ON THE SANDPILE GROUP OF A CYCLE GRAPH

LE MANH HA, VO THI PHUONG THAO
Hue University of Education
Email: lemanhha@dhsphue.edu.vn

Abstract: In this paper, we compute the sandpile group of a cycle graph by linear algebra approaching. Thereby we give a geometrical description of the cyclic group.

Keywords: Chip Firing Game, Sandpile Model, Sandpile group, Laplacian Matrix, The Smith Normal Form.

1 INTRODUCTION

The Abelian Sandpile Model is a model which displays self-organized criticality. It was first introduced by Bak, Tang and Wiesenfeld in 1987 as the simplest model to display such behavior [5]. The model was originally defined on a finite grid, but can be extended more generally to finite graphs as well. In the model, grains of sand are placed on the vertices of a finite graph. A vertex is said to be stable if the number of grains of sand at the vertex is strictly less than the degree of the vertex. If a vertex becomes unstable - that is, if the number of grains of sand is greater than or equal to the degree - the vertex will topple, sending one grain of sand along each of its adjacent edges. After a toppling of the vertex v_i, the number of grains in this cell decreases by its degree, while the number of those neighbors increases by the number of edges they are adjacent. To ensure a sandpile stabilizes in a finite number of steps, we distinguish a vertex s to be the 'sink.' The sink may collect any number of grains of sand and is never considered unstable. If the graph is connected it is easy to see that from any initial configuration the system reaches a stable configuration in which the number of grains in each vertex is less than its degree.

A stable configuration is recurrent if it is accessible from any other sandpile via a sequence of sand additions and topplings.

Dhar shows also that some configurations, the so-called recurrent configurations,

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play an important role and possess some interesting mathematical properties: they form a finite abelian group (called the *sandpile group*) whose order is equal to the number of spanning trees of the graph. The sandpile automaton was also studied by many authors, and it is referred as the chip-firing game or critical group on a graph [6, 7, 8]. The sandpile group of the graph G is denoted $SP(G)$. The sandpile group can be determined by the Laplacian matrix of G. The collection of recurrent sandpiles under stable addition form a group called the *sandpile group*.

The sandpile group of a connected graph which has no edge connecting the same vertex is closely connected with the graph Laplacian as follows: Let $G = (V, E)$ be a finite graph with n vertices. Then its Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, ..., d_n)$ is the degree matrix and $A(G)$ is the adjacency matrix of G.

In this paper, the main tools will be the Smith normal form for an integer matrix, which can be achieved by row and column operations that are invertible over the ring $\mathbb{Z}$ of integers. Given a square integer matrix A, its Smith normal form is the unique diagonal matrix $S(A) = diag(S_{11}, S_{22}, ..., S_{nn})$ whose entries are nonnegative and S_{ii} divides $S_{i+1,i+1}$. Note that, for each i, the product $S_{11}S_{22} \ldots S_{ii}$ is the greatest common divisor of all $i \times i$ minor determinants of A, this fact will also be used to determine the Smith normal form of an integer matrix. Two matrices $A, B \in \mathbb{Z}_{mn}$ are unimodular equivalent [10] (written $A \sim B$) if there exist matrices $P \in GL(m, \mathbb{Z}), Q \in GL(n, \mathbb{Z})$ such that $B = PAQ$. Equivalently, B is obtainable from A by a sequence of row and column operations mentioned above. It can be seen easily that $A \sim B$ implies $coker A \cong coker B$, and if $A = diag(a_1, a_2, \ldots, a_n)$ then

$$coker A \cong \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \ldots \oplus \mathbb{Z}_{a_n}$$

where $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$. (Of course, $\mathbb{Z}_1$ is the trivial group and $\mathbb{Z}_0 = \mathbb{Z}$.)

Definition 1. *Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The* adjacency matrix *of G is the $n \times n$ matrix $A = (a_{ij})_{n \times n}$ where*

$$a_{ij} = \begin{cases} 1 & \textit{if } \{v_i, v_j\} \in E \\ 0 & \textit{otherwise} \end{cases}$$

Definition 2. *Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The* Laplacian matrix *$L = L(G)$ of G is the $n \times n$ matrix defined by $L = D - A$, where $D = diag(deg(v_i) : i = 1, 2, ..., n)$ is the diagonnal matrix of vertex degrees, and A is the adjacency matrix of G.*

The degree *of the vertex* v_i*, denoted* $deg(v_i)$*, is the number of edges that are connected to the vertex* v_i*.*

Definition 3. *Let* $G = (V, E)$ *be a graph with vertex set* $V = \{v_1, v_2, \ldots, v_n\}$*. The* reduced Laplacian matrix *with respect to a vertex* v_i*, denoted* $L' = L'(G)$*, is the* $(n-1) \times (n-1)$ *matrix formed from the Laplacian matrix by removing the row and the column corresponding to the vertex* v_i*.*

Definition 4. *[9] Let* G *be a finite graph with vertices* $v_1, v_2, \ldots, v_n$*. We single out one vertex,* v_n*, called the* sink*. The sandpile group of* G *is defined as the quotient*

$$SP(G) = \mathbb{Z}^{n-1}/L'\mathbb{Z}^{n-1}.$$

This group was defined independently by Dhar [6], and motivated by the abelian sandpile model of self-organized criticality in statistical physics [5]. In the combinatorics literature, other common names for this group are the critical group [2] and the Jacobian [4].

The sandpile group can be understood combinatorially in terms of chip-firing [3, 2]. A nonnegative vector $u \in \mathbb{Z}^{n-1}$ may be thought of as a chip configuration on G with u_i chips at vertex v_i. A vertex v_i is unstable if $u_i \geq d_i$. An unstable vertex may fire, sending one chip along each incident edge. Note that the operation of firing the vertex v_i corresponds to adding the column vector L'_i to u. We say that a chip configuration u is stable if every non-sink vertex has fewer chips than its degree, so that no vertex can fire. If u is not stable, one can show that by successively firing unstable vertices, in finitely many steps we arrive at a stable configuration u^0. Note that firing one vertex may cause other vertices to become unstable, resulting in a cascade of firings in which a given vertex may fire many times. The order in which firings are performed does not affect the final configuration ; this is the "abelian property" of abelian sandpiles [6].

The structure of sandpile group of a cycle graph is clearly and was proved by Merris on 1992 [10]. In this paper, we give an elementary proof by transfering the reduced Laplacian of the cycle graph C_n to its the Smith normal form.

Theorem 1. *[11] Let* $G = (V, E)$ *be a graph. If* $A = dial(\alpha_1, ..., \alpha_{n-1})$ *is the Smith normal form of the reduced Laplacian matrix* $L'(G)$ *then*

$$SP(G) \cong \mathbb{Z}_{\alpha_1} \oplus ... \oplus \mathbb{Z}_{\alpha_n}$$

where $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$*,* $\mathbb{Z}_1$ *is the trivial group and* $\mathbb{Z}_0 = \mathbb{Z}$*.*

Example 1. *Let C_4 be the cycle of length 4. The adjacency matrix of C_4 is*

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Then the Laplacian matrix of C_4 is

$$L(C_4) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

and the reduce Laplacian matrix respect to the vertex v_4 is

$$L'(C_4) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

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We state now the main result of this paper.

Theorem 2. $SP(C_n) \cong \mathbb{Z}_n$

Proof. To show $SP(C_n) \cong \mathbb{Z}_n$, we must compute the Smith Normal Form of the reduced Laplacian for K_n, thus obtaining the invariant factors for $L(G)$. The reduced Laplacian matrix is for C_n is the $(n-1) \times (n-1)$ matrix:

$$L'(C_n) = \begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & ... & 0 \\ ... & & & & & \\ 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

The Smith Normal form can be obtained by the following process:

First, by adding to the 1^{st}-column by $n-2$ other column, we get

$$L'(C_n) \rightarrow \begin{pmatrix} 1 & -1 & 0 & ... & 0 & 0 \\ 0 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & ... & 0 \\ ... \\ 1 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

Then, by adding to the 2^{nd}-column by $n-2$ other column of the matrix, we obtain

$$L'(C_n) \rightarrow \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & -1 & 0 & ... & 0 \\ 0 & 0 & 2 & -1 & ... & 0 \\ ... \\ 1 & 2 & ... & 0 & -1 & 2 \end{pmatrix}$$

Next, by adding to the 3^{rd}-column by $n-3$ other columns from the 2^{nd}-colums to the $(n-1)^{th}$-columns of the matrix, we have:

$$L'(C_n) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & -1 & 0 & ... & 0 \\ ... \\ 1 & 2 & 3 & ... & 0 & -1 & 2 \end{pmatrix}$$

Similar, we add columns $n-2$ to column $n-1$:

$$L'(C_n) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & 0 & ... & 0 \\ ... \\ 1 & 2 & 3 & ... & n-3 & n-2 & n \end{pmatrix}$$

Lastly, we subtract k times row k from row $n-1$, that mean $r_{n-1} \rightarrow r_{n-1} - kr_k$, $k = 1, 2, ..., n-2$:

$$L'(C_n) \to \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & n \end{pmatrix}$$

So, $SP(C_n) \cong \mathbb{Z}_n$. □

3 CONCLUSION

In this paper, we compute the sandpile group of cycle graph by using the Smith Normal Form for an integer matrix. We confirm that, the sandpile group of cycle graph C_n is exactly the finite cyclic group of order n. This result gives a geometrical description of the cyclic group $\mathbb{Z}_n$.

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