ON THE EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS TO 2D G-BÉNARD PROBLEM IN UNBOUNDED DOMAINS

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ABSTRACT: In this paper, we consider the 2D g-Bénard problem in domains satisfying the Poincaré inequality with homogeneous Dirichlet boundary conditions. We show the existence and uniqueness of strong solutions. The obtained results particularly extend previous results for 2D g-Navier-Stokes equations and 2D Bénard problem.

Key words: g-Bénard problem, strong solution, existence, uniqueness.

SỰ TỒN TẠI VÀ DUY NHẤT NGHIỆM MẠNH ĐỐI VỚI BÀI TOÁN g-BÉNARD 2 CHIỀU TRONG MIỀN KHÔNG BỊ CHẶN

TÓM TẮT: Trong bài báo này, chúng tôi xét bài toán g-Bénard 2 chiều trong miền thỏa mãn bất đẳng thức Poincaré với các điều kiện biên Dirichlet thuần nhất. Chúng tôi chỉ ra sự tồn tại và tính duy nhất của nghiệm mạnh. Kết quả thu được đặc biệt mở rộng các kết quả trước đó cho phương trình g-Navier-Stokes 2 chiều và bài toán Bénard 2 chiều.

Từ khóa: Bài toán g-Bénard, nghiệm mạnh, tồn tại, duy nhất.

1. INTRODUCTION

Let Ω be a (not necessarily bounded) domain in $\mathbb{R}^2$ with boundary Γ. We consider the following two-dimensional (2D) g-Bénard problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \xi\theta + f_1,\ x\in\Omega,\ t>0, \\ \nabla\cdot(gu) = 0,\ x\in\Omega,\quad t>0, \\ \dfrac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta - \kappa\Delta\theta - \dfrac{2\kappa}{g}(\nabla g\cdot\nabla)\theta - \dfrac{\kappa\Delta g}{g}\theta = f_2,\ x\in\Omega, t>0, \\ u = 0,\ x\in\Gamma,\quad t>0, \\ \theta = 0,\ x\in\Gamma,\quad t>0, \\ u(x,0) = u_0(x),\quad x\in\Omega, \\ \theta(x,0) = \theta_0(x),\quad x\in\Omega, \end{cases} \tag{1.1}$$

where $u \equiv u(x,t) = (u_1, u_2)$ is the unknown velocity vector, $\theta \equiv \theta(x,t)$ is the temperature, $p \equiv p(x,t)$ is the unknown pressure, f_1 is the external force function, f_2 is the heat source function, $\nu > 0$ is the kinematic viscosity coefficient, ξ is a constant vector, $\kappa > 0$ is thermal diffusivity, u_0 is the initial velocity and θ_0 is the initial temperature.

The g-Bénard problem is a variation of the Boussinesq equations which consists in a system that couples Navier-Stokes and advection-diffusion heat in oderti model convection in a fluid. Moreover, when $g \equiv \text{const}$ we get the usual Bénard problem, and when $\theta \equiv 0$ we get the g-Navier-Stokes equations. The 2D g-Bénard problem arises when we study the usual 3D Boussinesq equations on thin domains $\Omega_g = \Omega \times (0, g)$. In what follows, we list some related results.

The existence and uniqueness of the weak solution of 2D g-Bénard problem has been studied in [2] for periodic time boundary conditions as well as Dirichlet boundary conditions on bounded domains. Then, in [3] M.Özlük and M. Kaya also study the existence of strong solutions for the 2D g-Bénard problem for periodic time boundary conditions. Thereafter, T.Q. Thinh and L.T. Thuy prove the existence and uniqueness of weak solutions in unbounded domains satisfying the Poincaré inequality with homogeneous Dirichlet boundary conditions, in [6].

The long-time behavior of the strong solutions are important because the numerical computaion of turbulent flows is connected with the computation of the solutions for large time and this will be a subject of a forthcoming work.

We will study the existence and uniqueness of strong solutions to 2D g-Bénard problem in domains that are not necessarily bounded but satisfy the Poincaré inequality. To do this, we assume that the domain Ω and functions f_1, f_2, g satisfy the following hypotheses:

(Ω) The domain Ω is an arbitrary (not necessarily bounded) domain in $\mathbb{R}^2$, provided that the Poincar´e inequality holds on Ω: There exist $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 g dx \le \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 \, g dx, \quad \text{for all } \phi \in C_0^\infty(\Omega); \tag{1.2}$$

(F) $f_1 \in L^2(0,T; H_g)$, $f_2 \in L^2(0,T; L^2(\Omega, g))$;

(G) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \le g(x) \le M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty^2 < m_0^2 \lambda_1, \tag{1.3}$$

where $\lambda_1 > 0$ is the constant in the inequality (1.2).

The article is organized as follows. In Section 2, for convenience of the reader, we recall the functional setting of the 2D g-Bénard problem. Section 3 we show the existence and uniqueness of strong solutions to the problem by combining the Galerkin method and the compactness lemma.

2. PRELIMINARIES

Let $\mathbb{L}^2(\Omega,g)=(L^2(\Omega,g))^2$ and $\mathbb{H}_0^1(\Omega,g)=(H_0^1(\Omega,g))^2$ be endowed with the usual inner products and associated norms. We define

$\mathcal{V}_1=\{u\in(C_0^\infty(\Omega,g))^2:\nabla\cdot(gu)=0\}$, $\mathcal{V}_2=\{\theta\in C_0^\infty(\Omega,g)\}$,

$H_g=$ the closure of $\mathcal{V}_1$ in $\mathbb{L}^2(\Omega,g)$, $V_g=$ the closure of $\mathcal{V}_1$ in $\mathbb{H}_0^1(\Omega,g)$,

$W_g=$ the closure of $\mathcal{V}_2$ in $H_0^1(\Omega,g)$, $V_g'=$ the dual space of V_g,

$W_g'=$ the dual space of W_g,

The inner products and norms in V_g,H_g are given by

$$(u,v)_g=\int_\Omega u\cdot vgdx,\quad u,v\in H_g \text{ and } ((u,v))_g=\int_\Omega\sum_{i,j=1}^{2}\nabla u_j\cdot\nabla v_i gdx,\quad u,v\in V_g$$

and norms $|u|_g^2=(u,u)_g$, $\|u\|_g^2=((u,u))_g$. The norms $|\cdot|_g$ and $\|\cdot\|_g$ are equivalent to the usual ones in $\mathbb{L}^2(\Omega,g)$ and $\mathbb{H}_0^1(\Omega,g)$. We also use $\|\cdot\|_*$ for the norm in V_g', and $\langle\cdot,\cdot\rangle$ for duality pairing between V_g and V_g'.

The inclusions $V_g\subset H_g\equiv H_{g'}\subset V_{g'}, W_g\subset L^2(\Omega,g)\subset W_{g'}$ are valid where each space is dense in the following one and the injections are continuous. By the Riesz representation theorem, it is possible to write $\langle f,u\rangle_g=(f,u)_g,\forall f\in H_g,\forall u\in V_g$.

Also, we define the orthogonal projection P_g as P_g: $H_g\to H_g$ and $\tilde{P}_g$ as $\tilde{P}_g$: $L^2(\Omega,g)\to W_g$. By taking into account the following equality

$$-\frac{1}{g}(\nabla\cdot g\nabla u)=-\Delta u-\frac{1}{g}(\nabla g\cdot\nabla)u,$$

we define the g-Laplace operator and g-Stokes operator as $-\Delta_g u=-\frac{1}{g}(\nabla\cdot g\nabla u)$ and $A_g u=P_g[-\Delta_g u]$, respectively. Since the operators A_g and P_g are self-adjoint, using integration by parts we have

$$\langle A_g u,u\rangle_g=\langle P_g[-\frac{1}{g}(\nabla\cdot g\nabla)u],u\rangle_g=\int_\Omega(\nabla u\cdot\nabla u)gdx=(\nabla u,\nabla u)_g.$$

Therefore, for $u\in V_g$, we can write $|A_g^{1/2}u|_g=|\nabla u|_g=\|u\|_g$.

Next, since the functional $\tau\in W_g\mapsto(\nabla\theta,\nabla\tau)_g\in\mathbb{R}$ is a continuous linear mapping on W_g, we can define a continuous linear mapping $\tilde{A}_g$ on W_g' such that

$$\forall \tau \in W_g, \langle \tilde{A}_g\theta, \tau\rangle_g = (\nabla\theta, \nabla\tau)_g, \text{ for all } \theta \in W_g.$$

We denote the bilinear operator $B_g(u,v) = P_g[(u\cdot\nabla)v]$ and the trilinear form

$$b_g(u,v,w) = \sum_{i,j=1}^{2}\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

where u, v, w lie in appropriate subspaces of V_g. Then, one obtains that $b_g(u,v,w) = -b_g(u,w,v)$, also b_g satisfies the inequality

$$|b_g(u,v,A_g w)| \le c|u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g^{1/2}|A_g v|_g^{1/2}|A_g w|_g. \tag{2.1}$$

where $u \in V_g, v, w \in D(A_g)$.

Similarly, for $u \in V_g$ and $\theta, \tau \in W_g$ we define $\tilde{B}_g(u,\theta) = \tilde{P}_g[(u\cdot\nabla)\theta]$ and

$$\tilde{b}_g(u,\theta,\tau) = \sum_{i,j=1}^{n}\int_\Omega u_i(x)\frac{\partial\theta(x)}{\partial x_j}\tau(x) g dx.$$

Then, one obtains that $\tilde{b}_g(u,\theta,\tau) = -\tilde{b}_g(u,\tau,\theta)$ and $\tilde{b}_g$ satisfies the inequality

$$|b_g(u,v,A_g w)| \le c|u|_g^{1/2}\|u\|_g^{1/2}\|v\|_g^{1/2}|A_g v|_g^{1/2}|A_g w|_g. \tag{2.2}$$

where $u \in V_g, \theta, \tau \in D(A_g)$.

We denote the operators $C_g u = P_g\big[\frac{1}{g}(\nabla g\cdot\nabla)u\big]$ and $\tilde{C}_g\theta = \tilde{P}_g\big[\frac{1}{g}(\nabla g\cdot\nabla)\theta\big]$ such that

$$\langle C_g u, v\rangle_g = b_g(\frac{\nabla g}{g}, u, v), \langle \tilde{C}_g\theta, \tau\rangle_g = \tilde{b}_g(\frac{\nabla g}{g}, \theta, \tau).$$

Finally, let $\tilde{D}_g\theta = \tilde{P}_g[\frac{\Delta g}{g}\theta]$ such that $\langle \tilde{D}_g\theta, \tau\rangle_g = -\tilde{b}_g(\frac{\nabla g}{g}, \theta, \tau) - \tilde{b}_g(\frac{\nabla g}{g}, \tau, \theta)$.

Using the above notations, we can rewrite the system (1.1) as abstract evolutionary equations

$$\begin{cases} \dfrac{du}{dt} + B_g(u,u) + \nu A_g u + \nu C_g u = \xi\theta + f_1, \\ \dfrac{d\theta}{dt} + \tilde{B}_g(u,\theta) + \kappa\tilde{A}_g\theta - \kappa\tilde{C}_g\theta - \kappa\tilde{D}_g\theta = f_2, \\ u(0) = u_0, \theta(0) = \theta_0. \end{cases} \tag{2.3}$$

3. EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS

Definition 3.1. *A pair of functions* (u,θ) *is called a strong solution of problem (2.3) on the interval* $(0,T)$ *if* $u \in L^2(0,T;D(A_g)) \cap L^\infty(0,T;V_g)$ *and*

$\theta \in L^2(0,T;D(\tilde{A}_g)) \cap L^\infty(0,T;W_g)$ *satisfy*

$$\begin{cases} \dfrac{d}{dt}(u,v)_g + b_g(u,u,v) + \nu(\nabla u, \nabla v)_g + \nu b_g(\dfrac{\nabla g}{g}, u, v) = (\xi\theta, v)_g + (f_1, v)_g, \\ \dfrac{d}{dt}(\theta,\tau)_g + \tilde{b}_g(u,\theta,\tau) + \kappa(\nabla\theta, \nabla\tau)_g + \kappa\tilde{b}_g(\dfrac{\nabla g}{g}, \tau, \theta) = (f_2, \tau)_g, \end{cases} \tag{3.1}$$

for all test functions $v \in V_g$ *and* $\tau \in W_g$ *for almost every* $t \in (0,T)$.

Theorem 3.1. *Let the initial data* $(u_0, \theta_0) \in V$ *be given, let the external forces* f_1, f_2 *satisfy hypothesis **(F)** and the function* g *satisfy hypothesis **(G)**. Then there exists a unique strong solution* (u,θ) *of problem (1.1) on the interval* $(0,T)$.

Proof. Existence. We use the standard Galerkin method. Let *m* be an arbitrary but fixed positive integer. For each *m* we define an approximate solution $(u^m(t), \theta^m(t))$ of (3.1) for $1 \le k \le m$ and $t \in [0,T]$ in the form,

$$u^{(m)}(t) = \sum_{j=1}^{m} f_j^{(m)}(t) u_j; \quad \theta^{(m)}(t) = \sum_{j=1}^{m} g_j^{(m)}(t)\theta_j,$$

$$u^{(m)}(0) = u_{m0} = \sum_{j=1}^{m} (a_0, u_j) u_j; \quad \theta^{(m)}(0) = \theta_{m0} = \sum_{j=1}^{m} (\tau_0, \theta_j)\theta_j,$$

$$\begin{aligned} &\frac{d}{dt}(u^{(m)}, A_g u_k)_g + b_g(u^{(m)}, u^{(m)}, A_g u_k) + \nu((u^{(m)}, A_g u_k))_g \\ &+ \nu b_g(\frac{\nabla g}{g}, u^{(m)}, A_g u_k) = (\xi\theta^{(m)}, A_g u_k)_g + (f_1, A_g u_k)_g, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\frac{d}{dt}(\theta^{(m)}, \tilde{A}_g \theta_k)_g + \tilde{b}_g(u^{(m)}, \theta^{(m)}, \tilde{A}_g \theta_k) + \kappa((\theta^{(m)}, \tilde{A}_g \theta_k))_g \\ &+ \kappa\tilde{b}_g(\frac{\nabla g}{g}, \tilde{A}_g \theta_k, \theta^{(m)}) = (f_2, \tilde{A}_g \theta_k)_g. \end{aligned} \tag{3.3}$$

This system forms a nonlinear first order system of ordinary differential equations for the functions $f_j^{(m)}(t)$ and $g_j^{(m)}(t)$ and has a solution on some maximal interval of existence $[0, T_m)$.

We multiply (3.2) and (3.3) by $f_j^{(m)}(t)$ and $g_j^{(m)}(t)$ respectively, then add these equations for $k = 1, \ldots, m$. Next, using (1.2), (1.3), (2.1), (2.1) and Cauchy-Schwarz's inequality, we obtain

$$\frac{d}{dt}\| u^{(m)}(t)\|_g^2 + \nu' | A_g u^{(m)}(t)|_g^2$$
$$\leq \frac{27c}{2\epsilon^3\nu^3}|u^{(m)}|_g^2 \| u^{(m)}\|_g^4 + \frac{2}{\epsilon\nu}\|\xi\|_\infty^2 |\theta^{(m)}|_g^2 + \frac{1}{\epsilon\nu}|f_1|_g^2, \tag{3.4}$$
$$\frac{d}{dt}\| \theta^{(m)}(t)\|_g^2 + \kappa' | \tilde{A}_g \theta^{(m)}(t)|_g^2$$
$$\leq \frac{27c}{2\epsilon^3\kappa^3}|u^{(m)}|_g^2 \| u^{(m)}\|_g^2 \|\theta^{(m)}\|_g^2 + \frac{2\kappa\|\Delta g\|_\infty^2}{\epsilon m_0^2}|\theta^{(m)}|_g^2 + \frac{1}{\epsilon\kappa}|f_2|_g^2, \tag{3.5}$$

where $\nu' = 2\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \epsilon\right), \kappa' = 2\kappa\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \epsilon\right)$

and $\epsilon > 0$ is chosen such that $\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} - \epsilon\right) > 0.$

Setting (see [6])

$$g(t) = \frac{27c}{2\epsilon^3\nu^3}|u^{(m)}|_g^2 \| u^{(m)}\|_g^2 \leq C_1,\ h(t) = \frac{2}{\epsilon\nu}\|\xi\|_\infty^2|\theta^{(m)}|_g^2 + \frac{1}{\epsilon\nu}|f_1|_g^2 \leq C_2,$$
$$\tilde{g}(t) = \frac{27c}{2\epsilon^3\kappa^3}|u^{(m)}|_g^2 \| u^{(m)}\|_g^2 + \frac{2\kappa\|\Delta g\|_\infty^2}{\lambda_1\epsilon m_0^2} \leq C_3,\ \tilde{h}(t) = \frac{1}{\epsilon\kappa}|f_2|_g^2 \leq C_4,$$

where C_1, C_2, C_3 and C_4 are positive constants.

We have

$$\frac{d}{dt}\| u^{(m)}(t)\|_g^2 + \nu' | A_g u^{(m)}(t)|_g^2 \leq g(t)\| u^{(m)}\|_g^2 + h(t), \tag{3.6}$$
$$\frac{d}{dt}\| \theta^{(m)}(t)\|_g^2 + \kappa' | \tilde{A}_g \theta^{(m)}(t)|_g^2 \leq \tilde{g}(t)\| \theta^{(m)}\|^2 + \tilde{h}(t). \tag{3.7}$$

Appyling the Gronwall's inequality to (3.6) and (3.7), we see that

$$\| u^{(m)}(t)\|_g^2 \leq \| u^{(m)}(0)\|_g^2\, e^{\int_0^t g(r)dr} + \int_0^t e^{\int_0^t g(r)dr - \int_0^t g(s)ds}\, h(s)ds,$$
$$\| \theta^{(m)}(t)\|_g^2 \leq \| \theta^{(m)}(0)\|_g^2\, e^{\int_0^t \tilde{g}(r)dr} + \int_0^t e^{\int_0^t \tilde{g}(r)dr - \int_0^t \tilde{g}(s)ds}\, \tilde{h}(s)ds,$$

with $0 \leq t \leq T$.

Then we have: $\sup\limits_{t\in[0,T]} \| u^{(m)}(t)\|_g^2 \leq C_5$ and $\sup\limits_{t\in[0,T]} \| \theta^{(m)}(t)\|_g^2 \leq C_6,$ (3.8)

where C_5 and C_6 are positive constants.

Integrating (3.4) and (3.5) from 0 to T, we obtain

$$\| u^{(m)}(T) \|_g^2 + \nu' \int_0^T |A_g u^{(m)}(t)|_g^2 \, dt \le \| u_0 \|_g^2 + \frac{27c}{2\epsilon^3 \nu^3} \int_0^T |u^{(m)}(t)|_g^2 \| u^{(m)}(t) \|_g^4 \, dt$$

$$+ \frac{2}{\epsilon \nu} \| \xi \|_\infty^2 \int_0^T |\theta^{(m)}(t)|_g^2 \, dt + \frac{1}{\epsilon \nu} \int_0^T |f_1(t)|_g^2 \, dt,$$

$$\| \theta^{(m)}(T) \|_g^2 + \kappa' \, | \tilde{A}_g \theta^{(m)}(t) |_g^2 \le \| \theta_0 \|_g^2 + \frac{27c}{2\epsilon^3 \kappa^3} \int_0^T |u^{(m)}(t)|_g^2 \| u^{(m)}(t) \|_g^2 \| \theta^{(m)}(t) \|_g^2 \, dt$$

$$+ \frac{2\kappa \| \Delta g \|_\infty^2}{\epsilon m_0^2} \int_0^T |\theta^{(m)}(t)|_g^2 \, dt + \frac{1}{\epsilon \kappa} \int_0^T |f_2(t)|_g^2 \, dt,$$

Furthermore, we have: $\int_0^T |A_g u^{(m)}(t)|_g^2 \, dt \le C_7$ and $\int_0^T |\tilde{A}_g \theta^{(m)}(t)|_g^2 \, dt \le C_8$, (3.9)

where the C_7 and C_8 are positive constants.

Hence, in particular, from (3.8) and (3.9)we see that

$\{u^m\}$ is bounded in $L^\infty(0,T;V_g)$, $\{\theta^m\}$ is bounded in $L^\infty(0,T;W_g)$.

$\{u^m\}$ is bounded in $L^2(0,T;D(A_g)$, $\{\theta^m\}$ is bounded in $L^2(0,T;D(\tilde{A}_g))$.

We establish uniform estimates, in m, for $\frac{du^{(m)}}{dt}$ and $\frac{d\theta^{(m)}}{dt}$. Let us recall (2.3), we have

$$\frac{du^m}{dt} = -B_g(u^m, u^m) - \nu A_g u - \nu C_g u^m + \xi \theta^m + f_1,$$

$$\frac{d\theta^m}{dt} = -\tilde{B}_g(u^m, \theta^m) - \kappa \, \tilde{B}_g(u^m, \theta^m) - \kappa \, \tilde{A}_g \theta^m + \kappa \, \tilde{C}_g \theta^m + \kappa \tilde{D}_g \theta^m + f_2.$$

Applying (2.1), we obtain

$$\int_0^T |B_g(u^{(m)}(t), u^{(m)}(t)|_g^4 \le c \int_0^T |u^{(m)}(t)|_g^2 \, | A_g u^{(m)}(t)|_g^2 \| u^{(m)}(t) \|_g^4 \, dt$$

$$\le c \int_0^T |A_g u^{(m)}(t)|_g^2 \| u^{(m)}(t) \|_g^6 \, dt \le c \, \| u^{(m)}(t) \|_{L^\infty(0,T;V_g)}^6 \int_0^T |A_g u^{(m)}(t)|_g^2 \, dt.$$

And $$\int_0^T |C_g(u^{(m)}(t))|_g^2 \, dt \le c \int_0^T \| \nabla g \|_\infty \| u \|_g^2 \, dt \le c \, \| \nabla g \|_\infty \int_0^T \| u \|_{V_g'} dt.$$

Therefore, $B_g(u^{(m)}, u^{(m)})$ belongs to the space $L^4(0,T;H_g)$ hence it belongs to $L^2(0,T;H_g)$ and $C_g u^{(m)}(t)$ also belongs to $L^2(0,T;H_g)$. As a result $\frac{du^{(m)}}{dt} \in L^2(0,T;H_g)$. Similarly, we also have $\frac{d\theta^{(m)}}{dt} \in L^2(0,T;L^2(\Omega, g))$.

Therefore, by the Aubin's compactness theorem (see, e.g., [1] or [4]) we conclude that there exist subsequences of $\{u^{(m)}\}$ and $\{\theta^{(m)}\}$, still denoted by $\{u^{(m)}\}$ and $\{\theta^{(m)}\}$ such that

$$u \in L^\infty(0,T;V_g) \cap L^2(0,T;D(A_g)), \frac{du}{dt} \in L^2(0,T;H_g),$$
$$\theta \in L^\infty(0,T;W_g) \cap L^2(0,T;D(\tilde{A}_g)), \frac{d\theta}{dt} \in L^2(0,T;L^2(\Omega,g)),$$

where

$u^{(m)} \rightharpoonup u$ in $L^2(0,T;D(A_g)), u^{(m)} \to u$ in $L^2(0,T;V_g)$ and $u^{(m)} \rightharpoonup^* u$ in $L^2(0,T;V_g)$,

$\theta^{(m)} \rightharpoonup \theta$ in $L^2(0,T;D(\tilde{A}_g)), \theta^{(m)} \to \theta$ in $L^2(0,T;W_g)$ and $\theta^{(m)} \rightharpoonup^* \theta$ in $L^\infty(0,T;V_g)$,

as $m \to \infty$.

Then we can pass to the limit in the equations. Let $w_1 \in D(A_g)$ and $w_2 \in D(\tilde{A}_g)$. We multiply (3.2) and (3.3) by $A_g w_1$, $\tilde{A}_g w_2$ respectively and then integrate by parts

$$(u^{(m)}, A_g w_1) + \int_{t_0}^t (b_g(u^{(m)}(s), u^{(m)}(s), P_m A_g w_1) ds + \nu \int_{t_0}^t (A_g u^{(m)}(s), A_g w_1) ds$$
$$+\nu \int_{t_0}^t (b_g(\frac{\nabla g}{g}, u^{(m)}(s), P_m A_g w_1) ds = (u^{(m)}(t_0), A_g w_1) + \int_{t_0}^t (\xi \theta^{(m)}(s), A_g w_1) ds + \int_0^t (f_1, A_g w_1)_g,$$
$$(\theta^{(m)}, \tilde{A}_g w_2) + \int_{t_0}^t (\tilde{b}_g(u^{(m)}(s), \theta^{(m)}(s)), P'_m \tilde{A}_g w_2) ds + \kappa \int_{t_0}^t (\tilde{A}_g \theta^{(m)}(s), \tilde{A}_g w_2) ds$$
$$+\kappa \int_{t_0}^t \tilde{b}_g(\frac{\nabla g}{g}, \theta^{(m)}, \tilde{A}_g w_2) ds = (\theta^{(m)}(t_0), \tilde{A}_g w_2) + \int_{t_0}^t (f_2, P'_m \tilde{A}_g w_2) ds,$$

for all $t_0, t \in [0,T]$.

Since $u^{(m)} \to u$ in $L^2(0,T;V_g)$ and $\theta^{(m)} \to \theta$ in $L^2(0,T;W_g)$ then

$$(u^{(m)}(t), A_g w_1) - (u^{(m)}(t_0), A_g w_1) \to (u(t), A_g w_1) - (u(t_0), A_g w_1),$$
$$(\theta^{(m)}(t), \tilde{A}_g w_2) - (\theta^{(m)}(t_0), \tilde{A}_g w_2) \to (\theta(t), \tilde{A}_g w_2) - (\theta(t_0), \tilde{A}_g w_2),$$

as $m \to \infty$.

For the nonlinear term

$$\left| \int_{t_0}^t (b_g(u^{(m)}(s), u^{(m)}(s), P_m A_g w_1) - (b_g(u(s), u(s), A_g w_1) ds \right|$$
$$\leq \left| \int_{t_0}^t (b_g(u^{(m)}(s), u^{(m)}(s), P_m A_g w_2 - A_g w_1) ds \right| + \left| \int_{t_0}^t (b_g(u^{(m)}(s) - u(s), u^{(m)}(s), A_g w_1) ds \right|$$
$$+ \left| \int_{t_0}^t (b_g(u(s), u^{(m)}(s) - u(s), A_g w_1) ds \right| := I_m^{(1)} + I_m^{(2)} + I_m^{(3)}.$$

Using Cauchy-Schwarz's inequality, Hölder inequality, estimates (1.2) - (2.1), we get

$$I_m^{(1)} \leq \frac{c}{\lambda_1^{1/2}} \| u^m \|_{L^2(0,T;V_g)} | A_g u |_{L^2(0,T;D(A_g))} | P_m A_g w_1 - A_g w_1 |.$$

$$I_m^{(2)} \le \frac{c}{\lambda_1^{1/2}} | A_g w_1 \| A_g u^m(s) |_{L^2(0,T;V_g)} \| u^m - u \|_{L^2(0,T;V_g)}.$$

$$I_m^{(3)} \le \frac{c}{\lambda_1^{1/8}} | A_g w_1 \| A_g u |_{L^2(0,T;V_g)} \| u^m(s) - u(s) \|_{L^2(0,T;V_g)}.$$

Since $u^{(m)}(s) \to u$ in $L^2(0,T;V_g)$ and $u^{(m)}(s) \rightharpoonup u$ in $L^2(0,T;D(A_g))$, we have

$$\lim_{m\to\infty} I_m^{(1)} = 0, \quad \lim_{m\to\infty} I_m^{(2)} = 0, \quad \lim_{m\to\infty} I_m^{(3)} = 0.$$

From the above result, we get

$$\lim_{m\to\infty} \int_{t_0}^{t} (b_g(u^{(m)}(s), u^{(m)}(s), P_m A_g w_1) ds = \int_{t_0}^{t} (b_g(u(s), u(s), A_g w_1) ds$$

Thus, $\lim_{m\to\infty} \int_{t_0}^{t} (\tilde{b}_g(u^{(m)}(s), \theta^{(m)}(s), P_m' \tilde{A}_g w_2) ds = \int_{t_0}^{t} (\tilde{b}_g(u(s), \theta(s), \tilde{A}_g w_2) ds,$

$$\lim_{m\to\infty} \int_{t_0}^{t} b_g\left(\frac{\nabla g}{g}, u^{(m)}(s), P_m A_g w_1\right) ds = \int_{t_0}^{t} b_g\left(\frac{\nabla g}{g}, u(s), A_g w_1\right) ds,$$

$$\lim_{m\to\infty} \int_{t_0}^{t} \tilde{b}_g\left(\frac{\nabla g}{g}, \theta^{(m)}(s), P_m' \tilde{A}_g w_2\right) ds = \int_{t_0}^{t} (\tilde{b}_g\left(\frac{\nabla g}{g}, \theta(s), \tilde{A}_g w_2\right) ds.$$

Following the technique given in [5], as $m \to \infty$ we obtain pass limit in the equations (3.2) and (3.3). Furthermore, applying similar techniques given in [5] it is easy to show that (u, θ) satisfies the initial conditions $u(0) = u_0$ and $\theta(0) = \theta_0$.

Uniqueness. Let be two system of equations of the g-Bénard problem on the interval $(0,T)$ with the given data $u_1(0), \theta_1(0), f_{11}, f_{12}$ and $u_2(0), \theta_2(0), f_{21}, f_{22}$ such that the systems have two strong solutions u_1, θ_1 and u_2, θ_2 respectively.

$$\frac{du_1}{dt} + B_g(u_1, u_1) + \nu A_g u_1 + \nu C_g u_1 = \xi \theta_1 + f_{11},$$

$$\frac{d\theta_1}{dt} + \tilde{B}_g(u_1, \theta_1) + \kappa \tilde{A}_g \theta_1 - \kappa \tilde{C}_g \theta_1 - \kappa \tilde{D}_g = f_{12},$$

$$\frac{du_2}{dt} + B_g(u_2, u_2) + \nu A_g u_1 + \nu C_g u_2 = \xi \theta_1 + f_{21},$$

$$\frac{d\theta_2}{dt} + \tilde{B}_g(u_2, \theta_2) + \kappa \tilde{A}_g \theta_2 - \kappa \tilde{C}_g \theta_2 - \kappa \tilde{D}_g = f_{22}.$$

Putting $u_1 - u_2 = u, \theta_1 - \theta_2 = \theta, f_{11} - f_{12} = f_1$ and $f_{21} - f_{22} = f_2$. Then, multiplying these two equations with $A_g u$ and $\tilde{A}_g \theta$ respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_g^2+b_g(u,u_1,A_gu)+b_g(u_2,u,A_gu)+\nu|A_gu|_g^2+\nu b_g\left(\frac{\nabla g}{g},u,A_gu\right)$$
$$=(\xi\theta)_g+(f_1,A_gu)_g,$$
$$\frac{1}{2}\frac{d}{dt}\|\theta\|_g^2+\tilde{b}_g(u,\theta_1,\tilde{A}_g)+\tilde{b}_g(u_2,\theta,\tilde{A}_g\theta)+\kappa|\tilde{A}_g\theta|_g^2+\kappa\tilde{b}_g\left(\frac{\nabla g}{g},\tilde{A}_g\theta,\theta\right)$$
$$=(f_2,\tilde{A}_g\theta)_g.$$

Next, the application of the Cauchy - Schwarz and Young inequalities results in the following inequality,

$$\frac{d}{dt}\left(\|u\|_g^2+\|\theta\|_g^2\right)\le K(t)\left(\|u\|_g^2+\|\theta\|_g^2\right)+\frac{1}{\nu\epsilon}\|f_1\|_\infty^2+\frac{1}{\kappa\epsilon}\|f_2\|_\infty^2,$$

where $K_1(t)=\frac{2}{\epsilon\nu\lambda^{1/2}}|u_1|_g|A_gu_1|_g+\frac{27c}{2\epsilon^3\nu^3}|u_2|_g^2\|u_2\|_g^2+\frac{2}{\epsilon\kappa\lambda^{1/2}}\|\theta_1\|_g|\tilde{A}_g\theta_1|_g,$

$$K_2(t)=\frac{27c}{2\epsilon^3\kappa^2}|u_2|_g^2\|u_2\|_g^2+\frac{2\kappa\|\Delta g\|_\infty^2}{\epsilon m_0^2\lambda_1}+\frac{2}{\epsilon\nu\lambda_1}\|\xi\|_\infty^2,$$

$$K(t)=\max_{t\in[0,T]}\{K_1(t),K_2(t)\}.$$

Thanks to the Gronwall inequality, we have

$$\|u(t)\|_g^2+\|\theta(t)\|_g^2\le e^{\int_0^t K(s)ds}\left(\|u(0)\|_g^2+\|\theta\|_g^2\right)+\frac{t}{\nu\epsilon}\|f_1\|_\infty^2+\frac{t}{\kappa\epsilon}\|f_2\|_\infty^2.$$

Hence, the continuous dependence of the strong solution on the initial data in any bounded interval for all $t \ge 0$. In particular, the solution is unique.

REFERENCES

1. P. Constantin, C. Foias (1988), *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago.
2. M.Özlük and M. Kaya (2018), *On the weak solutions and determining modes of the g-Bénard problem*, Hacet. J. Math. Stat. 47, 1453-1466.
3. M.Özlük and M. Kaya (2018), *On the strong solutions and the structural stability of the g-Bénard problem*, Numer. Funct. Anal. Optim. 39, 383-397
4. R. Temam (1979), *Navier-Stokes equations: Theory and numerical analysis*, second ed., NorthHolland, Amsterdam.
5. R. Temam (1984), *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, The Netherlands, 3rd edition.
6. T.Q. Thinh and L.T. Thuy (2020), *On the existence and uniqueness of solutions to 2D g-Bénard problem*, Hnue Journal of science. 62.