A NEW APPROACH TO ZERO DUALITY GAP OF VECTOR OPTIMIZATION PROBLEMS USING CHARACTERIZING SETS

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Abstract

In this paper we propose results on zero duality gap in vector optimization problems posed in a real locally convex Hausdorff topological vector space with a vector-valued objective function to be minimized under a set and a convex cone constraint. These results are then applied to linear programming.

Keywords: *Characterizing set, vector optimization problems, zero dualiy gap.*

MỘT CÁCH TIẾP CẬN MỚI CHO KHOẢNG CÁCH ĐỐI NGẪU BẰNG KHÔNG CỦA BÀI TOÁN TỐI ƯU VÉCTƠ SỬ DỤNG TẬP ĐẶC TRƯNG

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Tóm tắt

Trong bài viết này, chúng tôi đề xuất các kết quả về khoảng cách đối ngẫu bằng không trong bài toán tối ưu véctơ trên một không gian vectơ tôpô Hausdorff lồi địa phương với một hàm mục tiêu có giá trị vectơ được cực tiểu hóa dưới một tập và một ràng buộc nón lồi. Các kết quả này sau đó được áp dụng cho bài toán quy hoạch tuyến tính.

Từ khóa: *Tập đặc trưng, bài toán tối ưu véctơ, khoảng cách đối ngẫu bằng không.*

1. Introduction

Duality is one of the most important topics in optimization both from a theoretical and algorithmic point of view. In scalar optimization, the weak duality implies that the difference between the primal and dual optimal values is non-negative. This difference is called duality gap (Bigi and Papaplardo, 2005, Jeyakumar and Volkowicz, 1990). One says that a program has zero duality gap if the optimal value of the primal program and that of its dual are equal, i.e., the strong duality holds. There are many conditions guaranteeing zero duality gap (Jeyakumar and Volkowicz, 1990, Vinh *et al.*, 2016). We are interested in defining zero duality gap in vector optimization. However, such a definition cannot be applied to vector optimization easily, since a vector program has not just an optimal value but a set of optimal ones (Bigi and Papaplardo, 2005). Bigi and Pappalardo (2005) proposed some concepts of duality gap for a vector program with involving functions posed finite dimensional spaces, where concepts of duality gaps had been introduced but relying only on the relationships between the set of proper minima of the primal program and proper maxima of its dual. To the best of our knowledge, zero duality gap has not been generally studied in a large number of papers dealing with duality for vector optimization yet. Recently, zero duality gap for vector optimization problem was studied in Nguyen Dinh *et al.* (2020), where Farkas-type results for vector optimization under the weakest qualification condition involving the characterizing set for the primal vector optimization problem are applied to vector optimization problem to get results on zero duality gap between the primal and the Lagrange dual problems.

In this paper we are concerned with the *vector optimization problem* of the form

(VP) $\quad \mathrm{W}Inf\{F(x): x \in C, G(x) \in -S\}$,

where X, Y, Z are real locally convex Hausdorff topological vector spaces, S is nonempty convex cone in Z, $F: X \to Y^{\bullet}$, $G: X \to Z^{\bullet}$ are proper mappings, and $\emptyset \neq C \subset X$ (Here $WInfD$ is the set of all weak infimum of the set $D \subset Y$ by the weak ordering defined by a closed cone K in Y).

The aim of the paper is to establish results on zero duality gap between the problem (VP) and its Lagrange dual problem under the qualification conditions involving the characterizing set corresponding to the problem (VP). The principle of the weak zero duality gap (Theorem 1), to the best of the authors' knowledge, is new while the strong zero duality gap (Theorem 2) is nothing else but (Nguyen Dinh *et al.*, 2020, Theorem 6.1). The difference between ours and that of Nguyen Dinh *et al.* (2020) is the method of proof. Concretely, we do not use Farkas-type results to establish results on strong zero duality gap in our present paper.

The paper is organized as follows: In section 2 we recall some notations and introduce some preliminary results to be used in the rest of the paper. Section 3 provides some results on the value of (VP) and that of its dual problem. Section 4 is devoted to results on zero duality gap for the problem (VP) and its dual one. Finally, to illustrate the applicability of our main results, the linear programming problem will be considered in Section 5 and some interesting results related to this problem will be obtained.

2. Preliminaries

Let X, Y, Z be locally convex Hausdorff topological vector spaces (briefly, lcHtvs) with topological dual spaces denoted by X^*, Y^*, Z^*, respectively. The only topology considered on dual spaces is the weak*-topology. For a set $U \subset X$, we denote by $\overline{U}$ and $\overset{\circ}{U}$ the *closure* and the *interior* of U, respectively.

Let $K \subsetneqq Y$ be a closed and convex cone in Y with nonempty interior, i.e., $\overset{\circ}{K} \neq \emptyset$. The *weak ordering* generated by the cone K is defined by, for all $y_1, y_2 \in Y$,

$$y_1 <_K y_2 \iff y_1 - y_2 \in -\overset{\circ}{K},$$

or equivalently, $y_1 \not<_K y_2$ if and only if $y_1 - y_2 \notin -\overset{\circ}{K}$.

We enlarge Y by attaching a *greatest element* $+\infty_Y$ and a *smallest element* $-\infty_Y$ with respect to $<_K$, which do not belong to Y, and we denote $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$. By convention, $-\infty_Y <_K y$ and $y <_K (+\infty_Y)$ for any $y \in Y$. We also assume by convention that

$$-(+\infty_Y) = -\infty_Y, \qquad -(-\infty_Y) = +\infty_Y,$$

$$(+\infty_Y) + y = y + (+\infty_Y) = +\infty_Y, \quad \forall y \in Y \cup \{+\infty_Y\},$$

$$(-\infty_Y) + y = y + (-\infty_Y) = -\infty_Y, \forall y \in Y \cup \{-\infty_Y\}.$$

The sums $(-\infty_Y) + (+\infty_Y)$ and $(+\infty_Y) + (-\infty_Y)$ are not considered in this paper.

By convention, $\inf \emptyset = +\infty$, $+\infty k_0 = +\infty_Y$ and $-\infty k_0 = -\infty_Y$ for all $k_0 \in \overset{\circ}{K}$.

Given $\emptyset \neq M \subset Y^\bullet$, the following notions specified from Definition 7.4.1 of Bot *et al.* (2010) will be used throughout this paper.

• An element $\bar{v} \in Y^\bullet$ is said to be a *weakly infimal element* of M if for all $v \in M$ we have $v \not<_K \bar{v}$ and if for any $\tilde{v} \in Y^\bullet$ such that $\bar{v} <_K \tilde{v}$, then there exists some $v \in M$ satisfying $v <_K \tilde{v}$. The set of all weakly infimal elements of M is denoted by $\mathrm{W}InfM$ and is called the *weak infimum* of M.

• An element $\bar{v} \in Y^\bullet$ is said to be a *weakly supremal element* of M if for all $v \in M$ we have $\bar{v} \not<_K v$ and if for any $\tilde{v} \in Y^\bullet$ such that $\tilde{v} <_K \bar{v}$, then there exists some $v \in M$ satisfying $\tilde{v} <_K v$. The set of all weakly supremal elements of M is denoted by $\mathrm{W}SupM$ and is called the *weak supremum* of M.

• The *weak minimum* of M is the set $\mathrm{W}MinM = M \cap \mathrm{W}InfM$ and its elements are the *weakly minimal elements* of M. The *weak maximum* of M, $\mathrm{W}MaxM$, is defined similarly, $\mathrm{W}MaxM := M \cap \mathrm{W}SupM$.

Weak infimum and weak supremum of the empty set is defined by convention as $\mathrm{W}Sup\emptyset = \{-\infty_Y\}$ and $\mathrm{W}Inf\emptyset = \{+\infty_Y\}$, respectively.

Remark 1. *For all $M \subset Y^\bullet$ and $a \in Y$, the first three following properties can be easy to check while the last one comes from* (Tanino, 1992):

• $\mathrm{W}Inf(M + a) = a + \mathrm{W}InfM$,

• $\mathrm{W}InfM = \{-\infty_Y\} \iff \forall \tilde{v} \in Y$, $\exists v \in M: v <_K \tilde{v}$,

• $\mathrm{W}Inf(M + K) = \mathrm{W}InfM$,

• If $M \subset Y$ and $WInfM \subset Y$, then $\mathrm{W}InfM + \overset{\circ}{K} = M + \overset{\circ}{K}$.

Remark 2. *For all $M \subset Y^\bullet$, it holds $M \cap \left(WInfM - \overset{\circ}{K}\right) = \emptyset$. Indeed, assume that $M \cap (WInfM - \overset{\circ}{K}) \neq \emptyset$, then there is $v \in WInfM$ satisfying $v \in M + \overset{\circ}{K}$ which contradicts the first condition in definition of weak infimum.*

Proposition 1. *Assume that $\emptyset \neq M \subset Y$ and $WInfM \subset Y$. Then the following partitions of Y holds (The sets A, B, C form a partition of Y if $Y = A \cup B \cup C$ and they are pairwise disjoint sets):*

$$Y = \left(M + \overset{\circ}{K}\right) \cup \mathrm{W}InfM \cup \left(\mathrm{W}InfM - \overset{\circ}{K}\right)$$

$$= (M + \overset{\circ}{K}) \cup (\mathrm{W}InfM - K)$$

$$= (\mathrm{W}InfM + K) \cup (\mathrm{W}InfM - \overset{\circ}{K}).$$

Proof. The first partition is established by Dinh *et al.* (2017, Proposition 2.1). The others follow from the first one and the definition of $WInfM$.

Proposition 2. *Assume that* $\emptyset \neq M \subset N \subset Y$ *and* $WInfM \neq \{-\infty_Y\}$. *Then, one has* $WInfM \subset WInfN + K$.

Proof. As $\emptyset \neq M$ and $\emptyset \neq N$ we have $\mathrm{W}InfM \neq \{+\infty_Y\}$ and $\mathrm{W}InfN \neq \{+\infty_Y\}$. According to Proposition 1, one has $(\mathrm{W}InfN) \cap (N + \overset{\circ}{K}) = \emptyset$. Since $M \subset N$, it follows that $(\mathrm{W}InfN) \cap (M + \overset{\circ}{K}) = \emptyset$. On the other hand, one has $\mathrm{W}InfM + \overset{\circ}{K} = M + \overset{\circ}{K}$ (see Remark 1), we gain $(\mathrm{W}InfN) \cap (\mathrm{W}InfM + \overset{\circ}{K}) = \emptyset,$ which is equivalent to $(\mathrm{W}InfM) \cap (\mathrm{W}InfN - \overset{\circ}{K}) = \emptyset$.

The conclusion follows from the partition

$$Y = (\mathrm{W}InfM + K) \cup \left(\mathrm{W}InfM - \overset{\circ}{K}\right)$$

(see Proposition 1). □

Given a vector-valued mapping $F: X \to Y^{\bullet}$, the *effective domain* and the K-*epigraph* of F is defined by, respectively,

$$\mathrm{d}omF := \{x \in X: F(x) \neq +\infty_Y\}$$

$$\mathrm{e}piF := \{(x, y) \in X \times Y: y \in F(x) + K\}.$$

We say that F is *proper* if $\mathrm{d}omF \neq \emptyset$ and $-\infty_Y \notin F(X)$, and that F is K-*convex* if $\mathrm{e}piF$ is a convex subset of $X \times Y$.

Let $S \neq \emptyset$ be a convex cone in Z and $\leqq_S$ be the usual ordering on Z induced by the cone S, i.e., $z_1 \leqq_S z_2$ if and only if $z_2 - z_1 \in S$. We also enlarge Z by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$ which do not belong to Z, and define $Z^{\bullet} := Z \cup \{-\infty_Z, +\infty_Z\}$. The set,

$$\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y):\ T(S) \subset K\}$$

is called the *cone of positive operators* from Z to Y.

For $T \in \mathcal{L}(Z, Y)$ and $G: X \to Z \cup \{+\infty_Z\}$, the composite mapping $T \circ G: X \to Y^{\bullet}$ is defined by:

$$(T \circ G)(x) = \begin{cases} T(G(x)), & \text{if } G(x) \in Z, \\ +\infty_Y, & \text{if } G(x) = +\infty_Z. \end{cases}$$

Lemma 1 (Canovas *et al.*, 2020, Lemma 2.1(i))**.** *For all* $y, y' \in Y$ *and* $k_0 \in \overset{\circ}{K}$, *there is* $\mu > 0$ *such that* $y' \in y - \mu k_0 + K$.

Lemma 2. *Let* $\emptyset \neq M \subset Y$, $y_0 \in Y$, $k_0 \in \overset{\circ}{K}$, $\mu_0 = \inf\{\mu \in \mathbb{R}: y_0 + \mu k_0 \in M + K\}$. *The following assertions hold true:*

(i) $\mu_0 \neq +\infty$,

(ii) $y_0 + \mu k_0 \in \mathrm{W}InfM$ if and only if $\mu = \mu_0$.

Proof. Let us denote

$$\mathcal{M} := \{\mu \in \mathbb{R}: y_0 + \mu k_0 \in M + K\}.$$

(i) Take $\bar{m} \in M$. Let y_0 and $\bar{m}$ play the roles of y' and y in Lemma 1 respectively, one gets the existence of $\mu > 0$ such that

$$y_0 \in \bar{m} - \mu k_0 + K.$$

Then, $y_0 + \mu k_0 \in \bar{m} + K \subset M + K$, and hence, $\mathcal{M} \neq \emptyset$ which yields $\mu_0 \neq +\infty$.

(ii) Consider two following cases:

Case 1. $M + K = Y$: Then, $\mathcal{M} = \mathbb{R}$ and $\mu_0 = -\infty$. Furthermore, as $M + K = Y$, one has $M + \overset{\circ}{K} = M + K + \overset{\circ}{K} = Y + \overset{\circ}{K} = Y$, consequently,

$$\forall \tilde{v} \in Y,\ \exists v \in M: v <_K \tilde{v}, \tag{1}$$

which yields $\mathrm{W}InfM = \{-\infty_Y\}$ (see Remark 1). So, $y_0 + \mu k_0 \in \mathrm{W}InfM$ if and only if $\mu = -\infty = \mu_0$.

Case 2. $M + K \neq Y$: According to (i), one has $\mu_0 \neq +\infty$. We will prove that $\mu_0 \neq -\infty$. For this, it suffices to show that $\mathcal{M}$ is bounded from below. Firstly, it is worth noting that for an arbitrary $\tilde{y} \in Y$, there exists $\tilde{\mu} \in \mathbb{R}$ satisfying $\tilde{y} \in y_0 + \tilde{\mu} k_0 + K$ (apply Lemma 1 to $y' = \tilde{y}$ and $y = y_0$). So, if we assume that $\mathcal{M}$ is not bounded from below, then there is $\tilde{\mu}_1 \in \mathcal{M}$ (which also means $y_0 + \tilde{\mu}_1 k_0 \in M + K$) satisfying $\tilde{\mu}_1 < \tilde{\mu}$. This yields $\tilde{y} \in y_0 + \tilde{\mu} k_0 + K = (y_0 + \tilde{\mu}_1 k_0) + (\tilde{\mu} - \tilde{\mu}_1) k_0 + K \subset (M + K) + K + K = M + K$ and we get $Y \subset M + K$ (as $\tilde{y}$ is arbitrary), which contradicts the assumption $M + K \neq Y$.

Note now that $M+K\neq Y$, (1) does not hold true, $\mathrm{W}InfM\neq\{-\infty_Y\}$ (see Remark 1). As $M\neq\emptyset$ we have $\mathrm{W}InfM\neq\{+\infty_Y\}$.

We prove that $y_0+\mu_0k_0\in\mathrm{W}InfM$. *First*, we begin by proving $y_0+\mu_0k_0\notin M+\overset{\circ}{K}$. To obtain a contradiction, suppose that $y_0+\mu_0k_0\in M+\overset{\circ}{K}$. Then, there is a neighborhood U of 0_Y such that

$$y_0+\mu_0k_0+U\subset M+K.$$

Take $\epsilon>0$ such that $-\epsilon k_0\in U$, one gets $y_0+(\mu_0-\epsilon)k_0\in M+K$. This yields $\mu_0-\epsilon\in\mathcal{M}$, which contradicts the fact that $\mu_0=\inf\mathcal{M}$. So, $y_0+\mu_0k_0\notin M+\overset{\circ}{K}$, or equivalently, $v\nless_K y_0+\mu_0k_0$ for all $v\in M$. *Second,* let $\tilde{v}\in Y$ such that $y_0+\mu_0k_0<_K\tilde{v}$. Then, $y_0+\mu_0k_0\in\tilde{v}-\overset{\circ}{K}$, and hence, there is a neighborhood V of 0_Y such that $y_0+\mu_0k_0+V\in\tilde{v}-K$. Take $\nu>0$ such that $\nu k_0\in V$, one has $y_0+(\mu_0+\nu)k_0\in\tilde{v}-K$, which yields $\mu_0+\nu\in\ \mathcal{M}$. Since $\mu_0=\inf\mathcal{M}$, there is $\mu_2\in\mathcal{M}$ such that $\mu_2<\mu_0+\nu$. As $\mu_2\in\mathcal{M}$ one has $y_0+\mu_2k_0\in M+K$, or equivalently, there exists $k_1\in K$ such that $y_0+\mu_2k_0-k_1\in M$. On the other hand,

$$\begin{aligned}&y_0+\mu_2k_0-k_1\\ &=y_0+(\mu_0+\nu)k_0+(\mu_2-\mu_0\\ &-\nu)k_0-k_1\in\tilde{v}-K-\overset{\circ}{K}-K\\ &=\tilde{v}-\overset{\circ}{K},\end{aligned}$$

or equivalently, $y_0+\mu_2k_0-k_1<_K\tilde{v}$.

From what has already been proved we have $y_0+\mu_0k_0\in\mathrm{W}Inf\ M$.

It remains to prove that $\mu=\mu_0$ if $y_0+\mu k_0\in\mathrm{W}InfM$. It is easy to see that if $\mu>\mu_0$ then $y_0+\mu k_0=y_0+\mu_0k_0+(\mu-\mu_0)k_0\in\mathrm{W}InfM+\overset{\circ}{K}$ and if $\mu<\mu_0$ then $y_0+\mu k_0=y_0+\mu_0k_0+(\mu-\mu_0)k_0\in\mathrm{W}InfM-\overset{\circ}{K}$. So, it follows from the decomposition

$$Y=(\mathrm{W}InfM-\overset{\circ}{K})\cup\mathrm{W}InfM\cup(\mathrm{W}InfM+\overset{\circ}{K})$$

that $y_0+\mu k_0\notin\mathrm{W}InfM$ whenever $\mu\neq\mu_0$. □

We denote by $\mathcal{L}(X,Y)$ the space of linear continuous mappings from X to Y, and by $0_{\mathcal{L}}$ the zero element of $\mathcal{L}(X,Y)$ (i.e., $0_{\mathcal{L}}(x)=0_Y$ for all $x\in X$). The topology considered in $\mathcal{L}(X,Y)$ is the one defined by the point-wise convergence, i.e., for $(L_\alpha)_{\alpha\in D}\subset\mathcal{L}(X,Y)$ and $L\in\mathcal{L}(X,Y)$, $L_\alpha\to L$ means that $L_\alpha(x)\to L(x)$ in Y for all $x\in X$.

Let denote

$$K^+:=\{y^*\in Y^*:\langle y^*,k\rangle\geq 0,\forall k\in K\},$$

$$K_0^+:=\{y^*\in Y^*:\langle y^*,k\rangle>0,\forall k\in\overset{\circ}{K}\}.$$

The following basic properties are useful in the sequel.

Lemma 3 (Nguyen Dinh *et al.*, 2020, Lemma 2.3)**.** *It holds:*

(i) $K_0^+\neq\emptyset$;

(ii) $K^+\backslash\{0_{Y^*}\}=K_0^+$.

3. Vector optimization problem and its dual problem

Consider the *vector optimization problem* of the model

(VP) $\qquad\mathrm{W}Min\{F(x):x\in C,G(x)\in-S\}$,

where, as in previous sections, X,Y,Z are lcHtvs, K is a closed and convex cone in Y with nonempty interior, S is a closed, convex cone in Z, $F:X\to Y^\bullet$, $G:X\to Z^\bullet$ are proper mappings, and $\emptyset\neq C\subset X$. Let us denote $A:=C\cap G^{-1}(-S)$ and assume along this paper that $A\cap\mathrm{d}omF\neq\emptyset$, which also means that (VP) is feasible.

The infimum value of the problem (VP) is denoted by

$$\mathrm{val(VP)}:=\mathrm{W}Inf\{F(x):x\in C,G(x)\in-S\}\quad(2)$$

A vector $\bar{x}\in A$ such that $F(\bar{x})\in\mathrm{v}al(\mathrm{VP})$ is called a solution of (VP). The set of all solutions of (VP) is denoted by sol(VP). It is clear that $val(\mathrm{VP}) \cap F(A) = \mathrm{W}MinF(A)$.

The *characterizing set* corresponding to the problem (VP) is defined by Nguyen Dinh *et al.* (2020)

$$\mathbb{H} := \bigcup_{x \in C \cap \mathrm{d}omF \cap \mathrm{d}omG} (G(x) + S) \times (F(x) + K).$$

Let us denote p the conical projection from $Z \times Y$ to Y, i.e., $p(z,y) = y$ for all $(z,y) \in Z \times Y$, and consider the following sets

$$\mathbb{E}_1 := p(\mathbb{H} \cap (\{0_Z\} \times Y)), \tag{3}$$

$$\mathbb{E}_2 := p\left(\overline{\mathbb{H}} \cap (\{0_Z\} \times Y)\right). \tag{4}$$

Proposition 3 (Nguyen Dinh *et al.*, 2019, Propositions 3.3, 3.4)**.** *It holds:*

(i) $\mathbb{E}_1 = F(A \cap \mathrm{d}omF) + K$, *and consequently*, $\mathbb{E}_1 + K = \mathbb{E}_1$,

(ii) $\mathbb{E}_2 + K = \mathbb{E}_2$,

(iii) $\overset{\circ}{\mathbb{E}}_1 = \mathbb{E}_1 + \overset{\circ}{K}$ *and* $\overset{\circ}{\mathbb{E}}_2 = \mathbb{E}_2 + \overset{\circ}{K}$, *in particular*, $\overset{\circ}{\mathbb{E}}_1$ *and* $\overset{\circ}{\mathbb{E}}_2$ *are both nonempty*,

(iv) $\overline{\mathbb{H}} \cap (\{0_Z\} \times Y) = \{0_Z\} \times \mathbb{E}_2$ *and* $\overline{\mathbb{H} \cap (\{0_Z\} \times Y)} = \{0_Z\} \times \overline{\mathbb{E}_1}$.

Proposition 4. $\mathrm{val}(\mathrm{VP}) = WInf\mathbb{E}_1$.

Proof. It follows from Proposition 3(i), (2), and Remark 1

Nguyen Dinh and Dang Hai Long (2018) introduced the *Lagrangian dual problem* (VP*) of (VP) as follows

$$(\mathrm{VP}^*) \quad \mathrm{W}\underset{T \in \mathcal{L}_+(S,K)}{Sup} \mathrm{W}Inf\{F(x) + (T \circ G)(x) : x \in C\}.$$

The supremum value of (VP*) is defined as

$$val(\mathrm{VP}^*) := \mathrm{W}Sup\left(\bigcup_{T \in \mathcal{L}_+(S,K)} \mathrm{W}Inf\{F(x) + (T \circ G)(x) : x \in C\}\right).$$

For any $T \in \mathcal{L}_+(S,K)$, set

$$\boldsymbol{\mathcal{M}}(T) := \mathrm{W}Inf\{F(x) + (T \circ G)(x) : x \in C\}.$$

We say that an operator $T \in \mathcal{L}_+(S,K)$ is a solution of (VP*) if $\boldsymbol{\mathcal{M}}(T) \cap val(\mathrm{VP}^*) \neq \emptyset$ and the set of all solutions of (VP*) will be denoted by sol(VP*).

Remark 3. *Let* $D := \{(T,y) \in \mathcal{L}_+(S,K) \times Y : y \notin (F + T \circ G)(C) + \overset{\circ}{K}\}$ *and define* $\ell(T,y) = y$ *for all* $(T,y) \in D$. *According to* Nguyen Dinh *et al.* (2018, Remark 4), *one has*

$$\mathrm{val}(\mathrm{VP}^*) = \mathrm{W}Sup\ell(D).$$

Moreover, it follows from Nguyen Dinh and Dang Hai Long (2018, Theorem 5) *that weak duality holds for pair* (VP) $-$ (VP*). *Concretely, if* (VP) *is feasible and* $val(\mathrm{VP}^*) \neq \{-\infty_Y\}$ *then* $val(\mathrm{VP}) \subset val(\mathrm{VP}^*) + K$.

Proposition 5. *Assume that F is K-convex, that G is S-convex, and that C is a convex subset of X. Then, one has* $val(\mathrm{VP}^*) = WInf\mathbb{E}_2 = WInf\{y \in Y : (0_Z, y) \in \overline{\mathbb{H}}\}$, *where* $\mathbb{E}_2$ *is given in* (4).

Proof. [$\subset$] Take $\bar{y} \in val(VP^*)$, we will prove that $\bar{y} \in \mathrm{W}Inf\mathbb{E}_2$.

(α_1) *Firstly, prove that* $\bar{y} \in \mathbb{E}_2$. Assume the contrary, i.e., that $\bar{y} \notin \mathbb{E}_2$, or equivalently, $(0_Z, \bar{y}) \notin \overline{\mathbb{H}}$. Then, apply the convex separation theorem, there are $y_1^* \in Y^*$ and $z_1^* \in Z^*$ such that

$$\langle y_1^*, \bar{y} \rangle < \langle y_1^*, y \rangle + \langle z_1^*, z \rangle, \ \forall (z,y) \in \mathbb{H}. \tag{5}$$

• Prove that $y_1^* \in K_0^+$ and $z_1^* \in S^+$. Pick now $\bar{x} \in A \cap \mathrm{d}omF$. Take arbitrarily $k \in K$. It is easy to see that $(0_Z, F(\bar{x}) + \lambda k) \in \mathbb{H}$ for any $\lambda \geq 0$. So, by (5),

$$\langle y_1^*, \bar{y} \rangle < \langle y_1^*, F(\bar{x}) + \lambda k \rangle, \quad \forall \lambda \geq 0,$$

and hence,

$$\frac{1}{\lambda} \langle y_1^*, \bar{y} - F(\bar{x}) \rangle < \langle y_1^*, k \rangle, \quad \forall \lambda \geq 0.$$

Letting $\lambda \to +\infty$, one gains $\langle y_1^*, k \rangle \geq 0$. As k is arbitrarily, we have $y_1^* \in K^+$. To prove $y_1^* \in K_0^+$, in the light of Lemma 3, it is sufficient to show that $y_1^* \neq 0_{Y^*}$. On the contrary, suppose that $y_1^* = 0_{Y^*}$. According to (5), one has

$$\langle z_1^*, z\rangle > 0, \quad \forall (z,y) \in \mathbb{H}.$$

This, together with the fact that $(0_Z, F(\bar{x})) \in \mathbb{H}$, yields $\langle z_1^*, 0_Z\rangle > 0$, a contradiction.

We now show that $z_1^* \in S^+$. Indeed, take arbitrarily $s \in S$. For any $\lambda \geq 0$, one has $(G(\bar{x}) + \lambda s, F(\bar{x})) \in \mathbb{H}$, and hence, by (5)

$$\langle y_1^*, \bar{y}\rangle < \langle y_1^*, F(\bar{x})\rangle + \langle z_1^*, G(\bar{x}) + \lambda s\rangle,$$
$$\forall \lambda \geq 0,$$

which implies

$$\frac{1}{\lambda}(\langle y_1^*, \bar{y} - F(\bar{x})\rangle - \langle z_1^*, G(\bar{x})\rangle) < \langle z_1^*, s\rangle,$$
$$\forall \lambda \geq 0.$$

Letting $\lambda \to +\infty$, one gains $\langle z_1^*, s\rangle \geq 0$. Consequently, $z_1^* \in S^+$.

- We proceed to show that $\bar{y} \in \ell(D) - \mathring{K}$. Indeed, pick $k_0 \in \mathring{K}$. Since $y_1^* \in K_0^+$, it follows that $\langle y_1^*, k_0\rangle > 0$. Let $\bar{T}: Z \to Y$ defined by $\bar{T}(z) = \frac{\langle z_1^*, z\rangle}{\langle y_1^*, k_0\rangle} k_0$. Then, it is easy to check that $T \in \mathcal{L}_+(S,K)$ and $y_1^* \circ \bar{T} = z_1^*$.

Take $x \in C \cap \mathrm{d}omF \cap \mathrm{d}omG$. As $(G(x), F(x)) \in \mathbb{H}$ from (5), we have

$$\langle y_1^*, \bar{y}\rangle < \langle y_1^*, F(x)\rangle + \langle z_1^*, G(x)\rangle,$$

and hence, with the help of $\bar{T}$,

$$\langle y_1^*, \bar{y}\rangle < \langle y_1^*, F(x)\rangle + \langle y_1^*, (\bar{T} \circ G)(x)\rangle,$$

or equivalently,

$$\langle y_1^*, \bar{y} - F(x) - (\bar{T} \circ G)(x)\rangle < 0.$$

So, there is $\epsilon > 0$ such that

$$\langle y_1^*, \bar{y} - F(x) - (\bar{T} \circ G)(x)\rangle + \epsilon \leq 0,$$

or equivalently,

$$\left\langle y_1^*, \bar{y} - F(x) - (\bar{T} \circ G)(x) + \frac{\epsilon}{\langle y_1^*, k_0\rangle} k_0 \right\rangle \leq 0.$$

As $y_1^* \in K_0^+$, the last inequality entails

$$\bar{y} - F(x) - (\bar{T} \circ G)(x) + \frac{\epsilon}{\langle y_1^*, k_0\rangle} k_0 \notin \mathring{K},$$

or equivalently,

$$\bar{y} + \frac{\epsilon}{\langle y_1^*, k_0\rangle} k_0 \notin F(x) + (\bar{T} \circ G)(x) + \mathring{K}.$$

Hence, $\bar{y} + \frac{\epsilon}{\langle y_1^*, k_0\rangle} k_0 \in \ell(D)$ and we get $\bar{y} \in \ell(D) - \mathring{K}$. This contradicts the fact that $\bar{y} \in val(\mathrm{VP}^*) = \mathrm{W}Sup\ell(D)$. Consequently, $\bar{y} \in \mathbb{E}_2$.

(α_2) *Secondly, we next claim that* $\bar{y} \notin \mathbb{E}_2 + \mathring{K}$. For this purpose, we take arbitrarily $\tilde{k} \in \mathring{K}$ and show that $\bar{y} - \tilde{k} \notin \mathbb{E}_2$, or equivalently, $(0_Z, \bar{y} - \tilde{k}) \notin \overline{\mathbb{H}}$.

As $\bar{y} \in val(\mathrm{VP}^*) = \mathrm{W}Sup\ell(D)$ and $\bar{y} - \frac{1}{2}\tilde{k} <_K \bar{y}$, there is $\tilde{y} \in \ell(D)$ such that $\bar{y} - \frac{1}{2}\tilde{k} <_K \tilde{y}$, or equivalently,

$$\bar{y} - \frac{1}{2}\tilde{k} \in \tilde{y} - \mathring{K}. \tag{6}$$

As $\tilde{y} \in \ell(D)$, there exists $\tilde{T} \in \mathcal{L}_+(S,K)$ such that

$$\tilde{y} \notin F(x) + (\tilde{T} \circ G)(x) + \mathring{K},$$
$$\forall x \in C \cap \mathrm{dom}F \cap \mathrm{dom}G.$$

Moreover, by the convex assumption, $(F + \tilde{T} \circ G)(C \cap \mathrm{d}omF \cap \mathrm{d}omG) + \mathring{K}$ is a convex set of Y (Nguyen Dinh *et al.*, 2019, Remark 4.1). Hence, the convex separation theorem (Rudin, 1991, Theorem 3.4) ensures the existence of $y_0^* \in Y^*$ satisfying

$$\langle y_0^*, \tilde{y}\rangle < \langle y_0^*, v\rangle,$$
$$\forall v \in (F + \tilde{T} \circ G)(C \cap \mathrm{dom}F \cap \mathrm{dom}G) + \mathring{K}.$$

So, according to Nguyen Dinh *et al.* (2019, Lemma 3.3), one gets $y_0^* \in K_0^+$ and

$$\langle y_0^*, \tilde{y}\rangle \leq \langle y_0^*, (F + \tilde{T} \circ G)(x)\rangle,$$
$$\forall x \in C \cap \mathrm{dom}F \cap \mathrm{dom}G. \tag{7}$$

Take now $(z, y) \in \mathbb{H}$. Then, there is $\tilde{x} \in C \cap \mathrm{d}omF \cap \mathrm{d}omG$ such that

$$z \in G(\tilde{x}) + S \text{ and } y \in F(\tilde{x}) + K. \quad (8)$$

It is worth noting that $\tilde{T} \in \mathcal{L}_+(S, K)$, one gets from (8) that

$$\tilde{T}(z) - (\tilde{T} \circ G)(\tilde{x}) \in K \text{ and } y \in F(\tilde{x}) + K. \quad (9)$$

Since $y_0^* \in K_0^+$, it follows from (6), (7), and (9) that $\langle y_0^*, \bar{y} - \frac{1}{2}\tilde{k}\rangle < \langle y_0^*, \tilde{y}\rangle$, $\langle y_0^*, \tilde{y}\rangle \le \langle y_0^*, F(\tilde{x}) + (\tilde{T} \circ G)(\tilde{x})\rangle$, $\langle y_0^*, y\rangle \ge \langle y_0^*, F(\tilde{x})\rangle$, and $\langle y_0^*, \tilde{T}(z)\rangle \ge \langle y_0^*, (\tilde{T} \circ G)(\tilde{x})\rangle$. From these inequalities,

$$\langle y_0^*, \bar{y} - \tilde{k}\rangle < \langle y_0^*, \bar{y} - \tfrac{1}{2}\tilde{k}\rangle < \langle y_0^*, y\rangle + \langle y_0^* \circ \tilde{T}, z\rangle \quad (10)$$

(recall that $\langle y_0^*, \tilde{k}\rangle > 0$ as $y_0^* \in K_0^+$ and $\tilde{k} \in \mathring{K}$). Note that (10) holds for any $(z, y) \in \mathbb{H}$. This means that $(0_Z, \bar{y} - \tilde{k})$ is strictly separated from $\mathbb{H}$, and consequently, $(0_Z, \bar{y} - \tilde{k}) \notin \overline{\mathbb{H}}$ (see Zalinescu, 2002, Theorem 1.1.7).

(α_3) *Lastly*, we have just shown that $\bar{y} \in \mathbb{E}_2 \backslash (\mathbb{E}_2 + \mathring{K})$. So, $\bar{y} \in \mathrm{W}Min\mathbb{E}_2 \subset \mathrm{W}Inf\mathbb{E}_2$.

$[\supset]$ Take $\bar{y} \in \mathrm{W}Inf\mathbb{E}_2$, we will prove that $\bar{y} \in val(\mathrm{VP}^*)$.

(β_1) *Firstly*, take $\tilde{y} \in Y$ such that $\tilde{y} <_K \bar{y}$. Then, as $\bar{y} \in \mathrm{W}Inf\mathbb{E}_2$ one has $\tilde{y} \notin \mathbb{E}_2$. We now apply the argument in Step (α_1) again, with $\bar{y}$ replaced by $\tilde{y}$ to obtain $\tilde{y} \in \ell(D) - \mathring{K}$, or in the other words, there is $y' \in \ell(D)$ such that $\tilde{y} <_K y'$.

(β_2) *Secondly, prove that $\bar{y} \not<_K y$ for all* $y \in \ell(D)$. Suppose, contrary to our claim, that there is $\hat{y} \in \ell(D)$ such that $\bar{y} <_K \hat{y}$. Then, there is $\hat{k} \in \mathring{K}$ such that $\bar{y} + \hat{k} = \hat{y}$. Hence, $\hat{y} \in \ell(D)$ and $\bar{y} + \frac{1}{2}\hat{k} <_K \bar{y} + \hat{k} = \hat{y}$. Letting $\bar{y} + \frac{1}{2}\hat{k}$ and $\hat{y}$ play the roles of $\bar{y} - \tilde{k}$ and $\tilde{y}$ (respectively) in Step (α_2) and using the same argument as in this step, one gets $(0_Z, \bar{y} + \frac{1}{2}\hat{k}) \notin \overline{\mathbb{H}}$ which also means $\bar{y} + \frac{1}{2}\hat{k} \notin \mathbb{E}_2$. On the other hand, since $\bar{y} <_K \bar{y} + \frac{1}{2}\hat{k}$, there is $y_1 \in \mathbb{E}_2$ such that $y_1 <_K \bar{y} + \frac{1}{2}\hat{k}$, and consequently,

$$\bar{y} + \tfrac{1}{2}\hat{k} \in y_1 + \mathring{K} \subset \mathbb{E}_2 + K = \mathbb{E}_2.$$

We get a contradiction, and hence, $\bar{y} \not<_K y$ for all $y \in \ell(D)$.

(β_3) *Lastly*, it follows from Steps (β_1), (β_2) and the definition of weak supremum that $\bar{y} \in \mathrm{W}Sup\ell(D) = val(\mathrm{VP}^*)$. The proof is complete. □

Remark 4. *According to the proof of Proposition 5, we see that if all the assumptions of this proposition hold then one also has* $val(\mathrm{VP}^*) = WMin\mathbb{E}_2$.

4. Zero duality gap for vector optimization problem

Consider the pair of primal-dual problems (VP) and (VP^*) as in the previous section.

Definition 1. We say that (VP) has *weak zero duality gap* if $val(\mathrm{VP}) \cap val(\mathrm{VP}^*) \neq \emptyset$ and that (VP) has a *strong zero duality gap* if $val(\mathrm{VP}) = \mathrm{val}(\mathrm{VP}^*)$.

Theorem 1. *Assume that F is K-convex, that G is S-convex, and that C is a convex subset of X. Then, the following statements are equivalent:*

(i) $\overline{\mathbb{H} \cap [\{0_Z\} \times (y_0 + \mathbb{R}k_0)]} = \overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0 + \mathbb{R}k_0)]$

for some $y_0 \in Y$ and $k_0 \in \mathring{K}$,

(ii) (VP) has a weak zero duality gap.

Proof. [(i)⟹(ii)] Assume that there are $y_0 \in Y$ and $k_0 \in \mathring{K}$ satisfying

$$\overline{\mathbb{H} \cap [\{0_Z\} \times (y_0 + \mathbb{R}k_0)]} = \overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0 + \mathbb{R}k_0)]. \quad (11)$$

Let

$$\lambda_0 \coloneqq \inf\{\lambda \in \mathbb{R}: y_0 + \lambda k_0 \in \mathbb{E}_1\} = \inf\{\lambda \in \mathbb{R}: y_0 + \lambda k_0 \in \mathbb{E}_1 + K\}$$

(see Proposition 3). Then, according to Lemma 2, one has $y_0+\lambda_0 k_0 \in WInf\,\mathbb{E}_1$ which, together with Proposition 4, yields $y_0+\lambda_0 k_0 \in \mathrm{val(VP)}$.

We now prove that $y_0+\lambda_0 k_0 \in \mathrm{val(VP^*)}$. With the help of Lemma 2 and Proposition 5, we begin by proving

$$\lambda_0 = \inf\{\lambda \in \mathbb{R}: y_0+\lambda k_0 \in \mathbb{E}_2+K\}$$
$$= \inf\{\lambda \in \mathbb{R}: y_0+\lambda k_0 \in \mathbb{E}_2\}$$

(see Proposition 3).

Set $\lambda'_0 := \inf\{\lambda \in \mathbb{R}: y_0+\lambda k_0 \in \mathbb{E}_2\}$.

As $\mathbb{E}_1 \subset \mathbb{E}_2$, one has

$$\lambda_0 \geq \lambda'_0 \qquad (12)$$

Three following cases are possible:

Case 1. $\lambda_0 = -\infty$. Then, (12) yields $\lambda'_0 = -\infty = \lambda_0$.

Case 2. $\lambda_0 = +\infty$. Then, one has $\mathbb{E}_1 \cap (y_0+\mathbb{R}k_0) = \emptyset$, or equivalently, $\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)] = \emptyset$. This accounts for $\overline{\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]} = \emptyset$, and then, by (11), one gets $\overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)] = \emptyset$ which yields $\mathbb{E}_2 \cap (y_0+\mathbb{R}k_0) = \emptyset$. So, $\lambda'_0 = +\infty = \lambda_0$.

Case 3. $\lambda_0 \in \mathbb{R}$. We claim that $\lambda'_0 = \lambda_0$. Conversely, by (12), suppose that $\lambda_0 > \lambda'_0$. Then, there is $\lambda_1 < \lambda_0$ such that $y_0+\lambda_1 k_0 \in \mathbb{E}_2$, or equivalently,

$(0_Z, y_0+\lambda_1 k_0) \in \overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]$.

This, together with (11), leads to

$(0_Z, y_0+\lambda_1 k_0) \in \overline{\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]}$

and hence,

$\left(Z \times \left[y_0+\lambda_0 k_0 - \overset{\circ}{K}\right]\right) \cap (\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)\}]) \neq \emptyset$

(as $Z \times \left[y_0+\lambda_0 k_0 - \overset{\circ}{K}\right]$ is a neighborhood of $(0_Z, y_0+\lambda_1 k_0)$). Consequently, there is $\lambda_2 < \lambda_0$ such that $(0_Z, y_0+\lambda_2 k_0) \in \mathbb{H}$ which yields $y_0+\lambda_2 k_0 \in \mathbb{E}_1$. This contradicts the fact that $\lambda_0 = \inf\{\lambda \in \mathbb{R}: y_0+\lambda k_0 \in \mathbb{E}_1\}$.

So, $\lambda_0 = \lambda'_0$.

In brief, we have just proved that $y_0+\lambda_0 k_0 \in \mathrm{val(VP)} \cap \mathrm{val(VP^*)}$ which also means that $\mathrm{val(VP)} \cap \mathrm{val(VP^*)} \neq \emptyset$.

[(ii)⟹(i)] Assume that there is $y_0 \in val(\mathrm{VP}) \cap val(\mathrm{VP^*})$. Pick arbitrarily $k_0 \in \overset{\circ}{K}$. We now prove that

$$\overline{\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]}$$
$$= \overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]. \qquad (13)$$

It is easy to see that

$$\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)] \subset \overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]$$

and that $\overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]$ is a closed set. So, the inclusion "⊂" in (13) holds trivially. For the converse inclusion, take arbitrarily $(0_Z, y_0+\tilde{\lambda}k_0) \in \overline{\mathbb{H}} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]$ we will prove that

$$(0_Z, y_0+\tilde{\lambda}k_0) \in \overline{\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]}.$$

As $(0_Z, y_0+\tilde{\lambda}k_0) \in \overline{\mathbb{H}}$ we have $y_0+\tilde{\lambda}k_0 \in \mathbb{E}_2$, which implies that $\tilde{\lambda} \geq \inf\{\lambda \in \mathbb{R}: y_0+\lambda k_0 \in \mathbb{E}_2\}$. On the other hand, it holds $y_0 \in val(\mathrm{VP^*}) = WInf\,\mathbb{E}_2$ (see Proposition 5), and hence, $\inf\{\lambda \in \mathbb{R}: y_0+\lambda k_0 \in \mathbb{E}_2\} = 0$ (see Lemma 2). So, one gets $\tilde{\lambda} \geq 0$, which yields

$$y_0 <_K y_0 + \left(\tilde{\lambda}+\frac{1}{n}\right)k_0, \quad \forall n \in \mathbb{N}^*. \qquad (14)$$

Note that, one also has $y_0 \in \mathrm{val(P)} = WInf\,\mathbb{E}_1$. So, for each $n \in \mathbb{N}^*$, it follows from (14) and the definition of infimum that the existence of $y_n \in \mathbb{E}_1$ such that $y_n <_K y_0 + \left(\tilde{\lambda}+\frac{1}{n}\right)k_0$, and consequently, $y_0 + \left(\tilde{\lambda}+\frac{1}{n}\right)k_0 \in \mathbb{E}_1 + \overset{\circ}{K} \subset \mathbb{E}_1 + K = \mathbb{E}_1$ (see Proposition 3) which yields

$(0_Z, y_0 + \left(\tilde{\lambda}+\frac{1}{n}\right)k_0) \in \mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]$.

As $(0_Z, y_0 + \left(\tilde{\lambda}+\frac{1}{n}\right)k_0) \to (0_Z, y_0+\tilde{\lambda}k_0)$ we obtain

$(0_Z, y_0+\tilde{\lambda}k_0) \in \overline{\mathbb{H} \cap [\{0_Z\} \times (y_0+\mathbb{R}k_0)]}$.

The proof is complete. □

We now recall the qualification condition (Nguyen Dinh *et al.*, 2020)

$$(CQ) \quad \overline{\mathbb{H} \cap (\{0_Z\} \times Y)} = \overline{\mathbb{H}} \cap (\{0_Z\} \times Y).$$

We now study the results on a strong zero duality gap between the problem (VP) and its Lagrange dual problems, which are established under the condition (CQ) without using Farkas-type results while the such ones were established in Nguyen Dinh *et al.*, 2020, where the authors have used Farkas-type results for vector optimization under the condition (CQ) to obtain the ones (see Nguyen Dinh *et al.*, 2020, Theorem 6.1). We will show that it is possible to obtain the ones by using the convex separation theorem (through the use of Proposition 5 given in the previous section). The important point to note here is the use of the convex separation theorem to establish the Farkas-type results for vector optimization in Nguyen Dinh *et al.*, 2020 while the convex separation theorem to calculate the supremum value of (VP*) in this paper.

Theorem 2. *Assume that* $val(\mathrm{VP}^*) \neq \{-\infty_Y\}$. *Assume further that* F *is* K*-convex, that* G *is* S*-convex, and that* C *is convex. Then, the following statements are equivalent:*

(i) (CQ) *holds,*

(ii) (VP) *has a strong zero duality gap.*

Proof. $[(i) \Longrightarrow (ii)]$ Assume that (i) holds. Since p is continuous, we have

$$p\left(\overline{\mathbb{H} \cap [\{0_Z\} \times Y]}\right) \subset \overline{p(\mathbb{H} \cap [\{0_Z\} \times Y])} = \overline{\mathbb{E}_1}.$$

As (i) holds, it follows from Proposition 3(iv) that $\mathbb{E}_2 \subset \overline{\mathbb{E}_1}$. Recall that $\mathbb{E}_1, \mathbb{E}_2$ are nonempty subset of Y (by the definition of $\mathbb{E}_1, \mathbb{E}_2$ and Proposition 3(iii)). So, $\mathrm{W}Inf\,\mathbb{E}_2 \neq \{+\infty_Y\}$ and $\overline{\mathbb{E}_1} \subset Y$, and then, Proposition 2 shows that $\mathrm{W}Inf\,\mathbb{E}_2 \subset \mathrm{W}Inf(\overline{\mathbb{E}_1}) + K$. Noting that $\mathrm{W}Inf(\overline{\mathbb{E}_1}) = \mathrm{W}Inf\,\mathbb{E}_1$ (Nguyen Dinh *et al.*, 2017, Proposition 2.1(iv)). Hence, $\mathrm{W}Inf\,\mathbb{E}_2 \subset \mathrm{W}Inf\,\mathbb{E}_1 + K$. Combining this with the fact that $val(\mathrm{P}) = \mathrm{W}Inf\,\mathbb{E}_1$ and $val(\mathrm{VP}^*) = \mathrm{W}Inf\,\mathbb{E}_2$, we get $val(\mathrm{VP}^*) \subset val(\mathrm{VP}) + K$. As $K + \mathring{K} = \mathring{K}$ we have

$$\mathrm{val}(\mathrm{VP}^*) + \mathring{K} \subset \mathrm{val}(\mathrm{VP}) + \mathring{K}. \qquad (15)$$

On the other hand, by the weak duality (see Remark 3), one has $val(\mathrm{VP}) + \mathring{K} \subset \mathrm{val}(\mathrm{VP}^*) + \mathring{K}$, which, together with (15), gives $\mathrm{val}(\mathrm{VP}) + \mathring{K} = \mathrm{val}(\mathrm{VP}^*) + \mathring{K}$, and (ii) is achieved, taking (Lohne, 2011, Corollary 1.48) into account.

$[(ii) \Longrightarrow (i)]$ Assume that (ii) holds, we will prove that (i) holds. It is clear that

$$\overline{\mathbb{H} \cap [\{0_Z\} \times Y]} \subset \overline{\mathbb{H}} \cap (\{0_Z\} \times Y). \quad (16)$$

So, we only need to show that the converse inclusion of (16) holds. Take $(0_Z, \bar{y}) \in \overline{\mathbb{H}}$. Then, one has $\bar{y} \in \mathbb{E}_2$.

Assume that $val(\mathrm{VP}) = \{-\infty_Y\}$. Then, in the light of Proposition 4, one has $\mathrm{W}Inf\,\mathbb{E}_1 = \{-\infty\}$ which also means that $Y = \mathbb{E}_1 + K$ (see Remark 1). Observing that $\mathbb{E}_1 + K = \mathbb{E}_1$, consequently, $Y = \mathbb{E}_1$. This entails $\bar{y} \in \mathbb{E}_1$, or equivalently, $(0_Z, \bar{y}) \in \mathbb{H}$ showing that $(0_Z, \bar{y}) \in \overline{\mathbb{H} \cap [\{0_Z\} \times Y]}$.

Assume that $val(\mathrm{VP}) \neq \{-\infty_Y\}$. Then, as (ii) holds, from Propositions 4 and 5, $\mathrm{W}Inf\,\mathbb{E}_1 = \mathrm{val}(\mathrm{VP}) = \mathrm{val}(\mathrm{VP}^*) = \mathrm{W}Inf\,\mathbb{E}_2 \neq \{-\infty_Y\}$. By the decomposition
$Y = (\mathrm{W}Inf\,\mathbb{E}_2 - \mathring{K}) \cup (\mathrm{W}Inf\,\mathbb{E}_2 + K)$
(see Proposition 1) and the fact that
$\mathbb{E}_2 \cap (\mathrm{W}Inf\,\mathbb{E}_2 - \mathring{K}) = \emptyset$ (see Remark 2), one gets $\mathbb{E}_2 \subset \mathrm{W}Inf\,\mathbb{E}_2 + K$. So, there are $y_0 \in \mathrm{W}Inf\,\mathbb{E}_2$ and $\bar{k} \in K$ such that $\bar{y} = y_0 + \bar{k}$.

Pick $k_0 \in \mathring{K}$. For each $n \in \mathbb{N}^*$, one has

$$y_0 <_K y_0 + \bar{k} + \frac{1}{n}k_0 = \bar{y} + \frac{1}{n}k_0.$$

This, together with the fact that $y_0 \in \mathrm{W}Inf\,\mathbb{E}_2 = \mathrm{W}Inf\,\mathbb{E}_1$ yields the existence of sequence $\{y_n\}_{n\in\mathbb{N}^*} \subset \mathbb{E}_1$ such that $y_n <_K \bar{y} + \frac{1}{n}k_0$ for all $n \in \mathbb{N}^*$.

Then, $\bar{y} + \frac{1}{n}k_0 \in \mathbb{E}_1 + \mathring{K} \subset \mathbb{E}_1 + K = \mathbb{E}_1$ (see Proposition 3) which is equivalent to $(0_Z, \bar{y} + \frac{1}{n}k_0) \in \mathbb{H} \cap [\{0_Z\} \times Y]$. Here, note that $(0_Z, \bar{y} + \frac{1}{n}k_0) \to (0_Z, \bar{y})$, we obtain $(0_Z, \bar{y}) \in \overline{\mathbb{H} \cap [\{0_Z\} \times Y]}$, which is desired.

Remark 5. It is worth mentioning that when we take $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $C = P$, $f(\cdot) = \langle c, \cdot \rangle$, $G(\cdot) = -A(\cdot) + b$, the problem (VP) collapses to the problem (ILP) in Pham Duy Khanh *et al.* (2019). Then, the result on strong duality for the problem (ILP) in Pham Duy Khanh *et al.* (2019, Theorem 4.3) follows from Theorem 2.

Let us now introduce the second qualification condition, saying that $\mathbb{H}$ is closed regarding the set $\{0_Z\} \times Y$, concretely,

$$(CQ_{\mathrm{b}is}) \quad \mathbb{H} \cap (\{0_Z\} \times Y) = \overline{\mathbb{H}} \cap (\{0_Z\} \times Y).$$

Theorem 3. *Assume that the problem (VP) is feasible and $val(VP^*) \neq \{-\infty_Y\}$. Assume further that F is K-convex, that G is S-convex, and that C is a convex set of X. If the condition (CQ_{bis}) holds then the problem* (VP) *has a strong zero duality gap.*

Proof. According to Proposition 4 and Proposition 5, we have $\mathrm{v}al(\mathrm{VP}) = \mathrm{W}Inf\,\mathbb{E}_1$ and $\mathrm{v}al(\mathrm{VP}^*) = \mathrm{W}Inf\,\mathbb{E}_2$. As $(CQ_{\mathrm{b}is})$ holds, one finds that $\mathbb{E}_1 = \mathbb{E}_2$. Consequently, one has $\mathrm{v}al(\mathrm{VP}) = \mathrm{v}al(\mathrm{VP}^*)$. □

The following example shows that the converse implication in Theorem 3 does not hold.

Example 1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, and $C =]0,2[$. Let $F: \mathbb{R} \to \mathbb{R}^2$ and $G: \mathbb{R} \to \mathbb{R}$ be such that $F(x) = (x, 1-x)$ and $G(x) = x^2 - 1$ for all $x \in \mathbb{R}$. It is easy to see that F is K-convex, that G is S-convex, and C is convex. In this case, we have

$$\mathbb{H} = \bigcup_{x \in C \cap \mathrm{d}om F \cap \mathrm{d}om G} (G(x) + S) \times (F(x) + K)$$

$$= \bigcup_{x \in [0,2]} (x^2 - 1 + \mathbb{R}_+) \times ((x, 1-x) + \mathbb{R}^2_+).$$

By some calculations, we obtain

$$\mathbb{H} \cap (\{0_Z\} \times Y) = \{0\} \times (\{(x,y): 0 < x \leq 1, y \geq 1-x\} \cup ([1, +\infty[\times [0, +\infty)),$$

$$\overline{\mathbb{H}} \cap (\{0_Z\} \times Y) = \{0\} \times (\{(x,y): 0 \leq x \leq 1, y \geq 1-x\} \cup ([1, +\infty[\times [0, +\infty)).$$

On the other hand,

val(VP)

$= \mathrm{W}Inf\{F(x): x \in C, G(x) \in -S\}$

$= \mathrm{W}Inf\{(x, 1-x): x \in]0,1]\}$

$= (\{0\} \times [1, +\infty[\cup \{(x, 1-x): x \in [0,1]\} \cup ([1, +\infty[\times \{0\}),$

$\mathrm{W}MinF(A) = \{(x, 1-x): x \in]0,1]\}$, and

val(VP*)

$= \mathrm{W}Inf\,\mathbb{E}_2$

$= (\{0\} \times [1, +\infty) \cup \{(x, 1-x): x \in [0,1]\} \cup ([1, +\infty[\times \{0\}).$

It is clear that the converse implication in Theorem 3 does not hold.

5. A special case: Linear programming

In the this section, as an illustrate example for the results established above, we consider a special case of the problem (VP), that is the linear programming:

$$(\mathrm{LP}) \ \inf\langle c, x\rangle \ \ s.t\ x \in X, \qquad A(x) - \omega \in -S$$

where $c \in X^*$, $A \in \mathcal{L}(X, Z)$, and $\omega \in Z$. Observing that the problem (VP) collapses to the problem (LP) when we take $Y = \mathbb{R}$, $K = \mathbb{R}_+$ $C = X$, $f(\cdot) = \langle c, \cdot \rangle$, $G(\cdot) = A(\cdot) - w$. Then, the corresponding *characterizing set* of (LP) is

$$\mathbb{H}_L = \{(A(x), \langle c, x\rangle): x \in X\} + [S - \omega] \times \mathbb{R}_+.$$

The qualification condition (CQ) now is

$$(CQLP)\ \overline{\mathbb{H}_L \cap (\{0_Z\} \times \mathbb{R})} = \overline{\mathbb{H}_L} \cap (\{0_Z\} \times \mathbb{R}).$$

Recall that the *Lagrange dual problem* of (LP), denoted by $(\mathrm{LD^L})$, is

$$(\mathrm{LD^L}) \ \ \sup[-\langle \lambda, \omega\rangle] \ \ s.t. \lambda \in S^+, \ \lambda A = -c.$$

It is worth mentioning that the problem (LP) is a special case of the linear programming problem (IP) in Anderson (1983) and the problem (ILP) in Pham Duy Khanh *et al.* (2019) where $P = X$. The duality for the problem (ILP) was considered in Anderson (1983) under the closedness conditions. Recently, Pham Duy Khanh *et al.* (2019) had studied the duality for the problem (ILP) under some necessary and sufficient conditions.

We now introduce a new type of dual problem of (LP) called the *sequential dual problem* as follows:

$$(\mathrm{LD^S})\ \sup\left[-\limsup_{n\to\infty}\langle\lambda_n,\omega\rangle\right]\quad s.t.\ (\lambda_n)_{n\in\mathbb{N}^*}\subset S^+,\lambda_n A=-c,\forall n\in\mathbb{N}^*.$$

The relations between the values of the problem (LP) and its dual problems are given by the following proposition.

Proposition 6. *It holds:*

$sup(\mathrm{LD^L})\le sup(\mathrm{LD^S})\le inf(\mathrm{LP})$.

Proof.

• *Prove that $sup(\mathrm{LD^L})\le sup(\mathrm{LD^S})$:* It is easy to see that

$$\sup(\mathrm{LD^L})=\sup\{-\limsup_{n\to\infty}\langle\lambda_n,\omega\rangle:(\lambda_n)_{n\in\mathbb{N}^*}\in\mathcal{D}_L\}$$

$$\sup(\mathrm{LD^S})=\sup\{-\limsup_{n\to\infty}\langle\lambda_n,\omega\rangle:(\lambda_n)_{n\in\mathbb{N}^*}\in\mathcal{D}_S\}$$

where

$$\mathcal{D}_L:=\{(\lambda_n)_{n\in\mathbb{N}^*}\subset S^+:\lambda_n=\lambda\in S^+,\ \forall n\in\mathbb{N}^*,\lambda A=-c\}$$

$$\mathcal{D}_S:=\{(\lambda_n)_{n\in\mathbb{N}^*}\subset S^+:\lambda_n A=-c,\ \forall n\in\mathbb{N}^*\}$$

Obviously, $\mathcal{D}_L\subset\mathcal{D}_S$. So, $\sup(\mathrm{LD^L})\le\sup(\mathrm{LD^S})$.

• *Prove that $sup(\mathrm{LD^S})\le inf(\mathrm{LP})$:* Take $(\lambda_n)_{n\in\mathbb{N}^*}\in\mathcal{D}_S$ and $x\in X$ such that $A(x)-\omega\in -S$. Then, $\langle\lambda_n,\omega\rangle\ge\langle\lambda_n,A(x)\rangle=-\langle c,x\rangle$ for all $n\in\mathbb{N}^*$, and hence,

$\limsup_{n\to\infty}\langle\lambda_n,\omega\rangle\ge-\langle c,x\rangle$,

or, $-\limsup_{n\to\infty}\langle\lambda_n,\omega\rangle\le\langle c,x\rangle$.

The desired inequality follows from the definition of the problems (LP) and $(\mathrm{LD^S})$. □

The next result extends (Pham Duy Khanh *et al.*, 2019, Theorem 4.3) in the case when taking $P=X$.

Corollary 3. *The following statements are equivalent:*

(i) $(CQLP)$ holds,

(ii) $\sup(\mathrm{LD^L})=\sup(\mathrm{LD^S})=\inf(\mathrm{LP})$.

Proof. Firstly, by Proposition 6, (ii) is equivalent to (ii') $\sup(\mathrm{LD^L})=\inf(\mathrm{LP})$. The conclusion now follows from Theorem 2. □

We next introduce a sufficient condition, which ensures the fulfillment of the condition $(CQLP)$, and then, leads to the results on zero duality gap for the pairs $(\mathrm{LP})-(\mathrm{LD^L})$ and $(\mathrm{LP})-(\mathrm{LD^S})$.

Proposition 7. *Assume that there are $\lambda_0\in S^+$ and $x_0\in X$ such that*

$$\lambda_0 A=-c,\tag{14}$$

$$A(x_0)\in\omega-S\ \text{and}\ \lambda_0 A(x_0)=\langle\lambda_0,\omega\rangle\tag{15}$$

Then, $(CQLP)$ holds.

Proof. It is sufficient to prove that $\overline{\mathbb{H}_L}\cap(\{0_Z\}\times\mathbb{R})\subset\overline{\mathbb{H}_L\cap(\{0_Z\}\times\mathbb{R})}$. To do this, take $(0_Z,r)\in\overline{\mathbb{H}_L}$. We will show that $(0_Z,r)\in\overline{\mathbb{H}_L\cap(\{0_Z\}\times\mathbb{R})}$. Indeed, since $(0_Z,r)\in\overline{\mathbb{H}_L}$, it follows that there exists a net $(z_\alpha,r_\alpha,x_\alpha)_{\alpha\in I}\subset Z\times\mathbb{R}\times X$ such that

$$(z_\alpha,r_\alpha)\to(0_Z,r)\tag{16}$$

$$z_\alpha\in A(x_\alpha)-\omega+S\ \text{and}\ r_\alpha\ge\langle c,x_\alpha\rangle,\ \forall\alpha\in I.\tag{17}$$

Assume that there are $\lambda_0\in S^+$ and $x_0\in X$ such that (14) and (15) holds. This, together with (17), leads to the fact that $\langle\lambda_0,z_\alpha\rangle\ge-\langle c,x_\alpha\rangle-\langle\lambda_0,\omega\rangle\ge-r_\alpha-\langle\lambda_0,\omega\rangle$ for all $\alpha\in I$.

Since $z_\alpha\to 0_Z$ and $r_\alpha\to r$, it follows from the above inequality that $0\ge -r-\langle\lambda_0,\omega\rangle$. This, together with the last one of (15) and (14), one gets

$$0\ge -r-\langle\lambda_0,\omega\rangle$$

$$=-r-\lambda_0 A(x_0)=-r+\langle c,x_0\rangle,$$

or equivalently, $r\ge\langle c,x_0\rangle$. From this and the first one of (15), we obtain

$(0_Z, r) \in \mathbb{H}_L \cap (\{0_Z\} \times \mathbb{R})$,
and hence, $(0_Z, r) \in \overline{\mathbb{H}_L \cap (\{0_Z\} \times \mathbb{R})}$
as desired. □

The next result is a direct consequence of Proposition 7 and Corollary 3.

Corollary 4. *Assume all the assumptions of Proposition 7 hold. Then, one has*

$$\sup(\mathrm{LD^L}) = \sup(\mathrm{LD^S}) = \inf(\mathrm{LP}).$$

Corollary 5. *Assume that the following conditions hold:*

(C_1) *The problem* $(\mathrm{LD^L})$ *is feasible, i.e., there is* $\lambda_0 \in S^+$ *such that* $\lambda_0 A = -c$.

(C_2) $\omega \in A(X)$.

Then, $\sup(\mathrm{LD^L}) = \sup(\mathrm{LD^S}) = \inf(\mathrm{LP})$.

Proof. The fulfillment of (C_1) means that there is $\lambda_0 \in S^+$ such that (14). As (C_2) holds, there exists $x_0 \in X$ such that $\omega = A(x_0)$. This leads to the fact that (15) holds. The conclusion now follows from Corollary 4. □

Corollary 6. *Assume that* (C_1) *and one of the following condition holds:*

(C_3) $\omega = 0_Z$.

(C_4) *A is a surjection.*

Then, $\sup(\mathrm{LD^L}) = \sup(\mathrm{LD^S}) = \inf(\mathrm{LP})$.

Proof. It is easy to see that if at least one of the conditions (C_3) and (C_4) holds then (C_2) holds as well. So, Corollary 6 is a consequence of Corollary 5. □

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