

EQUIVALENT ELASTIC MODULI FOR BRICK MASONRY: A COMPARATIVE STUDY OF PERIODIC HOMOGENIZATION APPROACHES

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Abstract

The mechanical characterisation of masonry remains a central challenge in structural analysis due to its heterogeneous composition of bricks and mortar joints. This study conducts a comprehensive comparative assessment of four approaches for determining the equivalent elastic properties of masonry: finite element modelling (FEM), the two-step homogenization method, the one-step Fourier-based scheme, and the interface-joint model. FEM simulations of a two-dimensional brick–mortar assemblage with varying mortar thicknesses serve as the benchmark. Analytical predictions are compared with FEM results for longitudinal and transverse Young's moduli, shear modulus, and Poisson's ratio. The two-step method achieves excellent agreement with FEM, with errors typically below 3% across all parameters. In contrast, the one-step method consistently overestimates transverse and shear stiffnesses by up to 50-60%, while the interface approach underestimates them by 20-55%. These deviations grow with mortar thickness, reflecting the different modelling assumptions. Overall, the FEM results are bounded by the stiffer predictions of the one-step method and the more compliant interface model, while the two-step method aligns most closely with the reference values. The study concludes that the two-step scheme is most suitable for engineering design, while the interface model is valuable for nonlinear mortar analyses.

Keywords: *Periodic masonry homogenization; equivalent elastic properties; two-step method; one-step method; interface-joint approach; finite element method; mortar joint thickness; comparative analysis.*

1. Introduction

Brick masonry is one of the oldest and most widely used construction materials due to its availability, durability, and cost-effectiveness. From a mechanical perspective, masonry represents a heterogeneous composite material composed of bricks and mortar, whose overall response strongly depends on the interaction between its constituents. An accurate prediction of the equivalent elastic moduli of masonry is therefore essential for structural analysis, design, and assessment.

Traditionally, the mechanical characterisation of masonry has been carried out through experimental testing on representative specimens. Although experimental

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methods provide direct insights, they are time-consuming, costly, and often impractical when large-scale structures are considered. As an alternative, homogenization techniques have been developed to estimate the effective mechanical properties of masonry by modelling it as an equivalent continuum.

Among these techniques, periodic homogenization has gained significant attention. By assuming that masonry can be represented by a periodic unit cell, the method provides a rigorous framework to capture the effects of microstructural geometry and material contrast. Various approaches have been proposed in the literature, including:

- (i) the two-step homogenization method [1], which evaluates the equivalent stiffness in successive directions;
- (ii) the one-step homogenization method [2], often based on Eshelby's inclusion concepts;
- (iii) the interface-joint approach [3], [4], in which mortar joints are treated as elastic interfaces with finite stiffness; and
- (iv) numerical homogenization [5]-[7], relying on finite element simulations of the periodic cell.

Despite their widespread use, these methods differ significantly in terms of accuracy, computational complexity, and applicability to real masonry structures. A systematic comparison is therefore required to clarify their relative advantages and limitations.

The main objective of this study is to present a comparative review of the above-mentioned periodic homogenization methods for brick masonry in the elastic regime. Emphasis is placed on their theoretical foundations, modelling assumptions, and predictive capabilities. By highlighting their strengths and weaknesses, the article aims to provide guidance for researchers and engineers in selecting the most appropriate homogenization strategy for masonry structures.

2. Theoretical background

2.1. Equivalent elastic moduli of masonry

The concept of equivalent elastic moduli arises from the need to replace the heterogeneous brick–mortar composite with a homogeneous continuum that reproduces the average mechanical response of the real structure. In this framework, the effective stiffness tensor \mathbf{C}^* relates the average stress field $\bar{\boldsymbol{\Sigma}}$ to the average strain field $\bar{\boldsymbol{\varepsilon}}$ according to: $\bar{\boldsymbol{\Sigma}} = \mathbf{C}^* : \bar{\boldsymbol{\varepsilon}}$. Similarly, the average strain can be expressed in terms of the effective compliance tensor \mathbf{S}^* as: $\bar{\boldsymbol{\varepsilon}} = \mathbf{S}^* : \bar{\boldsymbol{\Sigma}}$. Here, \mathbf{C}^* and \mathbf{S}^* represent, respectively, the homogenised elasticity and compliance tensors. These tensors encapsulate the

combined contribution of bricks and mortar and are the primary quantities of interest when studying masonry in the elastic range.

2.2. Homogenization principle

Homogenization techniques rely on the concept of a periodic unit cell (PUC), which is a unit cell sufficiently large to contain the essential features of the masonry microstructure but small compared to the overall structure. By imposing suitable boundary conditions on the PUC, one can extract the effective elastic response. Two types of boundary conditions are commonly employed: prescribing uniform displacements on the boundaries, ensuring homogeneous strain or uniform stresses, ensuring homogeneous stress. In the linear elastic regime, the Hill-Mandel condition ensures energy equivalence between the heterogeneous microstructure and the homogenised medium, forming the theoretical foundation of all homogenization methods.

2.3. Periodic homogenization of masonry

Masonry, by construction, exhibits a periodic arrangement of bricks and mortar joints, making periodic homogenization particularly relevant. A periodic unit cell (PUC) is defined by repeating the basic pattern of masonry, such as the running bond configuration. The homogenised stiffness tensor is then computed as:

$$\mathbf{C}^* = \langle \mathbf{C} : \mathbf{A} \rangle, \mathbf{S}^* = \langle \mathbf{S} : \mathbf{B} \rangle \quad (1)$$

where \mathbf{A} and \mathbf{B} are localisation tensors for strain and stress, respectively, and $\langle \cdot \rangle$ denotes volume averaging over the PUC.

Different formulations of periodic homogenization exist, each introducing specific modelling assumptions. These variations give rise to the two-step, one-step, interface-joint, and numerical approaches, which will be described in detail in the following sections.

3. Methods of periodic homogenization

This section reviews the main periodic homogenization strategies that have been proposed in the literature for the evaluation of the equivalent elastic moduli of masonry in the elastic regime. Although they share the same theoretical foundation, their implementation differs in complexity, accuracy, and practical applicability.

3.1. Two-step homogenization (Pande)

The two-step homogenization method was originally developed by Pande, Liang, and Middleton [1] within the framework of the equivalent material approach. In their formulation, masonry is idealised as a composite system of bricks and mortar joints, modelled as alternating elastic layers. At the first stage, the equivalent properties of a stacked brick–mortar system with parallel bed joints are derived using energy equivalence principles. This provides homogenised elastic constants in the direction normal and

parallel to the mortar layers.

In the second stage, the method is extended to account for the presence of orthogonal head joints. By introducing a second set of layers perpendicular to the bed joints, the equivalent orthorhombic material properties of masonry are obtained. The resulting continuum is characterised by anisotropy, with three orthogonal directions of different stiffnesses. Explicit analytical expressions for the equivalent moduli are expressed as functions of:

- + the elastic modulus and Poisson's ratio of bricks and mortar,
- + the relative thickness of brick and mortar joints, and
- + the assumed continuity of head joints.

The approach is often referred to as "two-step homogenization" because it first homogenises along one direction (bed joints) and then incorporates the perpendicular set of joints (head joints).

The main advantages of this method are its analytical simplicity and the possibility of computing closed-form solutions for both 2D and 3D cases. However, due to its layered assumption, it tends to neglect local stress concentrations and discontinuities of head joints, leading to some limitations in accuracy when applied to irregular bond patterns. Despite these simplifications, the method provides valuable insights into the anisotropic nature of masonry and has laid the foundation for subsequent homogenization studies.

This method provides closed-form expressions for effective elastic constants:

$$\begin{aligned}
 E_{11}^{ef} &= p_{k1}E^m + p_{k2}E_{1step1}; & \nu_{12}^{ef} &= p_{k1}\nu^m + p_{k2}\nu_{12step1} \\
 E_{22}^{ef} &= \frac{1}{\frac{p_{k1}}{E^m}(1-2\nu^m) + \frac{p_{k2}}{E_{2step1}}\left(\frac{E_{1step1}}{E_{2step1}} - 2\nu_{12step1}\right) + \frac{2\nu_{12}^2}{E_{11}}} \\
 G_{12} &= \frac{1}{\frac{p_{k1}}{G^m} + \frac{p_{k2}}{G_{12step1}}}
 \end{aligned} \tag{2}$$

with

$$\begin{aligned}
 E_{1step1} &= \frac{1}{\frac{p_1}{E^m}[1-2\nu_m^2] + \frac{p_2}{E^b}[1-2\nu_b^2] + 2\frac{(p_1\nu_m + p_2\nu_b)^2}{p_1E^m + p_2E^b}} \\
 E_{2step1} &= p_1E^m + p_2E^b; & \nu_{12step1} &= (p_1\nu_m + p_2\nu_b)\frac{E_{11}}{E_{22}} \\
 G_{12step1} &= \frac{\mu^m\mu^b}{p_1\mu^m + p_2\mu^b}; & p_1 &= \frac{t^m}{t^m + 2b}; & p_2 &= \frac{2b}{t^m + 2b}
 \end{aligned} \tag{3}$$

The use of Eqs. (2) and (3) make it computationally efficient. However, it relies on simplifying assumptions such as isotropy of the phases and neglect of local stress concentrations, which may reduce accuracy in the presence of irregular geometries or strong anisotropy.

3.2. One-step homogenization (Wang)

An alternative homogenization strategy was introduced by Wang [2], often referred to as the one-step homogenization method. Unlike the two-step procedure of Pande *et al.*, which first homogenises the brick-mortar system in one direction and then incorporates the perpendicular joints, Wang's approach considers the entire PUC in a single stage. This PUC, typically defined by a running bond arrangement of bricks and mortar joints, is assumed to represent the infinite repetition of masonry microstructure.

The method is rigorously based on the Hill-Mandel energy equivalence condition, which states that the macroscopic strain energy density of the homogenised material must equal the average microscopic strain energy density of the heterogeneous unit cell. To achieve this, localisation tensors are introduced to relate the local strain and stress fields in bricks and mortar to the macroscopic quantities. The approach extensively uses Eshelby's inclusion theory, enabling the derivation of explicit analytical expressions for the effective elastic constants of masonry.

The effective stiffness of the masonry is obtained finally:

$$\mathbf{C}^{ef} = \mathbf{C}^m : [\mathbf{I} - \varphi(\mathbf{A}^b - \mathbf{S}_E^b)^{-1}] \quad (4)$$

where \mathbf{I} is the fourth-order identity tensor; φ is the volume fraction of inclusions; $\mathbf{A}^b = \mathbf{C}^m - \mathbf{C}^b^{-1} : \mathbf{C}^m$. \mathbf{S}_E^b is an expression for the Eshelby tensor. The base cell contains both brick and mortar, with geometry explicitly modelled.

Localisation tensors are introduced to capture stress and strain fluctuations within the cell. Effective stiffness in Eq. (4) is obtained analytically by evaluating series expansions of the inclusion response:

$$\mathbf{S}_{Eijkl}^b = \sum_{\xi \in \Lambda} \varphi g_0(\xi) g_0^{-\xi} g_{ijkl}(\xi) \quad (5)$$

where

$$g_{ijkl}(\xi) = \frac{1}{2\xi^2} \left[\xi_j (\delta_{in} \xi_m + \delta_{im} \xi_n) + \xi_i (\delta_{jn} \xi_m + \delta_{jm} \xi_n) \right] - \frac{1}{1-\nu} \frac{\xi_i \xi_j \xi_k \xi_l}{\xi^4} + \frac{\nu}{1-\nu} \frac{\xi_i \xi_j}{\xi^2} \delta_{mn} \quad (6)$$

$$g_0(\xi) = \frac{1}{2ab} \frac{1}{\xi_1 \xi_2} \left\{ \begin{aligned} & \sin(\xi_1 b) \sin(\xi_2 a) + [\sin(\xi_1 L) - \sin(\xi_1(L-b))] \cdot \\ & \times [\sin(\xi_2 H) - \sin(\xi_2(H-a))] \end{aligned} \right\}$$

$2L$, $2H$ are the basic cell dimensions and $2a$, $2b$ are the brick dimensions. For practice, the expressions for the constituents are:

$$\begin{aligned}
 S_{1111}^b &= \sum_{\xi \in \Lambda} \varphi g_0(\xi)^2 \left(\frac{(2-\nu^b)\xi_1^2}{(1-\nu^b)\xi^2} - \frac{\xi_1^4}{(1-\nu^b)\xi^4} \right) \\
 S_{2222}^b &= \sum_{\xi \in \Lambda} \varphi g_0(\xi)^2 \left(\frac{(2-\nu^b)\xi_2^2}{(1-\nu^b)\xi^2} - \frac{\xi_2^4}{(1-\nu^b)\xi^4} \right) \\
 S_{1122}^b &= \sum_{\xi \in \Lambda} \varphi g_0(\xi)^2 \left(\frac{\nu^b \xi_1^2}{(1-\nu^b)\xi^2} - \frac{\xi_1^2 \xi_2^2}{(1-\nu^b)\xi^4} \right) \\
 S_{2211}^b &= \sum_{\xi \in \Lambda} \varphi g_0(\xi)^2 \left(\frac{\nu^b \xi_2^2}{(1-\nu^b)\xi^2} - \frac{\xi_1^2 \xi_2^2}{(1-\nu^b)\xi^4} \right) \\
 S_{1212}^b &= \sum_{\xi \in \Lambda} \varphi g_0^2(\xi) \left(\frac{1}{2} - \frac{\xi_1^2 \xi_2^2}{(1-\nu^b)\xi^4} \right)
 \end{aligned} \tag{7}$$

or in the general form:

$$S_{ijkl}^b = \sum_{n_1=-N, n_1 \neq 0}^N \sum_{n_2=-N, n_2 \neq 0}^N \varphi [g_0(\xi)]^2 g_{ijkl}(\xi) \tag{8}$$

The approach is mathematically rigorous and able to account for the coupled influence of geometry and material contrast within a single framework. However, the derivation involves truncated series whose accuracy depends on the number of terms retained. While convergence is usually rapid for moderate truncation orders, the method systematically provides stiffer predictions than FE references, particularly for transverse and shear moduli. Its practical use is therefore more suited to theoretical investigations or parametric analyses than to direct engineering applications.

3.3. Interface-joint approach (Cecchi)

A further development in the homogenization of masonry was proposed by Cecchi and Sab [3], who introduced an interface-joint approach. In this method, masonry is modelled as an assemblage of rigid or nearly rigid bricks connected through mortar joints that are represented not as volumetric continua, but as elastic interfaces with finite stiffness. This assumption reflects the fact that, in many practical situations, deformation is concentrated primarily in the mortar layers, while bricks remain comparatively stiffer.

The theoretical foundation of this approach is rooted in asymptotic homogenization techniques combined with the Hill-Mandel condition of energy equivalence. The mortar joints are described by interface constitutive relations, in which the relative displacement (jump) across the joint is linearly related to the traction through interface stiffness tensors. By embedding these interface laws within a periodic framework, the effective elastic properties of masonry can be derived analytically.

An important feature of the interface-based method is its ability to capture the anisotropic behaviour of masonry while maintaining a reduced level of complexity compared with full volumetric modelling of mortar. For both stack bond and running bond masonry, the approach provides explicit expressions for the homogenised Young's moduli, shear moduli, and Poisson's ratios. These expressions highlight the critical role of mortar thickness and stiffness: increasing joint thickness or reducing its stiffness leads to a marked decrease in the equivalent elastic moduli.

An alternative representation considers mortar joints as elastic interfaces of finite thickness and stiffness rather than fully resolved volumes. In this framework, displacement jumps across interfaces are related to tractions through constitutive tensors:

$\mathbf{K} = \frac{1}{e} [\mu^m \mathbf{I} + (\mu^m + \lambda^m) \mathbf{n} \otimes \mathbf{n}]$ with \mathbf{n} unit normal vector outside the interface. \mathbf{K} for the horizontal/vertical interface are written:

$$\mathbf{K}^h = \frac{1}{e^h} \begin{pmatrix} K' & 0 & 0 \\ 0 & K'' & 0 \\ 0 & 0 & K'' \end{pmatrix}, \quad \mathbf{K}^v = \frac{1}{e^v} \begin{pmatrix} K' & 0 & 0 \\ 0 & K'' & 0 \\ 0 & 0 & K'' \end{pmatrix}$$

where $K' = \lambda^m + 2\mu^m; K'' = \mu^m$ in plane deformations and $K' = \lambda_1^m + 2\mu^m; K'' = \mu^m$ in plane stresses.

The expression for the homogenised elasticity constants in this case is:

$$\begin{aligned} \tilde{C}_{1111}^{el} &= \frac{\left(A_{1111}^b K' + \mathbf{B} \frac{e^h}{a} \right) \left(4K' \frac{e^h}{a} + \frac{b}{a} K'' \frac{e^v}{a} \right)}{4 \left(\frac{e^h}{a} \right)^2 \frac{e^v}{b} \mathbf{B} + A_{1111}^b \frac{e^h}{a} \mathbf{C} + K \mathbf{D}} \\ \tilde{C}_{2222}^{el} &= K' \frac{\mathbf{B} \frac{e^v}{b} + K A_{1111}^b}{\mathbf{B} \frac{e^v}{b} \frac{e^h}{a} + K A_{1111}^b \left(\frac{e^h}{a} + \frac{e^v}{b} \right) \mathbf{C} + K^2} \\ \tilde{C}_{1122}^{el} &= A_{1122}^b K' \frac{4K' \frac{e^h}{a} + \frac{b}{a} K'' \frac{e^v}{a}}{4 \left(\frac{e^h}{a} \right)^2 \frac{e^v}{b} \mathbf{B} + A_{1111}^b \frac{e^h}{a} \mathbf{C} + K \mathbf{D}} \\ \tilde{C}_{1212}^{el} &= A_{1212}^b K'' \frac{K' \frac{e^v}{b} + 4 \frac{a}{b} K'' \frac{e^h}{b}}{A_{1212}^b \frac{e^h}{a} \mathbf{F} + K'' \mathbf{G}} \end{aligned} \tag{9}$$

where

$$\mathbf{B} = (A_{1111}^B)^2 - (A_{1122}^B)^2; \mathbf{C} = 4K' \frac{e^h}{a} + \frac{b}{a} K'' \frac{e^v}{a} + 4K' \frac{e^v}{b}; \mathbf{D} = 4K' \frac{e^h}{a} + \frac{b}{a} K'' \frac{e^v}{a};$$

$$\mathbf{F} = K' \frac{e^v}{b} + 4 \frac{a}{b} K'' \frac{e^h}{b} + 4 \left(\frac{a}{b} \right)^2 K'' \frac{e^v}{b}; \mathbf{G} = K' \frac{e^v}{b} + 4 \frac{a}{b} K'' \frac{e^h}{b}$$

The main advantage of this method lies in its ability to explicitly describe the mechanical role of mortar joints, including imperfect bonding, sliding, or separation. As a result, it is well-suited to investigate degradation mechanisms and nonlinear responses. On the other hand, the predicted effective stiffness strongly depends on the calibration of interface parameters, and linear elastic estimates tend to underestimate transverse and shear stiffness compared with volumetric FEM.

3.4. Numerical homogenization on morphology (FEM)

Numerical homogenization is one of the most rigorous and versatile methods for evaluating the effective elastic properties of masonry. Unlike analytical approaches that rely on simplifying assumptions, such as sequential treatment of head and bed joints or plane stress/strain idealisations, numerical techniques apply homogenization theory directly to the actual geometry and material heterogeneity of masonry.

In this framework, masonry is modelled as a periodic composite medium, with a representative unit cell explicitly meshed and bricks and mortar assigned their respective properties. Periodic boundary conditions are imposed to ensure compatibility and equilibrium, and the local problem is solved using the finite element method (FEM). The macroscopic constitutive response is then obtained by averaging stresses and strains in accordance with Hill's energy equivalence condition.

The main advantages of this method include:

- + the ability to incorporate real geometry, including different bond patterns, joint thicknesses, and irregular layouts;
- + the possibility of capturing three-dimensional effects, which influence out-of-plane behaviour and failure mechanisms;
- + accurate resolution of local stress concentrations at brick–mortar interfaces, critical for predicting crack initiation and damage evolution.

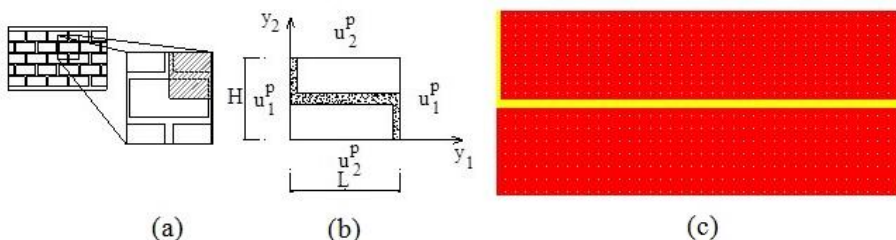


Fig. 1. Computational domain for numerical homogenization (one quarter of the periodic unit cell).

Periodic boundary conditions are imposed on the boundaries of the cell to ensure consistency with the homogenization framework. Equivalent elastic moduli are computed by averaging stresses and strains over the cell under different loading conditions.

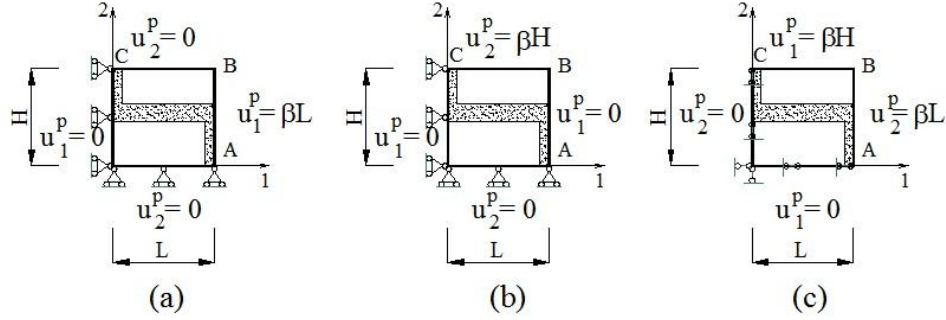


Fig. 2. Periodic boundary conditions applied to the computational cell for numerical homogenization.

Case 1: Simple deformation along y_1 : $C_{1111}^{ef} = \frac{\bar{\sigma}_{11}}{\beta}$, $C_{1122}^{ef} = \frac{\bar{\sigma}_{22}}{\beta}$

Case 2: Simple deformation along y_2 : $C_{1122}^{ef} = \frac{\bar{\sigma}_{11}}{\beta}$, $C_{2222}^{ef} = \frac{\bar{\sigma}_{22}}{\beta}$

Case 3: Simple shear: $C_{1212}^{ef} = \frac{\bar{\sigma}_{12}}{\beta}$.

4. Comparison in 2D case study

To illustrate the performance of the different periodic homogenization methods, a two-dimensional case study is considered. The reference unit cell consists of a brick of length $L = 250$ mm and height $H = 55$ mm, with a mortar joint of thickness t_m varying from 2 mm to 20 mm. The elastic properties of the constituents are [5]:

- Brick: $E_b = 11000$ MPa; $\nu_b = 0.20$.
- Mortar: $E_b = 2200$ MPa; $\nu_b = 0.25$.

This setup enables the evaluation of the influence of mortar thickness on the equivalent elastic moduli of masonry.

4.1. Finite element reference model (FEM)

Numerical homogenization using finite element analysis was employed as the reference solution. The PUC consists of one brick and the surrounding mortar layers, discretised with quadrilateral plane-stress elements. Periodic boundary conditions were applied to enforce compatibility of displacements and equilibrium of tractions across opposite boundaries, consistent with Hill's macro-homogeneity condition.

For each mortar thickness t_m , the PUC was subjected to unit strain states to extract

the corresponding average stress responses. The effective elastic constants - $E_{11}, E_{22}, G_{12}, \nu_{12}$ - were then obtained through volume-averaging procedures.

4.2. Analytical predictions

For comparison, the three analytical approaches described in Section 2 were applied to the same masonry configuration:

- + Two-step homogenization (Pande),
- + One-step homogenization (Wang), with truncation orders $N = 2, 50, 100, 1000$.
- + Interface-joint model (Cecchi).

This setup provides a consistent framework to evaluate the accuracy and limitations of each approach relative to the finite element benchmark.

4.3. Comparative results

Tables and figures compare the effective elastic constants $E_{11}, E_{22}, G_{12}, \nu_{12}$ obtained from FEM and the three analytical schemes for mortar thicknesses $t_m = 2$ mm, 4 mm, 10 mm, 14 mm, 20 mm. FEM results are used as a reference.

Tab. 1. Absolute values of effective elastic constants (for mortar thickness $t_m = 2, 4, 10, 14, 20$ mm)

| t_m (mm) | Method | E_{11} (MPa) | E_{22} (MPa) | G_{12} (MPa) | ν_{12} |
|---------------|--------------------|-------------------|-------------------|-------------------|------------|
| 2 | FEM (ref.) | 10400 | 9652 | 3897 | 0.1978 |
| | Pande (Two-step) | 10399.6 | 9532.7 | 3884.6 | 0.1980 |
| | Wang (One-step) | 10622.7 | 10620.5 | 4424.3 | 0.2005 |
| | Cecchi (Interface) | 10411.0 | 7972.3 | 3272.0 | 0.1894 |
| 4 | FEM (ref.) | 9875 | 8651 | 3428 | 0.1980 |
| | Pande (Two-step) | 9859.0 | 8479.9 | 3402.4 | 0.1964 |
| | Wang (One-step) | 10268.9 | 10260.6 | 4274.9 | 0.2011 |
| | Cecchi (Interface) | 9882.0 | 6252.5 | 2543.1 | 0.1798 |
| 10 | FEM (ref.) | 8609 | 6767 | 2614 | 0.2003 |
| | Pande (Two-step) | 8524.5 | 6574.3 | 2568.6 | 0.1935 |
| | Wang (One-step) | 9323.0 | 9278.1 | 3873.7 | 0.2034 |
| | Cecchi (Interface) | 8575.0 | 3797.1 | 1524.9 | 0.1561 |
| 14 | FEM (ref.) | 7947 | 6004 | 2306 | 0.2033 |
| | Pande (Two-step) | 7820.8 | 5829.3 | 2255.9 | 0.1926 |
| | Wang (One-step) | 8769.0 | 8690.3 | 3636.9 | 0.2056 |
| | Cecchi (Interface) | 7880.2 | 3009.6 | 1203.6 | 0.1434 |
| 20 | FEM (ref.) | 7139 | 5220 | 2003 | 0.2089 |
| | Pande (Two-step) | 6967.8 | 5081.6 | 1949.7 | 0.1923 |
| | Wang (One-step) | 8026.1 | 7893.7 | 3317.0 | 0.2098 |
| | Cecchi (Interface) | 7026.2 | 2295.7 | 914.6 | 0.1279 |

Table 1 summarises the absolute values of the effective elastic constants obtained by the four approaches for mortar thicknesses $t_m = 2$ mm, 4 mm, 10 mm, 14 mm, 20 mm. As expected, the FEM results decrease monotonically with increasing mortar thickness, reflecting the higher compliance of thicker joints.

The two-step method (Pande) provides values very close to FEM across all parameters, with only minor differences visible. The one-step method (Wang, $N = 2$) predicts significantly larger values of E_{22} and G_{12} while ν_{12} remains comparable to FEM. Conversely, the interface-joint model (Cecchi) matches E_{11} closely but yields substantially smaller E_{22} and G_{12} , particularly at larger mortar thicknesses.

Tab. 2. Relative error (%) of analytical methods compared with FEM
 (Error = (Method - FEM)/FEM × 100%)

| t_m (mm) | Method | ΔE_{11} | ΔE_{22} | ΔG_{12} | $\Delta \nu_{12}$ |
|---------------|--------------------|-----------------|-----------------|-----------------|-------------------|
| 2 | Pande (Two-step) | -0.004% | -1.23% | -0.32% | +0.10% |
| | Wang (One-step) | +2.14% | +10.03% | +13.53% | +1.36% |
| | Cecchi (Interface) | +0.11% | -17.40% | -16.05% | -4.23% |
| 4 | Pande (Two-step) | -0.16% | -1.98% | -0.75% | -0.79% |
| | Wang (One-step) | +4.00% | +18.60% | +24.70% | +1.54% |
| | Cecchi (Interface) | +0.07% | -27.73% | -25.82% | -9.17% |
| 10 | Pande (Two-step) | -0.98% | -2.85% | -1.74% | -3.40% |
| | Wang (One-step) | +8.30% | +37.11% | +48.22% | +1.54% |
| | Cecchi (Interface) | -0.39% | -43.88% | -41.68% | -22.07% |
| 14 | Pande (Two-step) | -1.59% | -2.91% | -2.18% | -5.28% |
| | Wang (One-step) | +10.34% | +44.74% | +57.73% | +1.12% |
| | Cecchi (Interface) | -0.84% | -49.88% | -47.79% | -29.47% |
| 20 | Pande (Two-step) | -2.39% | -2.65% | -2.67% | -7.94% |
| | Wang (One-step) | +12.43% | +51.21% | +65.64% | +0.43% |
| | Cecchi (Interface) | -1.58% | -56.04% | -54.33% | -38.82% |

Table 2 presents the relative errors of the analytical predictions compared with FEM. Several key observations can be made:

- Two-step (Pande): Errors remain consistently small, typically within $\pm 3\%$ for all parameters, confirming its robustness and accuracy in the elastic regime.
- One-step (Wang): While E_{11} differs by 2% - 12%, the method systematically overestimates E_{22} (+10% to +51%) and G_{12} (+14% to +66%). The discrepancy grows with mortar thickness, indicating that model assumptions dominate over truncation effects.
- Interface (Cecchi): The longitudinal modulus E_{11} stays within $\pm 2\%$ of FEM, but E_{22} and G_{12} are strongly underestimated (-17% to -56% and -16% to -54%, respectively). The underestimation increases with joint thickness. Poisson's ratio is also underestimated at larger t_m , with deviations up to -39%.

Overall, the FEM benchmark is bounded by the relatively stiff one-step predictions and the more compliant interface results, while the two-step method reproduces FEM most closely.

To complement the tabulated results, Figures 3a–d illustrate the variation of the effective elastic constants with mortar thickness. Each figure compares the four approaches (FEM, Pande, Wang, Cecchi) for one parameter.

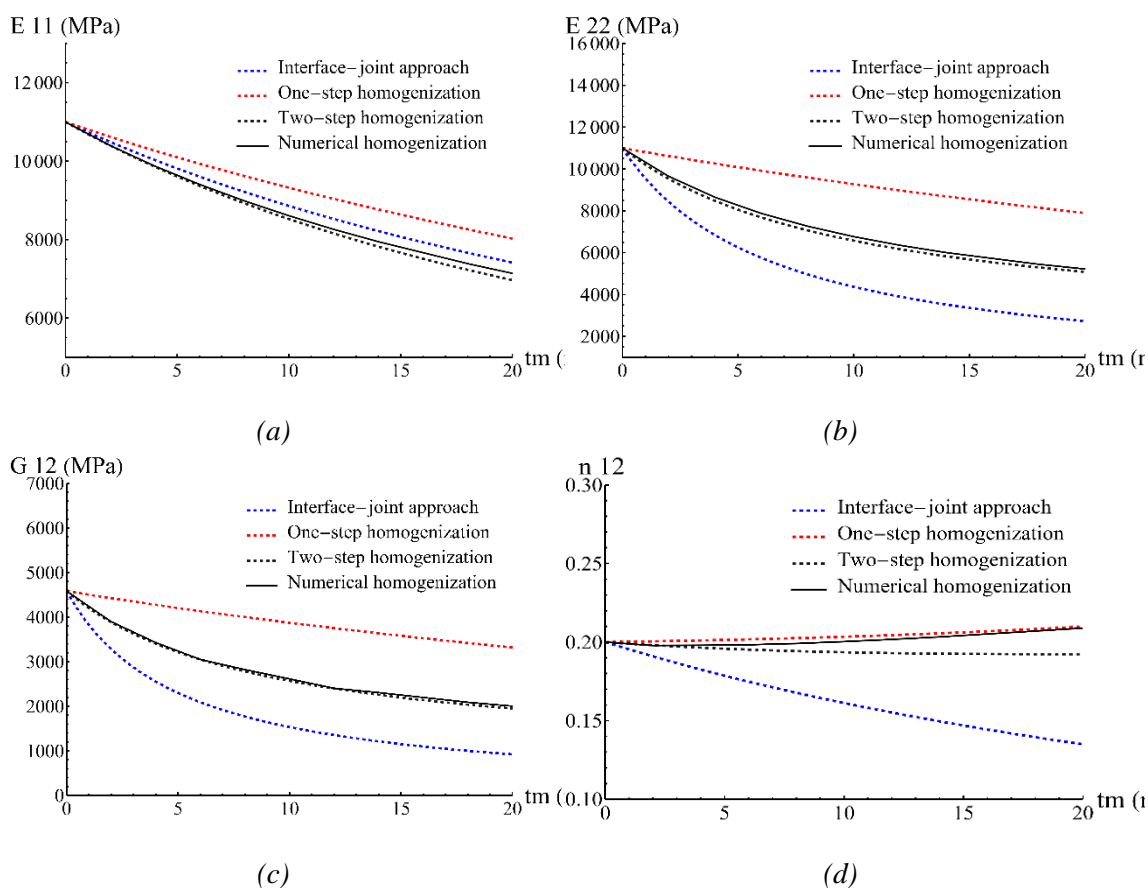


Fig. 3. Variation of equivalent elastic constants with mortar thickness: comparison between FEM and analytical homogenization methods. (a) Longitudinal modulus E_{11} ; (b) Transverse modulus E_{22} ; (c) Shear modulus G_{12} ; (d) Poisson's ratio ν_{12} .

All methods predict a monotonic decrease of E_{11} with increasing mortar thickness (Fig. 3a). The two-step method coincides almost exactly with FEM, while the interface model also follows the same trend with deviations below 2%. The one-step method consistently predicts higher values, with the gap widening as joints become thicker. A clear divergence is observed in Fig. 3b for the transverse modulus E_{22} . FEM and Pande agree closely, while Wang's one-step approach overestimates by 10-50%, and Cecchi's

interface model underestimates by 20-56%. This spread confirms that E_{22} is the most sensitive modulus to modelling assumptions. The FEM and Pande results remain close, showing a smooth decrease with t_m in the case of shear modulus G_{12} (Fig. 3c). Wang’s method produces significantly stiffer predictions (+15-66%), whereas Cecchi’s model underestimates by 16-54%. Figure 3d shows that FEM and Pande values also remain nearly constant (0.19-0.21) for Poisson’s ratio ν_{12} . The one-step method gives slightly higher values (+0.5-1.5%), while the interface model predicts markedly lower ratios, especially at larger t_m (down to 0.128 at 20 mm). This underestimation reflects reduced transverse coupling when joints are treated as weak interfaces.

Similar trends were also observed for other combinations of brick and mortar properties, for instance, for bricks with $E_b = 12800\text{MPa}$; $\nu_b = 0.30$, and mortar with $E_m = 4000\text{MPa}$; $\nu_m = 0.30$ (see Fig. 4). For brevity, detailed results for these additional cases are not presented in this article.

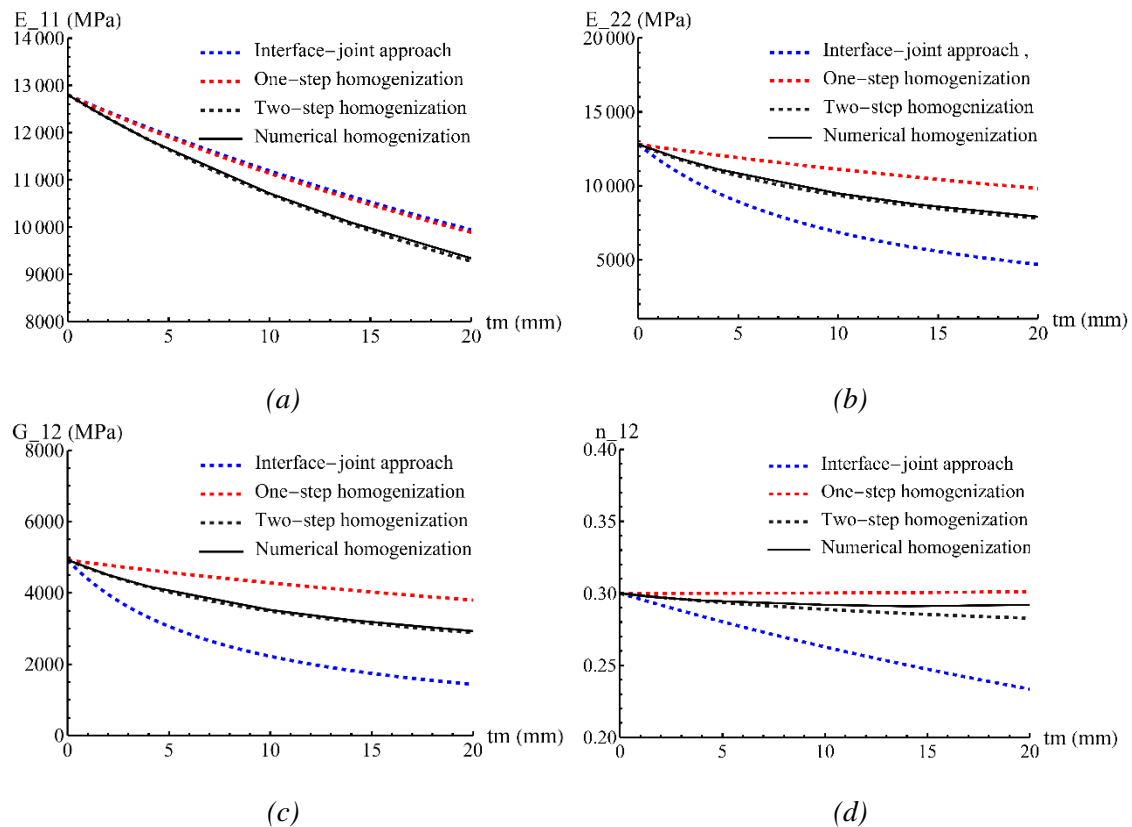


Fig. 4. Variation of equivalent elastic constants with mortar thickness: comparison between FEM and analytical homogenization methods. a) Longitudinal modulus E_{11} ; b) Transverse modulus E_{22} ; c) Shear modulus G_{12} ; d) Poisson’s ratio ν_{12} with $E_b = 12800$; $\nu_b = 0.30$ and $E_m = 4000$; $\nu_m = 0.30$.

These figures highlight that the two-step approach reproduces FEM with remarkable accuracy, the one-step method provides an upper bound, and the interface model a lower bound.

4.4. Discussion

The comparative results reveal that the performance of the homogenization schemes depends strongly on how mortar joints are represented.

First, the two-step approach (Pande) consistently provides the best agreement with FEM. Deviations are typically below 3% for all moduli, regardless of mortar thickness. This accuracy reflects the layered treatment of masonry, which reproduces the anisotropy introduced by both head and bed joints while maintaining a simple analytical structure. The method is therefore highly reliable for elastic design purposes.

Second, the one-step method (Wang) shows systematic overestimation of stiffness, particularly for the transverse and shear moduli. Although convergence with respect to the Fourier truncation parameter N is rapid ($N \geq 50$ gives stable results), the discrepancy with FEM remains large, reaching 50-60% for E_{22} and G_{12} at $t_m = 20$ mm. This indicates that the source of error is intrinsic to the model assumptions, not to truncation. In practice, the one-step approach may be regarded as providing an upper bound of stiffness.

Third, the interface-joint model (Cecchi) exhibits the opposite trend: E_{11} remains close to FEM (within 2%), but E_{22} and G_{12} are consistently underestimated, by up to 56% and 54% at large t_m . This behaviour arises because joints are treated as compliant interfaces, which localise deformation and reduce transverse and shear load transfer. While this limits the accuracy of global elastic predictions, it highlights the model's suitability for analysing joint-dominated mechanisms such as sliding, separation, or bond degradation. In this sense, Cecchi's method can be interpreted as providing a lower bound.

Taken together, the three analytical methods frame the FEM reference:

- Pande \approx FEM (accurate lower bound);
- Wang \approx stiff upper bound;
- Cecchi \approx compliant lower bound for transverse and shear response.

The FEM predictions lie within these bounds, confirming its role as the most

reliable tool. This bounded behaviour also provides valuable insight: analytical methods may be used not only for prediction but also to delineate the plausible range of elastic properties depending on the assumed role of mortar joints.

From a practical perspective, engineers may employ the two-step method for elastic analysis with confidence, while researchers interested in theoretical bounds may combine the Wang and Cecchi models to explore extreme stiffness scenarios. For nonlinear studies or failure modelling, the interface approach is indispensable.

5. Conclusions

This study compared four approaches – FEM, the two-step homogenization (Pande), the one-step scheme (Wang), and the interface-joint model (Cecchi) – for evaluating the equivalent elastic constants of brick masonry. Based on the numerical results and analytical predictions, the following conclusions can be drawn:

- Two-step homogenization (Pande) achieves excellent agreement with FEM, with errors generally below 3% across all parameters and mortar thicknesses. It is the most accurate analytical tool for practical elastic design;

- One-step homogenization (Wang) systematically overestimates transverse and shear stiffnesses, with errors growing to 50-60% for thick joints. Despite rapid convergence with Fourier terms, the model assumptions lead to overly stiff predictions, making it more suitable as an upper-bound estimate;

- Interface-joint homogenization (Cecchi) provides close predictions for E_{11} but significantly underestimates E_{22} and G_{12} . This reflects its emphasis on joint compliance and makes it particularly relevant for joint-dominated or nonlinear mechanisms, even if less accurate in global elastic constants;

- Together, the three analytical approaches bound the FEM results: Pande aligns with FEM, Wang defines a stiff upper bound, and Cecchi defines a compliant lower bound. This bounding interpretation provides both practical utility and theoretical insight;

- For engineering applications, the two-step method offers a reliable and efficient alternative to FEM. For advanced research into nonlinear failure or bond degradation, the

interface approach remains essential, while the one-step method serves as a theoretical complement for bounding studies.

Future work should extend the comparative framework to nonlinear regimes, where joint opening, sliding, and damage play a critical role. Incorporating three-dimensional geometries, irregular bond patterns, and experimental validation will further enhance the applicability of homogenization models. In addition, efficient multiscale strategies combining FEM benchmarks with simplified analytical bounds may provide a practical balance between accuracy and computational cost.

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MÔ ĐUN ĐÀN HỒI TƯƠNG ĐƯƠNG CỦA KẾT CẤU KHỐI XÂY: NGHIÊN CỨU SO SÁNH CÁC PHƯƠNG PHÁP ĐỒNG NHẤT KHỐI XÂY CÓ QUY CÁCH

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Tóm tắt: Việc xác định đặc tính cơ học của khối xây là một thách thức trọng tâm trong phân tích kết cấu do thành phần gạch và vữa không đồng nhất. Nghiên cứu này tiến hành đánh giá so sánh toàn diện bốn phương pháp để xác định các tính chất đàn hồi tương đương của khối xây: mô phỏng phần tử hữu hạn (FEM), phương pháp đồng nhất hai bước, phương pháp đồng nhất một bước dựa trên chuỗi Fourier và mô hình phần tử tiếp xúc. Mô phỏng FEM trên cấu trúc gạch-vữa 2D với chiều dày vữa thay đổi được sử dụng làm chuẩn tham chiếu. Các kết quả giải tích được so sánh với mô phỏng FEM cho mô đun đàn hồi phương dọc và phương ngang, mô đun cắt và hệ số Poisson. Phương pháp đồng nhất hai bước cho mức độ tương đồng rất cao so với kết quả FEM, với sai số thường dưới 3% cho mọi tham số. Ngược lại, phương pháp đồng nhất một bước cho mô đun đàn hồi phương ngang và mô đun trượt cao hơn (sai số tới 50-60%), trong khi phương pháp phần tử tiếp xúc lại đánh giá thấp chúng, từ 20-55%. Sai lệch này tăng lên khi chiều dày mạch vữa lớn hơn, phản ánh sự khác biệt trong giả thiết mô hình. Nhìn chung, kết quả FEM nằm trong khoảng được giới hạn bởi dự đoán cứng hơn của phương pháp đồng nhất một bước và dự đoán mềm hơn của phương pháp phần tử tiếp xúc, còn phương pháp đồng nhất hai bước bám sát nhất với giá trị tham chiếu. Nghiên cứu kết luận rằng phương pháp hai bước là phù hợp nhất cho thiết kế kỹ thuật, trong khi mô hình phần tử tiếp xúc lại có giá trị đối với các phân tích khi vữa phi tuyến.

Từ khóa: *Đồng nhất khối xây có quy cách; đặc tính đàn hồi tương đương; phương pháp hai bước; phương pháp một bước; mô hình phần tử tiếp xúc; phương pháp phần tử hữu hạn; chiều dày mạch vữa; phân tích so sánh.*

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