A MULTI-STAGE HOMOGENIZATION METHOD FOR DETERMINING THE EFFECTIVE ELASTIC PROPERTIES OF LAYERED MATERIALS

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Abstract

The paper presents a multi-stage homogenization method to determine the effective properties of layered materials which have *n* elastic isotropic layers or elastic transversely isotropic (in the direction of the layers) ones. The homogenization procedure uses (*n*-1) times an explicit analytical formula for the layered material with *n* layers. The normal stress and strain at the interface between two material layers are assumed to be continuous, i.e., no slip and no detachment. The results obtained from this method are compared with the existing analytical four-sub-matrix method and they show good agreements. The multi-stage homogenization method is a powerful tool that can be used to quickly identify the effective properties of the layered materials.

Keywords: *Multi-stage homogenization method; four-sub-matrix method; layered materials; transversely isotropic; effective elastic properties.*

1. Introduction

Layered material is a quite common form of material in civil engineering which has layers of material with different properties being stacked. Some examples of this type of material are floor panel, multi-layered wall panel, reinforced beam, bridge deck structure, road multi-layered pavement, stratified sedimentary rock. In computational practice, in order to simplify and reduce the computational cost, this layered material should be referred to an equivalent material with the effective properties determined from the properties of the component material layers. Besides, some methods such as experimental and simulation ones, the analytical method of homogenization has become a performance method, which gives explicitly results of overall properties of the material (effective properties) and proves its effectiveness [1-4].

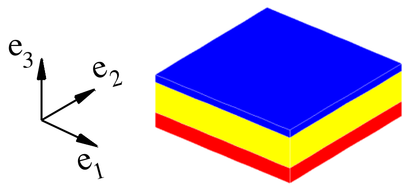

Figure 1. Multi-layered material.

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The homogenization approach for layered materials in [1, 5] relies on a technique which is based on the partially inversed and reversed behavior law technique. If the component materials are homogeneous and isotropic in each layer, the stiffness matrix of each layer of size 6×6 is divided into four sub-matrices of the form as follows:

$$\begin{cases} \mathbb{L}_{33} = \begin{bmatrix} L_{3333} & L_{3323} & L_{3313} \\ L_{2333} & L_{2323} & L_{2313} \\ L_{1333} & L_{1323} & L_{1313} \end{bmatrix}, \mathbb{L}_{31} = \begin{bmatrix} L_{3311} & L_{3322} & L_{3312} \\ L_{2311} & L_{2322} & L_{2312} \\ L_{1311} & L_{1322} & L_{1312} \end{bmatrix}, \\ \mathbb{L}_{13} = \begin{bmatrix} L_{1133} & L_{1123} & L_{1113} \\ L_{2233} & L_{2223} & L_{2213} \\ L_{1233} & L_{1223} & L_{1213} \end{bmatrix}, \mathbb{L}_{11} = \begin{bmatrix} L_{1111} & L_{1122} & L_{1112} \\ L_{2211} & L_{2222} & L_{2212} \\ L_{1211} & L_{1222} & L_{1212} \end{bmatrix} \end{cases} \tag{1}$$

The overall stiffness matrix is obtained from the solution of the equilibrium equation $\nabla.\sigma^{\alpha}(x) = 0$. If the interface between the material layers is perfect, i.e., no slip and no detachment at the contact between layers, the four sub-matrices are defined according to Eq. (2). Finally, the overall stiffness matrix (effective stiffness matrix) is that composed from these sub-matrices according to the principle of separation in Eq. (1).

$$\begin{cases} \overline{\mathbb{L}}_{33} = \left\langle \mathbb{L}_{33}^{-1} \right\rangle^{-1}, \\ \overline{\mathbb{L}}_{31} = \left\langle \mathbb{L}_{33}^{-1} \right\rangle^{-1} \left\langle \mathbb{L}_{33}^{-1} \mathbb{L}_{31} \right\rangle, \overline{\mathbb{L}}_{13} = \overline{\mathbb{L}}_{31}^{\ T}, \\ \overline{\mathbb{L}}_{11} = \left\langle \mathbb{L}_{11} - \mathbb{L}_{13} \mathbb{L}_{33}^{-1} \mathbb{L}_{31} \right\rangle + \left\langle \mathbb{L}_{13} \mathbb{L}_{11}^{-1} \right\rangle \left\langle \mathbb{L}_{33}^{-1} \right\rangle^{-1} \left\langle \mathbb{L}_{33}^{-1} \mathbb{L}_{31} \right\rangle \end{cases} \tag{2}$$

In Eq. (2) the symbol $\langle * \rangle$ denotes the average of * on a volume.

From this feature, one can refer this method as four-sub-matrix method. Using the technique of inversion and partial reverse, only the final results of the overall stiffness matrix are given in [5] without an explicit solution of the equilibrium equation. Note that resolving the equilibrium equation requires technique of changing the position of the invariants as well as the manner to use four sub-matrices. Therefore, it is also difficult in practice. Despite this, the correctness of the homogenization method has been proven by comparing the numerical results obtained with the Voigt - Reuss boundary in the case of materials composed of three isotropic elastic layers. It has been shown that some material layers in the layered material are transversely isotropic can be encountered, for example, the geological layer with cracks and pores distributed in parallel or the unidirectional reinforcement layer of the reinforced material. This type of material has not yet been mentioned in [5].

This study will present a multi-stage homogenization method for layered materials which gives an explicit analytical formula of the components in the effective compliance matrix. This method can be applied to both the elastic isotropic and transversely isotropic component materials. This procedure relies on the hypothesis that normal stress and strain at the contact surface between two material layers is continuous. In other words, the contact surface is considered perfect, i.e., no slip and no detachment at the contact between layers. Two cases of homogenization of two-layer and three-layer isotropic materials will be compared with the analytical method which is referred in [5] to prove the effectiveness of this multi-stage homogenization method.

2. Homogenization method for layered materials

2.1. Homogenization theory for 2 layers material

Let us consider two-layer materials with different elastic properties and they obey Hooke law [6-8]:

$$\varepsilon_{ij}^{\alpha} = \mathbb{S}_{ijkl}^{\alpha}\sigma_{kl}^{\alpha} \tag{3}$$

where ε^{α}, σ^{α}, $\mathbb{S}^{\alpha}$ are strain, stress, and compliance tensors of the α^{th} - material layer, respectively (α = 1.2).

Assume that the layers have the same surface size and differ only in thickness, the ratio of volume of each layer with respect to the total volume of the layered material is the one of the layer thickness. For the α^{th} layer, this ratio is denoted by φ^{α}. It is noted that $\sum_{\alpha=1}^{2}\varphi^{\alpha} = 1$. The homogenization method for layered materials is shown by the following overall relationship:

$$\bar{\varepsilon} = \bar{\mathbb{S}} : \bar{\sigma} \tag{4}$$

When $\varepsilon_{ij}^{\alpha}$, σ_{ij}^{α} are mean strain, stress tensors of the α^{th} - material layer, respectively, we have the equation as follows:

$$\bar{\varepsilon}_{ij} = \sum_{\alpha=1}^{2}\varphi^{\alpha}\varepsilon_{ij}^{\alpha} = \sum_{\alpha=1}^{2}\varphi^{\alpha}\mathbb{S}_{ijkl}^{\alpha}\sigma_{kl}^{\alpha} \tag{5}$$

Note that, in the compliance form, their constitutive law reads: $\varepsilon_{ij}^{\alpha} = \mathbb{S}_{ijkl}^{\alpha}\sigma_{kl}^{\alpha} = \dfrac{1+\nu^{\alpha}}{E^{\alpha}}\sigma_{ij}^{\alpha} - \dfrac{\nu^{\alpha}}{E^{\alpha}}\sigma_{kk}^{\alpha}\delta_{ij}$.

According to Eq. (4) and Eq. (5), $\bar{\varepsilon}$, $\bar{\sigma}$ are also mean strain, stress tensors by volume $(\bar{\varepsilon} = \bar{\varepsilon}_{ij},\ \bar{\sigma} = \bar{\sigma}_{ij})$. In other words, the mean stress is the overall stress determined by the relationship:

$$\bar{\sigma} = \sum_{\alpha=1,2} \varphi^{\alpha} \sigma^{\alpha} \tag{6}$$

When the layers are homogeneous, isotropic or transversely isotropic, $\bar{\mathbb{S}}$ is an effective compliance matrix that is also transversely isotropic. According to [7, 8], equation (4) is rewritten as:

$$\begin{pmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ \bar{\varepsilon}_{33} \\ \bar{\varepsilon}_{23} \\ \bar{\varepsilon}_{13} \\ \bar{\varepsilon}_{12} \end{pmatrix} = \begin{pmatrix} \bar{S}_{1111} & \bar{S}_{1122} & \bar{S}_{1133} & 0 & 0 & 0 \\ \bar{S}_{1122} & \bar{S}_{1111} & \bar{S}_{1133} & 0 & 0 & 0 \\ \bar{S}_{1133} & \bar{S}_{1133} & \bar{S}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{S}_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\left(\bar{S}_{1111} - \bar{S}_{1122}\right) \end{pmatrix} \times \begin{pmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{33} \\ \bar{\sigma}_{23} \\ \bar{\sigma}_{13} \\ \bar{\sigma}_{12} \end{pmatrix} \tag{7}$$

Therefore, $\bar{\mathbb{S}}$ has 5 unidentified components $\bar{S}_{1111}, \bar{S}_{1122}, \bar{S}_{1133}, \bar{S}_{3333}$ and $\bar{S}_{1313}$. For determination of these components, four cases of basic simple load are considered as illustrated in Fig. 1 [4, 7, 8].

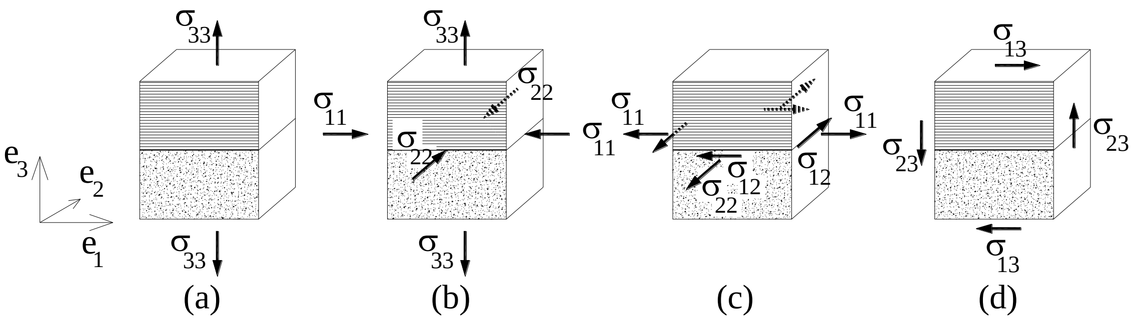

Figure 2. Load cases to be considered: axial tensile load (a), axial symmetry loading (b), tensile load combined with shear load in one plane (c), longitudinal shear load (d).

Case 1: Axial tensile load (see Fig. 2a)

In this case, the mean stress in Descartes co-ordinate (e_1, e_2, e_3) is written as follows:

$$\bar{\sigma} = \bar{\sigma}_{33}\; e_3 \otimes e_3 \tag{8}$$

From Eqs. (3), (4) and (8) one derives:

$$\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = \bar{S}^{0}_{1133}\bar{\sigma}_{33},\; \bar{\varepsilon}_{33} = \bar{S}_{3333}\bar{\sigma}_{33},\; \bar{\varepsilon}_{23} = \bar{\varepsilon}_{13} = \bar{\varepsilon}_{12} = 0 \tag{9}$$

Alternatively, under the macroscopic loading, the behavior of each layer is assumed to be transversely isotropic. Therefore, the stress is determined by the following equation:

$$\sigma^{\alpha} = \bar{\sigma}_{33}[e_3 \otimes e_3 + \beta^{\alpha}(e_1 \otimes e_1 + e_2 \otimes e_2)] \tag{10}$$

in which β^{α} are the coefficients identified with the continuous condition of stress and strain at the contact surface.

$$\sigma_{j3}^{\alpha=1} = \sigma_{j3}^{\alpha=2} \quad \text{and} \quad \varepsilon_{kj}^{\alpha=1} = \varepsilon_{kj}^{\alpha=2} \quad \text{with } (k, j) = (1,1),(2,2),(1,2) \tag{11}$$

Applying relations (8) and (10) to the Eq. (6) gives:

$$\sum_{\alpha=1}^{2} \varphi^{\alpha} \beta^{\alpha} = 0 \tag{12}$$

From the relationships (9), (11) and (12), one can obtain two components $\bar{S}_{1133}$ and $\bar{S}_{3333}$ by

$$\begin{cases} \bar{S}_{1133} = \sum_{\alpha=1}^{2} \varphi^{\alpha} \left[S_{1133}^{\alpha} + \beta^{\alpha} \left(S_{1111}^{\alpha} + S_{1122}^{\alpha} \right) \right] \\ \bar{S}_{3333} = \sum_{\alpha=1}^{2} \varphi^{\alpha} \left[S_{3333}^{\alpha} + \beta^{\alpha} \left(S_{3311}^{\alpha} + S_{3322}^{\alpha} \right) \right] \end{cases} \tag{13}$$

with

$$\beta^{(2)} = \frac{\varphi^{(1)}(S_{1133}^{(1)} - S_{1133}^{(2)})/}{\varphi^{(2)}(S_{1111}^{(1)} + S_{1122}^{(1)}) + \varphi^{(1)}(S_{1111}^{(2)} + S_{1122}^{(2)})}; \beta^{(1)} = -\frac{\varphi^{(2)}}{\varphi^{(1)}} \beta^{(2)} \tag{14}$$

Case 2: Axial symmetry loading (see Fig. 2b)

The overall stress is determined by the formula:

$$\bar{\sigma} = \bar{\sigma}_{33}[e_3 \otimes e_3 - \frac{1}{2}(e_1 \otimes e_1 + e_2 \otimes e_2)] \tag{15}$$

According to that, the relationship between the overall stress and strain is presented by Eq. (16):

$$\begin{cases} \bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = \bar{S}_{11kl} \bar{\sigma}_{kl} = \left[\bar{S}_{1133} - \frac{1}{2}(\bar{S}_{1111} + \bar{S}_{1122}) \right] \bar{\sigma}_{33} \\ \bar{\varepsilon}_{33} = \bar{S}_{33kl} \bar{\sigma}_{kl} = \left[\bar{S}_{3333} - \frac{1}{2}(\bar{S}_{3311} + \bar{S}_{3322}) \right] \bar{\sigma}_{33} \\ \bar{\varepsilon}_{23} = \bar{\varepsilon}_{13} = \bar{\varepsilon}_{12} = 0 \end{cases} \tag{16}$$

In each layer, the local stress satisfies Eq. (10) by replacing β by γ whereas the overall stress is determined from Eq. (6). Thus, Eq. (15) can be rewritten by:

$$\bar{\sigma}=\sum_{\alpha=1}^{2}\varphi^{\alpha}\bar{\sigma}_{33}e_3\otimes e_3+\sum_{\alpha=1}^{2}\varphi^{\alpha}\gamma^{\alpha}\bar{\sigma}_{33}(e_1\otimes e_1+e_2\otimes e_2) \tag{17}$$

The mean stress in expressions (6), (15) and (17) gives the expression:

$$\sum_{\alpha=1}^{2}\varphi^{\alpha}\gamma^{\alpha}=-\frac{1}{2} \tag{18}$$

Once again apply the continuous condition of strain at the contact surface, i.e., $\varepsilon_{11}^{\alpha=1}=\varepsilon_{11}^{\alpha=2}$ with the local strain determined by:

$$\varepsilon_{11}^{\alpha}=\mathbb{S}_{11kl}^{\alpha}\sigma_{kl}^{\alpha}=\left[S_{1133}^{\alpha}+\gamma^{\alpha}(S_{1111}^{\alpha}+S_{1122}^{\alpha})\right]\bar{\sigma}_{33} \tag{19}$$

one obtains:

$$\left[S_{1133}^{(1)}+\gamma^{(1)}(S_{1111}^{(1)}+S_{1122}^{(1)})\right]=\left[S_{1133}^{(2)}+\gamma^{(2)}(S_{1111}^{(2)}+S_{1122}^{(2)})\right] \tag{20}$$

The coefficients γ^{α} are calculated from Eqs. (18), (20) and they were given as follows:

$$\gamma^{(2)}=\frac{\varphi^{(1)}\left(S_{1133}^{(1)}-S_{1133}^{(2)}\right)-\frac{1}{2}\left(S_{1111}^{(1)}+S_{1122}^{(1)}\right)}{\varphi^{(2)}\left(S_{1111}^{(1)}+S_{1122}^{(1)}\right)+\varphi^{(1)}\left(S_{1111}^{(2)}+S_{1122}^{(2)}\right)};\gamma^{(1)}=\frac{-\frac{1}{2}-\varphi^{(2)}\gamma^{(2)}}{\varphi^{(1)}} \tag{21}$$

Substituting Eq. (18) in Eq. (19) and considering Eq. (16) one has the result:

$$\bar{S}_{1111}+\bar{S}_{1122}=2\bar{S}_{1133}-2\sum_{\alpha=1}^{2}\varphi^{\alpha}\left[S_{1133}^{\alpha}+\gamma^{\alpha}(S_{1111}^{\alpha}+S_{1122}^{\alpha})\right] \tag{22}$$

Case 3: Tensile load combined with shear load in plane (e_1, e_2) ***(see Fig. 2c)***

The overall stress in this case is written by the form:

$$\bar{\sigma}=\bar{\sigma}_{12}(e_1\otimes e_2+e_2\otimes e_1)+\frac{1}{2}(\bar{\sigma}_{11}-\bar{\sigma}_{22})(e_1\otimes e_1-e_2\otimes e_2) \tag{23}$$

Assuming that $\sigma^{\alpha}=\zeta^{\alpha}\bar{\sigma}$ with the condition of the continuity of stress, it is obtained from Eq. (6):

$$\sum_{\alpha=1}^{2}\varphi^{\alpha}\zeta^{\alpha}=1 \tag{24}$$

Using the continuous condition of strain $\varepsilon_{11}^{\alpha=1}=\varepsilon_{11}^{\alpha=2}$ and $\varepsilon_{22}^{\alpha=1}=\varepsilon_{22}^{\alpha=2}$ at the contact surface and taking into account Eq. (23), one derives Eq. (25) below:

$$\begin{cases}\bar{S}_{1111}-\bar{S}_{1122}=\sum_{\alpha=1}^{2}\varphi^{\alpha}\zeta^{\alpha}(S_{1111}^{\alpha}-S_{1122}^{\alpha})\\ \bar{S}_{1122}-\bar{S}_{2222}=\sum_{\alpha=1}^{2}\varphi^{\alpha}\zeta^{\alpha}(S_{2211}^{\alpha}-S_{2222}^{\alpha})\end{cases} \tag{25}$$

with $\zeta^{(2)} = \dfrac{S_{1111}^{(1)} - S_{1122}^{(1)}}{\varphi^{(1)}\left(S_{1111}^{(2)} - S_{1122}^{(2)}\right) + \varphi^{(2)}\left(S_{1111}^{(1)} - S_{1122}^{(1)}\right)}; \zeta^{(1)} = \dfrac{1-\varphi^{(2)}\zeta^{(2)}}{\varphi^{(1)}}$ (26)

Case 4: Longitudinal shear load (see Fig. 2d)

The expression of the overall stress in this case is:

$$\bar{\sigma} = \bar{\sigma}_{13}(e_1 \otimes e_3 + e_3 \otimes e_1) + \bar{\sigma}_{23}(e_2 \otimes e_3 + e_3 \otimes e_2) \tag{27}$$

Supposing the local stress in each layer written by $\sigma^\alpha = \delta^\alpha \bar{\sigma}$, considering the mean stress to be the overall stress, i.e., $\bar{\sigma} = \sum_{\alpha=1}^{2} \varphi^\alpha \bar{\sigma}^\alpha$, brings the result:

$$\sum_{\alpha=1}^{2} \varphi^\alpha \delta^\alpha = 1 \tag{28}$$

Combining with the continuous condition of stress at the contact surface $\sigma_{13}^{\alpha=1} = \delta^{\alpha=1}\bar{\sigma}_{13}$, $\sigma_{13}^{\alpha=2} = \delta^{\alpha=2}\bar{\sigma}_{13}$ one has:

$$\delta^\alpha = 1, \quad \alpha = 1,2 \tag{29}$$

Owing to the hypothesis of homogeneous, isotropic or transversely isotropic layers, substituting Eq. (29) into Eq. (3) and Eq. (27) gives the following equation:

$$\bar{\varepsilon}_{13} = \sum_{\alpha=1}^{n} \varphi^\alpha \left[(S_{1313}^\alpha + S_{1331}^\alpha)\bar{\sigma}_{13} + (S_{1323}^\alpha + S_{1332}^\alpha)\bar{\sigma}_{23} \right] = \bar{S}_{1313}\bar{\sigma}_{13} \tag{30}$$

and thus, it is obtained the result:

$$\bar{S}_{1313} = \sum_{\alpha=1}^{n} \varphi^\alpha S_{1313}^\alpha \tag{31}$$

The equations (13), (22), (25) and (31) are sufficient to determine 5 independent components of the compliance matrix $\bar{\mathbb{S}}$ of the homogenization problem of two material layers (32):

$$\begin{cases} \bar{S}_{1133} = \sum_{\alpha=1}^{2} \varphi^\alpha \left[S_{1133}^\alpha + \beta^\alpha \left(S_{1111}^\alpha + S_{1122}^\alpha \right) \right] \\ \bar{S}_{3333} = \sum_{\alpha=1}^{2} \varphi^\alpha \left[S_{3333}^\alpha + \beta^\alpha \left(S_{3311}^\alpha + S_{3322}^\alpha \right) \right] \\ \bar{S}_{1111} = \sum_{\alpha=1}^{2} \varphi^\alpha \left[(\beta^\alpha - \gamma^\alpha)(S_{1111}^\alpha + \mathbb{S}_{1122}^\alpha) + \frac{1}{2}\zeta^\alpha \left(S_{1111}^\alpha - S_{1122}^\alpha \right) \right] \\ \bar{S}_{1122} = \sum_{\alpha=1}^{2} \varphi^\alpha \left[(\beta^\alpha - \gamma^\alpha)(S_{1111}^\alpha + \mathbb{S}_{1122}^\alpha) + \frac{1}{2}\zeta^\alpha \left(-S_{1111}^\alpha + S_{1122}^\alpha \right) \right] \\ \bar{S}_{1313} = \sum_{\alpha=1}^{n} \varphi^\alpha S_{1313}^\alpha \end{cases} \tag{32}$$

where $\beta^{\alpha}, \gamma^{\alpha}, \zeta^{\alpha}$ are calculated from Eqs. (14), (21) and (26).

It should be noted that, if the materials are isotropic, $\bar{S}_{1122} = \bar{S}_{2211} = \bar{S}_{1133} = \bar{S}_{3311} = \bar{S}_{3322} = \bar{S}_{2233}$ whereas if the materials are transversely isotropic, for example in plane (e_1, e_2), $\bar{S}_{1122} = \bar{S}_{2211} \neq \bar{S}_{1133} = \bar{S}_{3311} = \bar{S}_{3322} = \bar{S}_{2233}$.

2.2. Proposal of a homogenization method for layered materials with n layers

With respect to the problem of determining effective elastic properties of layered materials with *n* layers of material, the multi-stage homogeneous method is proposed as shown in the diagram in Fig. 3. Concretely, in Step 1, homogenizing the first 2 layers (layer 1 and layer 2) according to Eq. (32) one obtains a new layer, a first homogeneous equivalent medium (HEM_1). This HEM_1 is considered the first layer of Step 2, layer 3 becomes now the second layer in this step. Formula (32) gives us the homogeneous layer HEM_2. This layer becomes to the first layer in Step 3 while layer 4 becomes the second layer of this step and so on until the end.

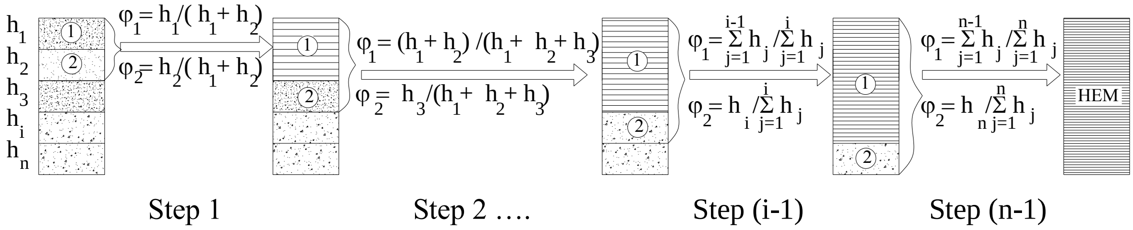

Figure 3. The process of homogenizing the layered materials with n layers.

It should be noted that at the homogenization step (*i*-1), the value of the volume ratio φ_i (i = 1, 2) of layer 1 and layer 2 is redefined by the following formula with h_i the thickness of the i^{th} original layer:

$$\varphi_1 = \sum_{j=1}^{i-1} h_j / \sum_{j=1}^{i} h_j ; \; \varphi_2 = h_i / \sum_{j=1}^{i} h_j \tag{33}$$

By doing so (*n*-1) steps to the last layer one obtains HEM_{n-1}, which is also homogeneous layer of *n* different layers of HEM.

3. Comparison of the proposed method with the four-sub-matrix method

3.1. Case of two-layer material

Considering a two-layer material which consists of two material layers with the properties given in Tab. 1.

Table 1. Properties of two layers in the layered material

Layer	Young modulus E (MPa)	Poisson ratio ν	h (cm)	φ
1	8000	0.3	1	0.2
2	2000	0.13	4	0.8

Applying formula (32) then take the inversion of the effective compliance matrix $\overline{\mathbb{S}}$ one obtains the effective stiffness matrix $\overline{\mathbb{L}}$:

$$\overline{\mathbb{L}} = \begin{pmatrix} 3490,28 & 843,58 & 509,277 & 0 & 0 & 0 \\ 843,58 & 3490,28 & 509,277 & 0 & 0 & 0 \\ 509,277 & 509,277 & 2481,2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1031,99 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1031,99 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1323,35 \end{pmatrix} \tag{34}$$

According to the four-sub-matrix method [5], the stiffness matrix $\mathbb{L}$ is divided into 4 sub-matrices as shown in formula (1), and then applying formula (2) the effective stiffness matrix obtained by this method as follows:

$$\overline{\mathbb{L}} = \begin{pmatrix} 3501,49 & 854,79 & 509,277 & 0 & 0 & 0 \\ 854,79 & 3501,49 & 509,277 & 0 & 0 & 0 \\ 509,277 & 509,277 & 2481,2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1031,99 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1031,99 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1323,35 \end{pmatrix} \tag{35}$$

From the results presented in Eqs. (34) and (35), it is seen that the differences are only appeared in L_{1111}, L_{1122} and with the tiny values of 0.3% and 1.3%, respectively.

3.2. Case of three-layer material

The properties of component layers of a layered material with three layers are given in Tab. 2.

Table 2. Properties of three layers in the layered material

Layer	Young modulus E (MPa)	Poisson ratio ν	h (cm)	φ
1	8000	0.3	1	1/7
2	2000	0.13	4	4/7
3	3000	0.17	2	2/7

At Step 1 of the present method, the effective stiffness matrix $\overline{\mathbb{L}}$ obtained is the one presented in 2.1, i.e., expressed by Eq. (34). At Step 2, the first material now has the properties within the effective stiffness matrix of Step 1 with the volume ratio $\varphi_1 = 1/7 + 4/7 = 5/7$. The third layer in Tab. 2 is the second material at this step with a volume ratio $\varphi_2 = 2/7$. Applying the formula (32) gives:

$$\overline{\mathbb{L}} = \begin{pmatrix} 3412,81 & 789,709 & 544,856 & 0 & 0 & 0 \\ 789,709 & 3412,81 & 544,856 & 0 & 0 & 0 \\ 544,856 & 544,856 & 2656,15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1092,9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1092,9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1311,55 \end{pmatrix} \quad (36)$$

Meanwhile, the result derived from the four-sub-matrix method in [5] is shown as Equation (37):

$$\overline{\mathbb{L}} = \begin{pmatrix} 3422,77 & 799,673 & 544,856 & 0 & 0 & 0 \\ 799,673 & 3422,77 & 544,856 & 0 & 0 & 0 \\ 544,856 & 544,856 & 2656,15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1092,9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1092,9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1311,55 \end{pmatrix} \quad (37)$$

It is observed from Eqs. (36) and (37) that, as also the previous case, the differences in result between two methods are slight. Precisely, they take the values of 0.3% and 1.2% for L_{1111}, L_{1122}, respectively. Furthers, other components of the effective stiffness matrix determined according to two methods are coincided. It should be highlighted that similar results are also obtained when increasing the number of material layers to 4, 5, or 6 layers, etc. Due to the limit of presentation in this paper, the results of these cases will not be expressed here.

From the results obtained for two cases considered above, it should be noted that the method of the four-sub-matrix method in [5] is similar to the multi-stage homogenization method with consideration of 4 independent load cases. Appearance of the small errors can be explained by the influence of the coefficients $\beta^\alpha, \gamma^\alpha, \zeta^\alpha$ in the multi-stage homogenization method which is rounded to 5 numbers after commas. However, it can be highlighted that the multi-stage homogenization method is powerful because when the number of layers increases, the error is still only in two quantities

L_{1111}, L_{1122} and inferior to 1.2%. In addition, it can be used not only for layered materials where the material layer is isotropic but also transversely isotropic.

4. Conclusion

The paper presents the techniques of the multi-stage homogenization method for determination of effective properties of the layered material, i.e., using the formula (32) with (n-1) times for the layered material with n layers. Two cases of two and three layers of layered materials were studied and the results were compared with the ones of the existing analytical method, four-sub-matrix method. The comparison provided a good accordance. The multi-stage homogenization method is a simple, powerful tool that can be used to quickly identify the effective properties of the layered materials. Furthers, this method can be useful for the layered materials with transversely isotropic behavior of component layers.

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PHƯƠNG PHÁP ĐỒNG NHẤT HÓA TỪNG BƯỚC XÁC ĐỊNH ĐẶC TÍNH ĐÀN HỒI CỦA VẬT LIỆU XẾP LỚP

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Tóm tắt: *Bài báo trình bày phương pháp đồng nhất hóa từng bước nhằm xác định tính chất đặc trưng của vật liệu xếp lớp có n lớp đàn hồi, đẳng hướng hoặc đẳng hướng ngang trong phương của lớp. Thủ tục đồng nhất hóa sử dụng (n-1) lần một công thức giải tích cho vật liệu xếp lớp với n lớp. Ứng suất pháp tuyến và biến dạng tại mặt phân giới giữa các lớp được giả thiết là liên tục, có nghĩa là không có sự trượt và tách tại vị trí này. Các kết quả của phương pháp này đã được so sánh với kết quả của lời giải giải tích trong phương pháp bốn ma trận con và cho kết quả tương đồng giữa hai phương pháp. Phương pháp đồng nhất hóa từng bước đã cung cấp một công cụ hiệu quả trong việc đồng nhất hóa vật liệu xếp lớp nhằm xác định nhanh chóng tính chất đặc trưng của vật liệu này.*

Từ khóa: Phương pháp đồng nhất hóa từng bước; phương pháp bốn ma trận con; vật liệu xếp lớp; đẳng hướng ngang; đặc tính đàn hồi.

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