A NEUTRAL VERSION OF THE DIFFUSIVE HUTCHINSON EQUATION: THE ADMISSIBLY STABLE MANIFOLDS AND ASYMPTOTIC BEHAVIOR

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ABSTRACT:

Consider the neutral evolution equation $\frac{\mathrm{d}}{\mathrm{d}t}Fu_t = B(t)u(t) + \Phi(t,u_t),$ under the conditions that the family of linear partial differential operators $\left(B(t)\right)_{t\geq 0}$ generates the evolutionary process $\left(U(t,s)\right)_{t\geq s\geq 0}$ having an exponential dichotomy on the half-line and the nonlinear delay operator $\Phi(t,\cdot)$ satisfies the φ-Lipschitz condition, where φ belongs to some admissible space on the half-line. The existence of admissibly stable manifolds of $\mathcal{E}$-class is obtained in Nguyen - Trinh [Taiwan. J. Math. 23 (2019), 897-923]. This short paper discusses the application of the theoretical result to study asymptotic behavior of a neutral version of the diffusive Hutchinson equation..

Keywords: Exponential dichotomy, neutral evolution equations, stable manifolds, admissibly stable manifolds, admissible spacesg

MỘT PHIÊN BẢN TRUNG TÍNH CỦA PHƯƠNG TRÌNH HUTCHINSON CÓ KHUẾCH TÁN: ĐA TẠP ỔN ĐỊNH CHẤP NHẬN ĐƯỢC VÀ DÁNG ĐIỆU TIỆM CẬN

ABSTRACT:

Xét phương trình tiến hoá trung tính $\frac{\mathrm{d}}{\mathrm{d}t}Fu_t = B(t)u(t) + \Phi(t,u_t),$ với các điều kiện họ toán tử đạo hàm riêng tuyến tính $\left(B(t)\right)_{t\geq 0}$ sinh ra quá trình tiến hóa $\left(U(t,s)\right)_{t\geq s\geq 0}$ có nhị phân mũ và toán tử phi tuyến $\Phi(t,\cdot)$ thỏa mãn điều kiện φ-Lipschitz, trong đó φ thuộc vào một không gian chấp nhận được trên nửa đường thẳng. Sự tồn tại của đa tạp ổn định chấp nhận được $\mathcal{E}$-lớp đã nhận được trong công trình Nguyen - Trinh [Taiwan. J. Math. 23 (2019), 897-923]. Bài báo này thảo luận về áp dụng của kết quả lý thuyết đó và nghiên cứu dáng điệu tiệm cận của một phiên bản trung tính của phương trình Hutchinson có khuếch tán.

Từ khoá: Nhị phân mũ, phương trình tiến hoá trung tính, đa tạp ổn định, đa tạp ổn định chấp nhận được, không gian chấp nhận được.

1. INTRODUCTION

Consider the following neutral evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Fu_t = B(t)u(t)+\Phi(t,u_t), \tag{1.1}$$

with the initial datum $u_0=\phi\in\mathcal{C}:=C([-r,0],X)$ where X is a Banach space, $B(t):(D(B),\|\cdot\|_{D(B)})\subset X\to X$ is a (possibly unbounded) linear operator for every fixed $t\geqslant 0$ and $\|B(t)x\|\leqslant K\|x\|_{D(B)}$ for $x\in D(B)$. That is to say, $B(t)$ has the same domain of definition denoted by $D(B)$ for all $t>0$. Furthermore, $F:\mathcal{C}\to D(B)$ is a bounded linear operator called a difference operator; $\Phi:\mathbb{R}_+\times\mathcal{C}\to X$ is a continuous nonlinear operator called a delay operator, and u_t is the history function defined by $u_t(\theta):=u(t+\theta)$ for $\theta\in[-r,0]$.

The existence of invariant manifolds to differential equations has been known for a century. It is a central object and an extremely powerful tool in the study of nonlinear systems. The reader is referred to any books on nonlinear differential equations for the history of the problem (see, e.g., [3, 5, 8, 10] and the references therein). Recently, Nguyen [5] has proved the existence of a new type of invariant manifolds, namely the *invariant stable manifolds of admissible classes*. Such manifolds consist of solutions' trajectories belonging to wide classes of admissible Banach spaces which can be L_p-spaces, Lorentz $L_{p,q}$ or certain interpolation spaces.

The motivation of this paper is as follows. Petzeltová – Staffans [9] studied integrodifferential convolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(x+\mu*x)-Ax-\nu*x=f,$$

where A is the generator of an analytic semigroup on a Hilbert space H. Hernández – Trofimchuk [2] (see also Hernández – Wu [1]) consider a neutral version of the diffusive Hutchinson equation

$$\frac{\partial}{\partial t}[w(t,x)-kw(t-\tau,x)]-\Delta w(t,x)=w(1-w(t-\tau,x)).$$

These observations suggest that we should consider the neutral evolution equations

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}Fu_t &= B(t)u(t)+\Phi(t,u_t), \quad t>s,\\ u_s &= \phi\in\mathcal{C}.\end{cases} \tag{NEE}$$

For the neutral evolution equations (NEE), Nguyen – Trinh [8] studied the existence of *admissibly stable manifolds* (Nguyen [5] called it invariant manifolds of admissible classes, invariant manifolds of $\mathcal{E}$-class) (see Definition 2.4 below) which are constituted by trajectories of solutions belonging to certain Banach space $\mathcal{E}$. The existence of these manifolds is obtained in the case that the linear part $(B(t))_{t\geqslant 0}$ generates the evolutionary process having an exponential dichotomy on the half-line, and its nonlinear term is φ-Lipshitz, i.e, $\|\Phi(t,\phi)-\Phi(t,\psi)\|\leqslant\varphi(t)\|\phi-\psi\|_{\mathcal{C}}$, where $\phi,\psi\in\mathcal{C}$ and $\varphi(\cdot)$ is a real and positive function which belong to an admissible space. Besides, regarding the admissibly stable manifolds, Nguyen – Bui [7] recently discussed the existence of an admissibly inertial manifold for semi-linear evolution equations involving sectorial operators.

This short paper discusses the application of the theoretical result Nguyen – Trinh [8, Theorem 2.8] to study asymptotic behavior of a neutral version of the diffusive Hutchinson equation.

2. EXPONENTIAL DICHOTOMY ANDINVARIANT STABLE MANIFOLDS OF $\mathcal{E}$-CLASS

Let X be a Banach space (with a norm $\|\cdot\|$) and for a given $r>0$ we denote by $\mathcal{C}:=C([-r,0],X)$ the Banach space of all continuous functions from $[-r,0]$ into X, equipped with the norm $\|\phi\|_{\mathcal{C}}=\sup_{\theta\in[-r,0]}\|\phi(\theta)\|$ for $\phi\in\mathcal{C}$. For a continuous function $w:[-r,\infty)\to X$ the *history function* $w_t\in\mathcal{C}$ is defined by $w_t(\theta)=w(t+\theta)$ for all $\theta\in[-r,0]$.

Definition 2.1 *A family of bounded linear operators* $(U(t,s))_{t\geqslant s\geqslant 0}$ *on Banach space* X *is a (strongly continuous, exponentially bounded) evolutionary process if [* (1)]

1. $U(t,t)=I$ and $U(t,r)U(r,s)=U(t,s)$ for all $t\geqslant r\geqslant s\geqslant 0$;
2. the map $(t,s)\mapsto U(t,s)x$ is continuous for every $x\in X$;
3. there are constants $K,\nu\geqslant 0$ such that $\|U(t,s)x\|\leqslant Ke^{\nu(t-s)}\|x\|$ for all $t\geqslant s\geqslant 0$ and $x\in X$.

Definition 2.2 *An evolutionary process* $(U(t,s))_{t\geqslant s\geqslant 0}$ *on the Banach space* X *is said to have an exponential dichotomy on* $[0,\infty)$ *if there exist bounded linear projections* $P(t), t\geqslant 0$*, on* X *and positive constants* N,ν *such that [* (1)]

1. $U(t,s)P(s)=P(t)U(t,s)$, for $t\geqslant s\geqslant 0$;
2. the restriction $U(t,s)_{|}:\ker P(s)\to\ker P(t)$, for $t\geqslant s\geqslant 0$, is an isomorphism, and we denote its inverse by $U(s,t)_{|}:=(U(t,s)_{|})^{-1}$, $0\leqslant s\leqslant t$;
3. $\|U(t,s)x\|\leqslant Ne^{-\nu(t-s)}\|x\|$ for $x\in P(s)X$ and $t\geqslant s\geqslant 0$;
4. $\|U(s,t)_{|}x\|\leqslant Ne^{-\nu(t-s)}\|x\|$ for $x\in\ker P(t)$ and $t\geqslant s\geqslant 0$.

The projections $P(t)$, for $t\geqslant 0$, are called the *dichotomy projections*, and the constants N,ν are the *dichotomy constants*.

Note that the exponential dichotomy of the evolutionary process $(U(t,s))_{t\geqslant s\geqslant 0}$ implies that $H:=\sup_{t\geqslant 0}\|P(t)\|<\infty$ and the map $t\mapsto P(t)$ is strongly continuous (see [4, Lemma 4.2]).

Next, using the projections $(P(t))_{t\geqslant 0}$ on X, we can define the family of operators $\left(\mathcal{P}(t)\right)_{t\geqslant 0}$ on $\mathcal{C}$, $\mathcal{P}(t):\mathcal{C}\to\mathcal{C}$ defined by

$$\left(\mathcal{P}(t)\phi\right)(\theta)=U(t-\theta,t)P(t)\phi(0),\qquad \textit{for all } \theta\in[-r,0]. \tag{2.1}$$

Then, we have that $\mathcal{P}(t)^2=\mathcal{P}(t)$, and therefore the operators $\mathcal{P}(t)$, $t\geqslant 0$, are projections on $\mathcal{C}$. Moreover,

$$\mathcal{P}(t)\mathcal{C}=\left\{\phi\in\mathcal{C}:\left[(\forall\theta\in[-r,0])(\exists\nu_0\in P(t)X):\phi(\theta)=U(t-\theta,t)\right]\right\}. \tag{2.2}$$

To obtain the existence of invariant stable manifolds we also need the following notion of the φ-Lipschitz of the nonlinear delay term Φ.

Definition 2.3 ($$-Lipschitz function)

Let E be an admissible space and φ be a positive function belonging to E. A function $\Phi:[0,\infty)\times\mathcal{C}\to X$ is said to be φ*-Lipschitz* if Φ satisfies [(1)]

1. $\|\Phi(t,0)\|\leqslant\varphi(t)$ for all $t\in\mathbb{R}_+$
2. $\|\Phi(t,\phi_1)-\Phi(t,\phi_2)\|\leqslant\varphi(t)\|\phi_1-\phi_2\|_{\mathcal{C}}$ for all $t\in\mathbb{R}_+$ and all $\phi_1,\phi_2\in\mathcal{C}$.

Standing Hypothesis 1 *We assume the following hypotheses on family of linear operators* $B(t)$*, the difference operator* F*, and the nonlinear term* Φ*: [(1)]*

1. The family of linear operators $(B(t))_{t\geqslant 0}$ generates an evolutionary process $(U(t,s))_{t\geqslant s\geqslant 0}$ as defined in Definition 2.1. Furthermore, the domain of each $B(t)$ is independent of t, and is denoted by $D(B)$ which is a Banach space with norm $\|\cdot\|_{D(B)}$ such that

$$\|B(t)x\|\leqslant K\|x\|_{D(B)},\quad \textit{for all } x\in D(B). \tag{2.3}$$

2. The difference operator $F:\mathcal{C}\to D(B)$ is of the form $F\phi=\phi(0)-\Psi\phi$ for all $\phi\in\mathcal{C}$ where $\Psi\in\mathcal{L}(\mathcal{C},D(B))$ satisfies $\|\Psi\|<1$.

3. The nonlinear term Φ is φ-Lipschitz as defined in Definition 2.3.

Definition 2.4 (see Nguyen) *Let* E *be a Banach function space and let* X *be a Banach space endowed with the norm* $\|\cdot\|$*. Denote by* $\mathrm{SM}(\mathbb{R}_+)$ *of strongly measurable functions on* $\mathbb{R}_+$*. We set*

$$\mathcal{E}:=\mathcal{E}(\mathbb{R}_+,\mathcal{C})=\{f:\mathbb{R}_+\to\mathcal{C}: f\in\mathrm{SM}(\mathbb{R}_+),\|f(\cdot)\|_{\mathcal{C}}\in E\}$$

(modulo λ-nullfunctions) endowed with the norm $\|f\|_{\mathcal{E}}=\|\|f(\cdot)\|_{\mathcal{C}}\|_E$. One can easily see that $\mathcal{E}$ is a Banach space. We call it *the Banach space corresponding to the Banach function space* E.

In order to study the invariant manifolds of $\mathcal{E}$-class for semi-linear evolution equations we need some restrictions on the admissible spaces and assume the following hypothesis.

Standing Hypothesis 2 *Consider the admissible space* E *such that its associate space* E' *is also an admissible space. Moreover, for such an admissible space* E*, we suppose that* E' *contains an* ν*-exponentially* E*-invariant function, that is, the function* $\varphi\geqslant 0$ *having the property that, for any fixed* $\nu>0$ *the function* h_ν *defined by*

$$h_\nu(t):=\|e^{-\nu|t-\cdot|}\varphi(\cdot)\|_{E'},\quad \textit{for all } t\geqslant 0, \tag{2.4}$$

belongs to E.

Note that if $\Phi(t,\phi)$ is φ-Lipschitz, then $\|\Phi(t,\phi)\|\leqslant\varphi(t)(1+\|\phi\|_{\mathcal{C}})$ for all $\phi\in\mathcal{C}$ and $t\geqslant 0$. Using the operators F, $B(t)$, and Φ we can now define the nonlinear mapping $\Phi:\mathbb{R}_+\times C([-r,\infty),D(B))\times\mathcal{C}$ by

$$\Phi(t,v,\phi)=-B(t)Fv_t+B(t)v(t)+\Phi(t,\phi). \tag{2.5}$$

Then, the operator Φ satisfies

$$\|\Phi(t,0,0)\|\leqslant\varphi(t), \tag{2.6}$$

$$\|\Phi(t,u,\phi)-\Phi(t,v,\psi)\|\leqslant K\|\Psi\|\|u_t-v_t\|_{\mathcal{C}}+\varphi(t)\|\phi-\psi\|_{\mathcal{C}}, \tag{2.7}$$

for all $t\in\mathbb{R}_+$, for all $\phi,\psi\in\mathcal{C}$ and for all $u,v\in C([-r,\infty),D(B))$.

We next rewrite the equation (1.1) in the form

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}Fu_t &= B(t)Fu_t+\Phi(t,u,u_t),\quad t\in(0,\infty),\\ u_0 &= \phi\in\mathcal{C}:=C([-r,0],X).\end{cases} \tag{2.8}$$

Along with Equation (2.8) we consider the integral equation

$$\begin{cases}Fu_t &= U(t,0)F\phi+\int_0^t U(t,\xi)\Phi(\xi,u,u_\xi)d\xi,\quad \textit{for all } t>0,\\ u_0 &= \phi\in\mathcal{C}.\end{cases} \tag{2.9}$$

We note that, if the evolutionary process $(U(t,s))_{t\geqslant s\geqslant 0}$ is generated by $(B(t))_{t>0}$, then the function $u:[-r,\infty)\to X$, which satisfies (2.9) for some given function ϕ, is called a *mild solution* of the semi-linear equation (2.8). Also, the direct calculations yield that the function u satisfies (2.9) if and only if it satisfies

$$Fu_t = U(t,s)Fu_s + \int_s^t U(t,\xi)\Phi(\xi,u,u_\xi)d\xi, \quad \text{for all } t>s\geqslant 0. \tag{2.10}$$

Definition 2.5 *A set $\mathcal{S}\subset\mathbb{R}_+\times\mathcal{C}$ is said to be an invariant stable manifold of $\mathcal{E}-class$ for the solutions to Equation (2.9) if for every $t\in\mathbb{R}_+$ the phase spaces $\mathcal{C}$ splits into a direct sum $\mathcal{C}=P(t)\mathcal{C}\oplus\ker P(t)$ with corresponding projections $P(t)$ and there exists a family of Lipschitz continuous mappings*

$$y_t : P(t)\mathcal{C}\to\ker P(t), \quad t\in\mathbb{R}_+$$

with the Lipschitz constants independent of t such that [(1)]

1. The collection

$$\mathcal{S}=\left\{(t,\psi+y_t(\psi))\in\mathbb{R}_+\times P(t)\mathcal{C}\oplus\ker P(t) : t\in\mathbb{R}_+,\psi\in P(t)\mathcal{C}\right\},$$

and we denote by

$$\mathcal{S}_t := \left\{\psi+y_t(\psi) : (t,\psi+y_t(\psi))\in\mathcal{S}\right\};$$

2. the manifold $\mathcal{S}_t$ is homeomorphic to $P(t)\mathcal{C}$ for all $t\geqslant 0$;

3. to each $\phi\in\mathcal{S}_s$ there corresponds one and only one solution $u(t)$ to Equation (2.10) on $[s-r,\infty)$ satisfying the conditions that $\tilde{u}_s=\phi$, and the function $\chi_{[s,\infty)}(t)u_t$, $t\in\mathbb{R}_+$, belongs to $\mathcal{E}$ where the function $\tilde{u}_s$ is defined by $\tilde{u}_s(\theta):=Fu_{s-\theta}$ for all $-r\leqslant\theta\leqslant 0$;

4. the collection $\mathcal{S}$ is positively F-invariant under Equation (2.9) in the sense that if $u(t)$, $t\geqslant s-r$, is a solution to Equation (2.9) satisfying conditions that $u_s\in\mathcal{S}_s$ and the function $\chi_{[s,\infty)}(t)u_t$, $t\in\mathbb{R}_+$, belongs to $\mathcal{E}$, then we have $\tilde{u}_t\in\mathcal{S}_t$ for all $t\geqslant s$, where the function $\tilde{u}_t$ is defined by

$$\tilde{u}_t(\theta)=Fu_{t-\theta}, \qquad \text{for all } -r\leqslant\theta\leqslant 0 \quad \text{and } t\geqslant 0. \tag{2.11}$$

Note that if we identify $P(t)\mathcal{C}\oplus\ker P(t)$ with $P(t)\mathcal{C}\times\ker P(t)$, then we can write $\mathcal{S}_t=\mathrm{graph}(y_t)$.

Theorem 2.1 (see Nguyen -- Trinh) *Suppose that the family of linear operators $B(t)$, the difference operator F, and the nonlinear term Φ satisfy* Standing Hypothesis 1 *and we consider function h_ν defined as in* Standing Hypothesis 2. *Put*

$$k := N(1+H)e^{\nu r}\max\Big\{\frac{2K\|\Psi\|}{\nu}+\frac{N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}},$$
$$K\|\Psi\|\|\tilde{e}_\nu(\cdot)\|_E + \|h_\nu(\cdot)\|_E\Big\}, \tag{2.12}$$

$$k_1 := N(1+H)e^{\nu r}\left[\frac{2K\|\Psi\|}{\nu}+\frac{N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty}{1-e^{-\nu}}\right], \tag{2.13}$$

where $\tilde{e}_\nu(t)=\|e^{-\nu|t-\cdot|}\|_{E'}$. If $\max\left\{\frac{Nk_1e^{\nu r}}{1-k_1-\|\Psi\|},\frac{k}{1-\|\Psi\|}\right\}<1$ then there exists an admissibly stable manifold $\mathcal{S}$ of $\mathcal{E}$-class for the solutions to Equation (2.9).

Moreover, every two solutions $u(t)$ and $v(t)$ on the manifold $\mathcal{S}$ of $\mathcal{E}$-class for the solutions to Equation (2.9) corresponding to diffrent functions ϕ and $\psi \in \mathcal{S}_s$ attract each other exponentially in sense that, there exists positive constants μ and C_μ independent of $s \geqslant 0$ such that

$$\| u_t - v_t \|_{\mathcal{C}} \leqslant C_\mu e^{-\mu(t-s)} \left\| P(s)\phi - P(s)\psi \right\|_{\mathcal{C}}, \quad \text{for all } t \geqslant s, \tag{2.14}$$

where $P(t)$, $t \geqslant 0$, are defined as in (2.1) and $\mathcal{S}_s$ is defined as in Definition 2.5.

3. THE EXISTENCE OF ADMISSIBLY STABLE MANIFOLDS FOR A DIFFUSIVE HUTCHINSON EQUATION

Motivated by Hernández – Trofimchuk [2], see also Hernández – Wu [1], this paper deals with the following neutral version of the diffusive Hutchinson equation. Consider the Hutchinson equation

$$\begin{cases} \dfrac{\partial}{\partial t}[w(t,x) - lw(t-1,x)] = a(t)\left[\Delta w(t,x) + \alpha w(t,x)\right] \quad + \psi(t) w(t-1,x) w(t,x), \quad t \geqslant 0, \quad x \in [0,\pi], \\ w(t,0) = w(t,\pi) = 0, \quad t \geqslant 0 \\ w_s(\theta, x) = w(s+\theta, x), \quad \theta \in [-1,0], \quad x \in [0,\pi], \end{cases} \tag{3.1}$$

where l and α are real constants with $|l| < 1$, $\alpha > 1$ and $\alpha \neq n^2$, for all $n \in \mathbb{N}$. The function $a(\cdot) \in L_{1,loc}(\mathbb{R}_+)$ and satisfies the condition $\gamma_1 \geqslant a(t) \geqslant \gamma_0 > 0$ for fixed constants γ_0, γ_1 and a.e. $t \geqslant 0$.

Put $\Omega = [0,\pi]$. We choose the Hilbert space $X := L_2(\Omega)$, Banach space $\mathcal{C} := C([-1,0], X)$ and let $B : D(B) \subset X \to X$ be defined by

$$B(f) = f'' + \alpha f$$

with the domain $D(B) = H_0^2(\Omega) := \{ f \in W^{2,2}[0,\pi] : f(0) = f(\pi) = 0 \}$.

Also define the difference and delay operators F and Φ as

$$F : \mathcal{C} \to D(B), \quad F(f) := f(0) - lf(-1) \tag{3.2}$$

and $\Phi : \mathbb{R}_+ \times \mathcal{C} \to X$ by

$$\Phi(t,\phi) := \psi(t)[(\delta_{-1}\phi)(\phi)], \quad \text{for all } \phi \in \mathcal{C}, \tag{3.3}$$

where δ_{-1} is the Dirac delta function concentrated at -1.

Putting now $B(t) := a(t)B$ the equation (3.1) can now be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} F u_t(\cdot) &= B(t)u(t) + \Phi(t, u_t(\cdot,\theta)), \quad t \geqslant s \geqslant 0, \\ u_s(\cdot,\theta) &= \phi(\cdot,\theta) \in \mathcal{C}, \end{cases} \tag{3.4}$$

where B is the generator of an analytic semigroup $(T(t))_{t>0}$, with

$$\sigma(B) = \{-1+\alpha, -4+\alpha, \ldots -n^2+\alpha, \ldots\}.$$

Since $\alpha \neq n^2$, for all $n \in \mathbb{N}$, we have that $\sigma(B) \cap i\mathbb{R} = \varnothing$. Applying the Spectral Mapping Theorem for analytic semigroups we get

$$\sigma(T(t)) = e^{t\sigma(B)} = \left\{e^{t(\alpha-1)}, e^{t(\alpha-4)}, \dots e^{t(\alpha-n^2)}, \dots\right\}$$

and

$$\sigma(T(t)) \cap \{z \in \mathbb{C} : |z| = 1\} = \varnothing, \quad \text{for all } t > 0.$$

Therefore, spectrum of the operator $T(1)$ splits into two disjoint sets σ_0 and σ_1, where $\sigma_0 \subset \{z \in \mathbb{C} : |z| < 1\}, \sigma_1 \subset \{z \in \mathbb{C} : |z| > 1\}$. Next, we choose $P = P(1)$ to be the Riesz projection corresponding to the spectral set σ_0, and $Q := I - P$. Clearly, P and Q commute with $T(t)$ for all $t > 0$. We denote by $T_Q(t) := T(t)Q$ the restriction of $T(t)$ on QX. As well-known in Semigroup Theory, the semigroup $(T(t))_{t>0}$ is called having an exponential dichotomy and the restriction $T_Q(t)$ is invertible. Moreover, there are positive constants N and γ such that

$$\begin{aligned} &\| T(t)|_{PX} \| \leqslant Ne^{-\gamma t}, \quad \text{for all } t \geqslant 0, \\ &\| T_Q(-t) \| = \| T_Q(t)^{-1} \| \leqslant Ne^{-\gamma t}, \quad \text{for all } t \geqslant 0. \end{aligned} \tag{3.5}$$

Clearly, the family $(B(t))_{t\geqslant 0} = (a(t)B)_{t\geqslant 0}$ generates the evolutionary process $(U(t,s))_{t\geqslant s\geqslant 0}$ defined by the formula

$$U(t,s) := T\left(\int_s^t a(\tau)d\tau\right).$$

From the dichotomy estimates of $(T(t))_{t\geqslant 0}$ in (3.5), it is straightforward to check that the evolutionary process $(U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with projection P and constants $N = \nu := \gamma\gamma_0$ by the following estimates

$$\| U(t,s)|_{PX} \| = \left\| T\left(\int_s^t a(\tau)d\tau\right)|_{PX} \right\| \leqslant Ne^{-\nu(t-s)},$$

$$\| U(s,t)_| \| = \| \left(U(t,s)|_{\ker P}\right)^{-1} \| = \left\| T_Q\left(-\int_s^t a(\tau)d\tau\right) \right\| \leqslant Ne^{-\nu(t-s)},$$

for all $t \geqslant s \geqslant 0$.

Clearly, the difference operator F be of form $F = \delta_0 - \Psi$ for $\Psi = l\delta_{-1}$ and $\| \Psi \| \leqslant |l| < 1$. The linear operator $B(t)$ for every fixed $t \geqslant 0$ with norm $\| \cdot \|_{H_0^2(\Omega)}$ and $\| B(t)u \| \leqslant K \| u \|_{H_0^2(\Omega)}$.

We fix any $\rho > 0$ and denote by $B_\rho := \{\phi \in \mathcal{C} : |\phi|_\beta \leqslant \rho\} \subset \mathcal{C}$ the ball with radius ρ in $\mathcal{C}$. By [6, Section 5], Φ is $(2\rho\psi(t))$-Lipschitz. We now use the cut-off technique as follows. Let Φ be ϕ-Lipschitz, and $\chi(s)$ be an infinitely differentiable function on $[0, +\infty)$ such that $\chi(s) = 1$ for $0 \leqslant s \leqslant 1$; $\chi(s) = 0$ for $s \geqslant 2$; $0 \leqslant \chi(s) \leqslant 1$ and $|\chi'(s)| \leqslant 2$ for $s \in [0, +\infty)$. We define the mapping by assuming that

$$\Phi_R(t, u_t) := \chi\left(\frac{|u|_{\mathcal{C}}}{R}\right)\Phi(t, u_t) \quad \text{for all } u_t \in \mathcal{C}. \tag{3.6}$$

By [6, Section 5], Φ_R is φ-Lipschitz, where $\varphi(t) := 2\rho\psi(t)\dfrac{2R^2 + 5R + 2}{R}$, for $t \in \mathbb{R}_+$.

Next, we consider the following abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} Fu_t(\cdot) & = B(t)u(t) + \Phi_R(t, u_t(\cdot,\theta)), \quad t \geqslant s \geqslant 0, \\ u_s(\cdot,\theta) & = \phi(\cdot,\theta) \in \mathcal{C}. \end{cases} \tag{3.7}$$

We now take $E := L_p(R_+)$ for $1 \leqslant p \leqslant +\infty$ and choose $\varphi(t) := |b| e^{-\alpha t} \in E := L_p(\mathbb{R}_+)$, for $p \geqslant 1$. In the case of Lebesgue space $E := L_p(\mathbb{R}_+)$, we have $N_1 = N_2 = 1$. Hence, we have

$$\frac{N_1 \| \Lambda_1 T_1^+ \varphi \|_\infty + N_2 \| \Lambda_1 \varphi \|}{1 - e^{-\nu}} \leqslant \frac{|b|(e^\alpha - e^{-\alpha})}{\alpha(1 - e^{-\nu})}.$$

We have

$$h_\nu(t) = \| e^{-\nu|t-\cdot|} \varphi(\cdot) \|_{L_q} = |b| \left(\frac{e^{-\nu qt} - e^{-\alpha qt}}{q(\alpha - \nu)} + \frac{e^{-\alpha qt}}{q(\alpha + \nu)} \right)^{1/q}.$$

Assume that $\alpha > \nu$, we have

$$h_\nu(t) \leqslant \frac{|b| e^{-\nu t}}{(q(\alpha - \nu))^{1/q}}.$$

Thus, $h_\nu(t) \in L_p$ and

$$\| h_\nu \|_{L_p} \leqslant \frac{|b|}{(\nu p)^{1/p} (q(\alpha - \nu))^{1/q}}.$$

We now have

$$\begin{aligned} h_1 &:= \frac{k}{1 - \| \Psi \|} \\ &\leqslant \frac{N(1+H)e^\nu}{1 - |l|} \max\left\{ \frac{2K|l|}{\nu} + \frac{|b|(e^\alpha - e^{-\alpha})}{\alpha(1 - e^{-\nu})}, \frac{K|l|}{(\nu q)^{1/q}} + \frac{|b|}{(\nu p)^{1/p}(q(\alpha - \nu))^{1/q}} \right\} \end{aligned}$$

and

$$\begin{aligned} h_2 &:= \frac{N k_1 e^{\nu r}}{1 - k_1 - \| \Psi \|} \\ &\leqslant \frac{N^2 (1+H) e^{2\nu} [2K|l|\alpha(1 - e^{-\nu}) + |b|\nu(e^\alpha - e^{-\alpha})]}{\alpha\nu(1 - |l|)(1 - e^{-\nu}) - N(1+H)e^\nu [2K|l|\alpha(1 - e^{-\nu}) + |b|\nu(e^\alpha - e^{-\alpha})]}. \end{aligned}$$

If $\max\{h_1, h_2\} < 1$ then there is an admissibly stable manifold $\mathcal{S}$ of L_p-class for the Equation (2.9). Moreover, this admissibly stable manifold is finite dimension.

4. CONCLUSIONS

Motivated by a neutral version of the diffusive Hutchinson equation, the paper deals with the neutral evolution equation $\frac{\mathrm{d}}{\mathrm{d}t} Fu_t = B(t)u(t) + \Phi(t, u_t)$, where the family of linear operators $B(t)$, the difference operator F, and the nonlinear term Φ satisfy Standing Hypothesis 1 and the function h_ν defined as in Standing Hypothesis 2. Using the result Nguyen – Trinh [8, Theorem 2.8] (see also Theorem 2.1) on the existence of an admissibly stable manifolds of $\mathcal{E}$-class (see Definition 2.5), the present paper studies the asymptotic behavior of the above-mentioned diffusive Hutchinson equation. Future work will study admissibly (un)stable manifolds/inertial manifolds for the diffusive Hutchinson equation with infinite delay.

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