the discontinuities may have a significant influence on the reduction of the structural stiffness. In this work, we consider the effect of multiple random holes and inclusions in functionally graded material (FGM) plate and apply the extended finite element method with enrichment functions to simulate the mechanical behavior of those discontinuous interfaces. The inclusions also have FGM properties. Numerical examples are considered and their obtained results are compared with the COMSOL, the finite element method software.**

Index Terms— **XFEM, hole, inclusion, multiple.**

1 INTRODUCTION

In many engineering applications, a structure may contain defects such as holes and inclusions. Therefore, analysis of mechanical behavior of a structure containing defects plays a vital role in ensuring reliability when operating and avoiding the resonance phenomenon that can cause large displacements, may lead to structural failure. Due to the complexity of physical and mechanical characteristics of those structure, analytical and experimental methods are difficult in obtaining the solution. The analytical methods are limited to models with simple boundary conditions and configuration, while the experimental method is often time-consuming, costly and only be done for a certain number of models. Moreover, the accuracy of result depends on samples, machine tools, experience, and so on. Therefore, some efficient numerical methods have been developed to simulate complex structures.

When modeling the interface problems by means of the finite element methods (FEM), the defects face must be coincided with the edge of the elements and the FEM has encountered many difficulties. To overcome these difficulties, the extended finite element method (XFEM) was developed to solve those problems. Some advantages of the XFEM over FEM may be pointed out as follows: conformal mesh when modeling discontinuity such as hole and inclusion is not required, discontinuities are represented by enrichment functions, any special element is no longer required.

Functionally graded materials (FGM) is a composite material whose mechanical properties change with a mathematical function. This function can contain a variable that is the coordinate of a point on an object. Because the material properties change throughout the body, FGM is of great interest in various technical fields. The main advantage of FGM is that there is no boundary between two different materials and therefore will not lead to stress in the body, despite the fact that the material properties change drastically.

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Most of the problem of discontinuous interfaces such as holes and inclusion is investigated with homogeneous material [1-5], or the FGM structure contains only one type of discontinuity [7-8]. In this paper, we present the extended finite element method for simulating the mechanical behavior of multiple random holes and inclusions in a finite FGM plate with varying elastic properties in the transverse direction. The inclusions also have FGM properties. Poisson's ratio is held constant and Young's modulus is considered to vary across the radius and x-axis.

2 EXTENED FINITE ELEMENT METHOD FOR DISCONTINUITY PROBLEMS

2.1 Level set method for holes and inclusions detection

In XFEM, the level set method is used to detect discontinuous boundaries. According to [2], a boundary of a hole or an inclusion can be considered a discontinuous boundary. The level set value on the boundary of a hole or an inclusion will be zero. The physical description of the elements containing the boundary of a hole or an inclusion will be described by the enrichment function.

To compute the normal level set function ϕ, consider Γ is the geometry of the hole or inclusion. At any point x, we define the scattering point x_Γ on the boundary so that the distance $\| x - x_\Gamma \|$ is the smallest. The level set function ϕ can be expressed as follows

$$\phi(x) = \pm \min_{x_\Gamma \in \Gamma} \|x - x_\Gamma\| \tag{1}$$

The appearance of a hole or an inclusion with a particular boundary Γ can be detected by the level set value ϕ. In the whole body, $\phi < 0$ at any point located inside the domain bounded by Γ and $\phi > 0$ at any point located outside the domain.

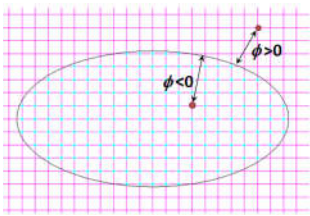


Figure 1 Signed distance function ϕ

2.2 Enrichment functions for discontinuities

To describe the physical properties of a material discontinuity element such as an inclusion, we will use the absolute enrichment function. According to [2], this function can be defined as the absolute value of the signed distance function as follow

$$\chi(x) = |\phi(x)| \tag{2}$$

$\phi(x)$ can be calculated by interpolating the nodal signed distances within an element

$$\chi(x) = \left| \sum_I N_I(x) \phi_I(x) \right| \tag{3}$$

Where N_i is shape function at node i

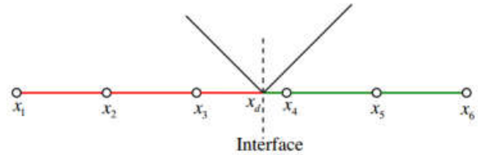


Figure 2. Jump function

To describe the physical properties of a geometric discontinuity element such as hole, we use Heaviside function. According to [2], a node that lies outside the hole will have $H(x) = 1$ and a node that lies inside the hole will have $H(x) = 0$.

$$H(x) = \begin{cases} 1 & \phi(x) > 0 \\ 0 & \phi(x) < 0 \end{cases} \tag{4}$$

2.3 XFEM for discontinuity problems

According to [2], the displacement field of a two-dimensional element with discontinuity will be of the following form

$$u(\mathbf{x}) = \sum_{j=1}^{n} N_j(\mathbf{x}) \left[u_j + \underbrace{g(\mathbf{x}) a_j}_{j \in n_r} \right] \tag{5}$$

N is the total number of nodes and n is the number of nodes under the element; u is the transposing element at the nodes of the element as in the finite element method, a is the degree of freedom added at the enriched nodes and $g(x)$ is the enrichment function.

With the displacement field approximated to be enriched, the linear system of equations has the following form

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{f}\} \tag{6}$$

With the stiffness matrix K of the enriched elements will be calculated according to the formula

$$K_{ij}^{e} = \begin{bmatrix} K_{ij}^{uu} & K_{ij}^{ua} \\ K_{ij}^{au} & K_{ij}^{aa} \end{bmatrix} \tag{7}$$

$$K_{ij}^{rs} = \int_{\Omega^e} \left(B_i^r\right)^T \mathbf{D} B_j^s d\Omega \tag{8}$$

$$r, s = u, a$$

$$B_i^u = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \tag{9}$$

$$B_i^b = \begin{bmatrix} \left(N_i g(x)\right)_{,x} & 0 \\ 0 & \left(N_i g(x)\right)_{,y} \\ \left(N_i g(x)\right)_{,y} & \left(N_i g(x)\right)_{,x} \end{bmatrix} \tag{10}$$

If an element contains boundary of a hole, $g(x) = H(x)$ and $g(x) = \chi(x)$ if boundary of an inclusion passes through an element.

As the FGM has the properties changing throughout the body, it is important to implement the following critical steps in the calculation process for XFEM. Divide the elements into multiple triangular child domains to locate the points that take the Gaussian integral. Repeat on each Gauss point:

- Calculate the deformation matrix B at the Gauss point under consideration
- Calculate the material matrix D at the point Gauss is considering
- Calculate the element's stiffness matrix
- Assemble the element stiffness matrix into the global stiffness matrix
- Finish the iteration on Gaussian points.

3 NUMERICAL SIMULATIONS

3.1 Problem 1

Considers a rectangular isotropic FGM plate with material variation in the Cartesian x-direction, the dimensions and are depicted in Fig. 3. The external load is q = 1 N/m^2. The lower edge of the plate is clamped and plane strain state is assumed. The plate contains an four circular holes and two circular inclusions. All holes and inclusions have equal radius R = 0.085 m and different positions as depicted in table 1. The Poisson's ratio of the matrix is assumed to be constant ν = 0.3 and the elastic modulus E of the matrix varies exponentially from the left to the right edge as follow

$$E(x) = E_0 e^{\beta x} \text{ where } E_0 = 10^3 \text{ Pa and } \beta = 2$$

The Poisson's ratio of the inclusion is assumed to be constant ν = 0.35 and the elastic modulus E of the matrix varies exponentially from the left to the right edge as follow

$$E(x) = E_0 e^{\beta x} \text{ where } E_0 = 2.10^3 \text{ Pa and } \beta = 2$$

TABLE 1. POSITIONS OF HOLE AND INCLUSION

Number	Type	X-position (m)	Y-position (m)
1	hole	0.2	0.785
2	inclusion	0.4	0.4
3	hole	0.7	0.3
4	inclusion	0.2	1.215
5	hole	0.5	1.55
6	hole	0.8	1.215

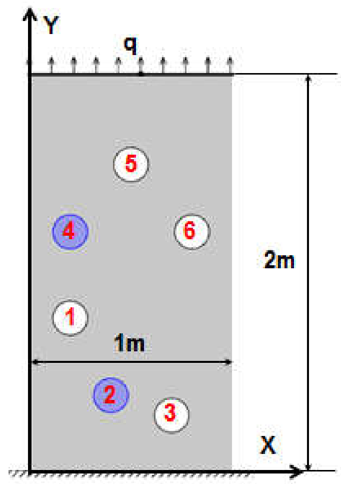
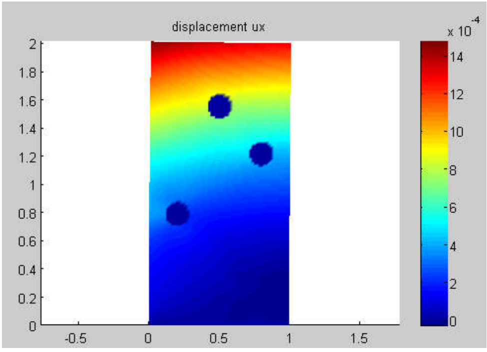
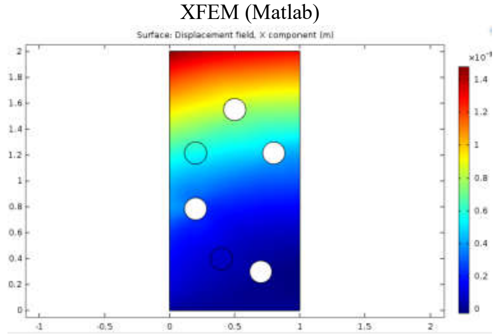


Figure 3. FGM plate with material variation in the x-direction with four circular holes and two circular inclusions

In order to assess the accuracy and ef ciency of the XFEM for modeling arbitrary discontinuities, we compare the finite element method (FEM) solution to that obtained by XFEM. The mesh size for the nite element solution was chosen so that further re ning the mesh does not produce signi cant change in the solution, thus the FEM solution could be taken as the reference solution, with which we compare the accuracy of the XFEM solution

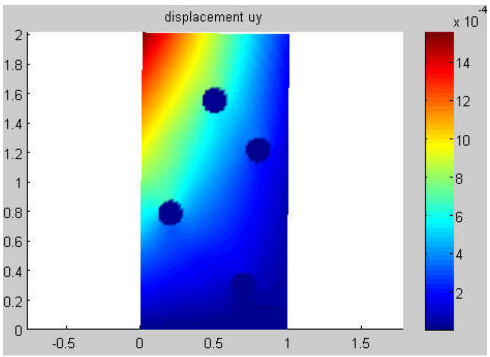


XFEM (Matlab)

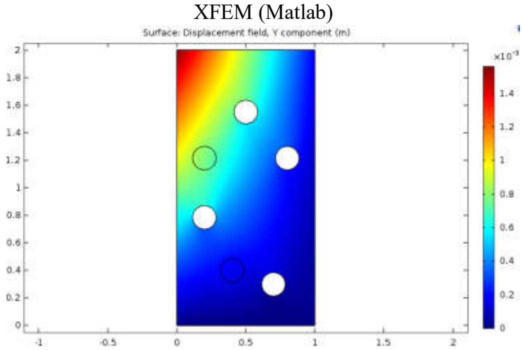


FEM (Comsol)

Figure 4. Comparison of displacement results u_x between XFEM and FEM

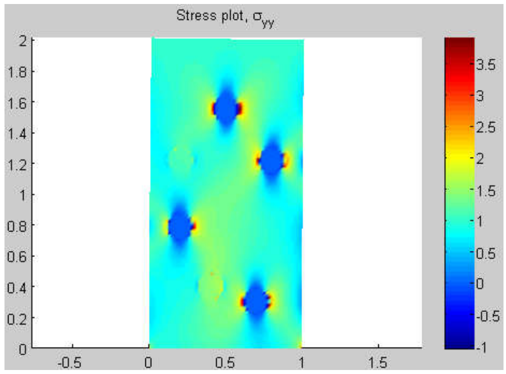


XFEM (Matlab)

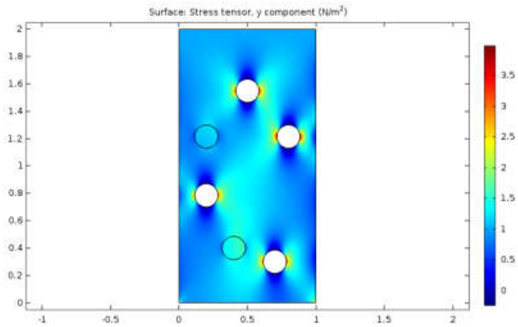


FEM (Comsol)

Figure 5. Comparison of displacement results u_y between XFEM and FEM



XFEM (Matlab)



FEM (Comsol)

Figure 6. Comparison of stress results σ_{yy} between XFEM and FEM

TABLE 2. COMPARISON OF RESULTS BETWEEN XFEM AND FEM

Displacement (m)	XFEM	FEM	%ERROR
Max(u_y)	0.001560	0.001563	0.19%
Min(u_y)	0	0	0
Max(u_x)	0.001478	0.001480	0.14%
Min(u_x)	$-2.58.\ 10^{-5}$	$-2.61.\ 10^{-5}$	1.14%

The computed results obtained by the XFEM and the FEM are listed in Table 2 including the percentage errors. As expected, the minimum and the maximum displacement obtained by the XFEM matches well with those derived from the FEM. The stress and displacement field of the plate are sketched in subsequent Figs. 4-6.

3.2 Problem 2

In the next problem, the dimensions, boundary condition, loading are taken from the previous case. The plate now contains an three circular holes and four circular inclusions. All holes and inclusions have different radius and different positions as depicted in table 3. The Poisson's ratio of the matrix is assumed to be constant $\nu = 0.3$ and the elastic modulus E of the matrix varies exponentially from the left to the right edge as follow

$E(x) = E_0 e^{\beta x}$ where $E_0 = 10^3$ Pa and $\beta = 2$

The Poisson's ratio of the inclusion is assumed to be constant ν = 0.4 and the elastic modulus E of the matrix varies exponentially from the left to the right edge as follow

$E(x) = E_0 e^{\beta x}$ where $E_0 = 2.10^3$ Pa and $\beta = 2$

TABLE 3. POSITIONS OF HOLE AND INCLUSION

Number	Type	X-position (m)	Y-position (m)	Radius (m)
1	hole	0.25	0.5	0.07
2	inclusion	0.5	0.25	0.085
3	inclusion	0.5	1	0.2
4	inclusion	0.75	0.5	0.12
5	hole	0.25	1.4	0.12
6	hole	0.5	1.8	0.08
7	inclusion	0.75	1.5	0.13

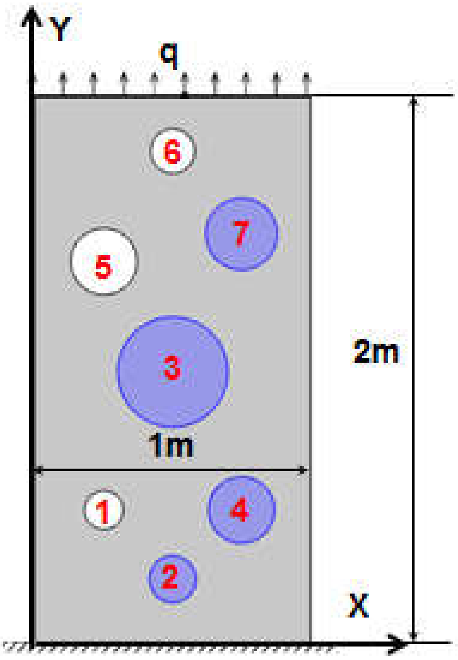
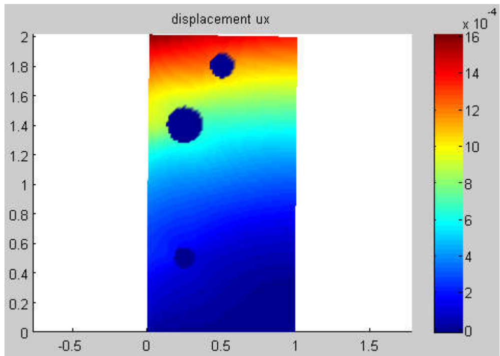
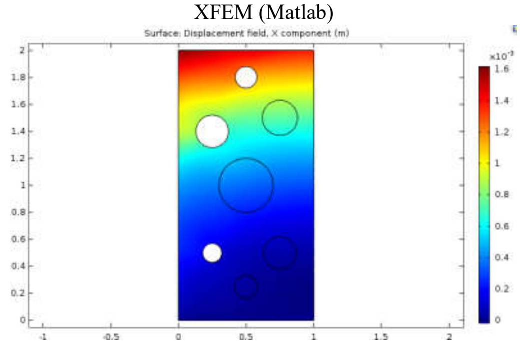


Figure 7. FGM plate with material variation in the x-direction with three circular holes and four circular inclusions

We compare the finite element method (FEM) solution to that obtained by XFEM.

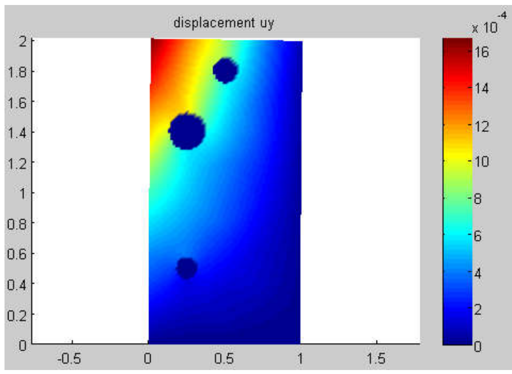


XFEM (Matlab)

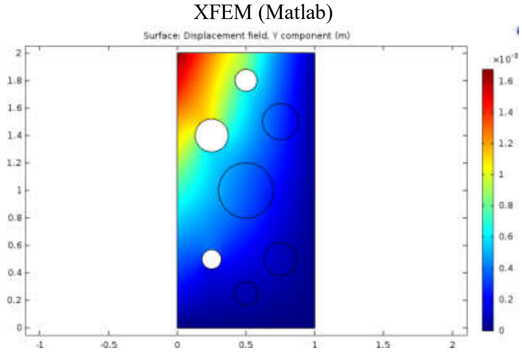


FEM (Comsol)

Figure 8. Comparison of displacement results u_x between XFEM and FEM

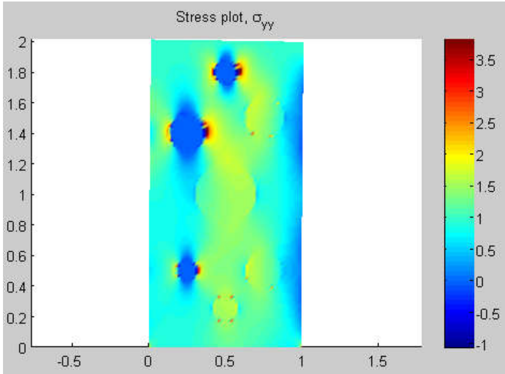


XFEM (Matlab)

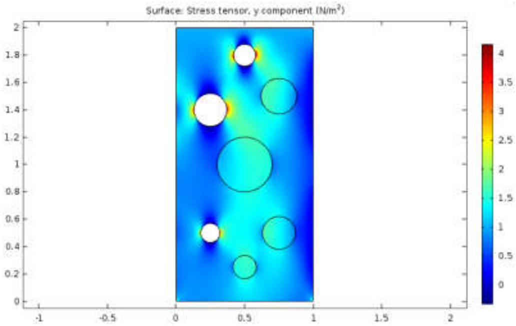


FEM (Comsol)

Figure 9. Comparison of displacement results u_y between XFEM and FEM



XFEM (Matlab)



FEM (Comsol)

Figure 10. Comparison of stress results σ_{yy} between XFEM and FEM

TABLE 4. COMPARISON OF RESULTS BETWEEN XFEM AND FEM

Displacement (m)	XFEM	FEM	%ERROR
Max(u_y)	0.001672	0.001676	0.23%
Min(u_y)	0	0	0%
Max(u_x)	0.001615	0.001619	0.24%
Min(u_x)	$-1.636.10^{-5}$	$-1.638.10^{-5}$	0.2%

The computed results obtained by the XFEM and the FEM are listed in Table 4 including the percentage errors. The minimum and the maximum displacement obtained by the XFEM matches well with those derived from the FEM. The stress and displacement field of the plate are sketched in subsequent Figs. 8-10.

4 CONCLUSION

A extended finite element method (XFEM) has been proposed for discontinuity problem analysis under the effect of multiple random holes and inclusions in functionally graded material. This method is convenient in treating those defects without meshing the internal boundaries because of the enrichment function. Several numerical examples are considered when many holes and inclusions with different sizes appear in the body. The obtained solutions show a good agreement of between the extended finite element method and the traditional finite element method. The presented approach has shown several advantages and it is promising to be extended to more complicated problems such as modeling the body containing different discontinuity boundaries.

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Mô phỏng ứng xử cơ học của tấm vật liệu phân lớp chức năng có chứa nhiều khuyết tật và tạp chất ngẫu nhiên bằng phương pháp phần tử hữu hạn mở rộng

Trần Kim Bằng, Trần Thế Huy, Bùi Quốc Tính, Trương Tích Thiện

Tóm tắt - **Việc phân tích ứng xử cơ học của cấu trúc có chứa nhiều tạp chất và khuyết tật như lỗ tròng đóng vai trò khá quan trọng trong các lĩnh vực kỹ thuật. Sự hiện diện của khuyết tật và các hạt tạp chất sẽ tạo nên những biên bất liên tục trong vật thể và gây ảnh hưởng đáng kể đến độ cứng của vật thể. Trong bài báo này, ảnh hưởng của những hạt tạp chất và khuyết tật phân bố ngẫu nhiên trong tấm vật liệu phân lớp chức năng sẽ được phân tích bằng phương pháp phần tử hữu hạn mở rộng. Trong đó, các hàm làm giàu sẽ được sử dụng để mô tả tính chất vật lý của các hạt tạp chất và khuyết tật. Các hạt tạp chất cũng được xét là vật liệu phân lớp chức năng. Các kết quả tính toán sẽ được so sánh với kết quả thu được từ phần mềm COMSOL, dựa trên nền tảng phương pháp phần tử hữu hạn truyền thống.**

Từ khóa - **Phương pháp phần tử hữu hạn mở rộng, lỗ tròn, tạp chất, ngẫu nhiên, phân lớp chức năng.**