ERROR BOUNDS FOR A CLASS OF MIXED PARAMETRIC VECTOR QUASI-EQUILIBRIUM PROBLEMS

Nguyen Huynh Vu Duy[1], Nguyen Ngoc Hien[2], and Vo Minh Tam[3]*

[1]*Office of Academic Affairs, Ho Chi Minh City Open University*

[2]*Academic Affairs Office, Dong Thap University*

[3]*Faculty of Mathematics - Informatics Teacher Education, Dong Thap University*

**Corresponding author: vmatm@dthu.edu.vn*

Article history

Received:11/06/2021; Received in revised form: 02/12/2021; Accepted: 09/12/2021

Abstract

In this paper, we establish the regularized gap function for a class of mixed parametric vector quasi-equilibrium problems (briefly, (MPVQEP)$_\gamma$). Then an error bound is also provided for (MPVQEP)$_\gamma$ via this gap function under suitable assumptions. Some examples are given to illustrate our results. Our main results extend and differ from those corresponding ones in the current literatures.

Keywords: *Error bound, mixed parametric vector quasi-equilibrium problems, regularized gap function, strongly monotone.*

CẬN SAI SỐ CHO MỘT LỚP BÀI TOÁN TỰA CÂN BẰNG VÉCTƠ THAM SỐ HỖN HỢP

Nguyễn Huỳnh Vũ Duy[1], Nguyễn Ngọc Hiền[2] và Võ Minh Tâm[3]*

[1]*Phòng Quản lý Đào tạo, Trường Đại học Mở Thành phố Hồ Chí Minh*

[2]*Phòng Đào tạo, Trường Đại học Đồng Tháp*

[3]*Khoa Sư phạm Toán - Tin, Trường Đại học Đồng Tháp*

**Tác giả liên hệ: vmtam@dthu.edu.vn*

Lịch sử bài báo

Ngày nhận: 11/06/2021; Ngày nhận chỉnh sửa: 02/12/2021; Ngày duyệt đăng: 09/12/2021

Tóm tắt

Trong bài báo này, chúng tôi thiết lập hàm gap chỉnh hóa cho một lớp bài toán tựa cân bằng véctơ tham số hỗn hợp (viết tắt là (MPVQEP)$_\gamma$). Khi đó, một cận sai số cũng thu được cho bài toán (MPVQEP)$_\gamma$ thông qua hàm gap chỉnh hóa được xem xét bởi một số giả thiết phù hợp. Một số ví dụ được đưa ra để mô tả các kết quả đạt được. Các kết quả chính của chúng tôi trong bài báo này mở rộng và khác với các kết quả tương ứng đã được nghiên cứu trong những công trình gần đây.

Từ khóa: *Cận sai số, bài toán tựa cân bằng véctơ tham số hỗn hợp, hàm gap chỉnh hóa, đơn điệu mạnh.*

DOI: https://doi.org/10.52714/dthu.11.5.2022.974

Cite: Nguyen Huynh Vu Duy, Nguyen Ngoc Hien, and Vo Minh Tam. (2022). Error bounds for a class of mixed parametric vector quasi-equilibrium problems. *Dong Thap University Journal of Science, 11*(5), 3-8.

1. Introduction and preliminaries

In 1997, Yamashita and Fukushima introduced a class of merit functions for variational inequality problems:

$$\Omega(x,\theta)=\sup_{y\in K}\{\langle h(x),x-y\rangle-\theta\|x-y\|^2\},$$

where θ is a nonnegative parameter, $\Omega(\cdot,\theta):\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$, $K\subset\mathbb{R}^n$, $h:\mathbb{R}^n\to\mathbb{R}^n$. This function was first introduced by Auslender (1976) for $\theta=0$ and by Fukushima (1992) for $\theta>0$. The function $\Omega(\cdot,0)$ is called the gap function, while the function $\Omega(\cdot,\theta)$ is called the regularized gap function, with $\theta>0$. One of the many useful applications of gap functions is to derive the so-called error bounds as an upper estimation of the distance between the solution set and an arbitrary feasible point. Since then, many authors investigated the regularized gap functions and error bounds for various kinds of optimization problems, variational inequality problems and equilibrium problems (see, for example, Anh *et al.* (2018), Bigi and Passacantando (2016), Gupta and Mehra (2012), Hung *et al.* (2020a, 2020b, 2021), Khan and Chen (2015a, 2015b), Mastroeni (2003) and the references therein).

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively.

Let $\mathbb{R}^m_+=\{(y_1,...,y_m)\in\mathbb{R}^m: y_i\geq 0, i=1,2,...,m\}$ be the nonnegative orthant of $\mathbb{R}^m, A\subset\mathbb{R}^n$ be a nonempty, closed and convex set in $\mathbb{R}^n$ and Γ be nonempty subsets of a finite dimensional space. For each $i\in\{1,2,...,m\}$, let $T_i:\Gamma\times A\to\mathbb{R}^n$, $H_i:A\times A\to\mathbb{R}$ be continuous bifunctions such that $H_i(x,x)=0$ for all $x\in A$ and $\kappa:A\times A\to\mathbb{R}^n$ be a continuous bifunction such that $\kappa(x,y)+\kappa(y,x)=0_{\mathbb{R}^n}$ for all $x,y\in A$. Let $K:A\times\Gamma\to 2^A, H:=(H_1,H_2,...,H_m), T:=(T_1,T_2,...,T_m)$ and for any $x,v\in\mathbb{R}^n$,

$$\langle T(\gamma,x),v\rangle:=(\langle T_1(\gamma,x),v\rangle,\langle T_2(\gamma,x),v\rangle,...,\langle T_m(\gamma,x),v\rangle).$$

We now consider the following generalized parametric vector quasi-equilibrium problem (briefly, $(MPVQEP)_\gamma$) in finding $x\in K(x,\gamma)$ for each parameter $\gamma\in\Gamma$ fixed such that

$$H(x,y)+\langle T(\gamma,x),\kappa(y,x)\rangle\notin-\mathrm{int}\mathbb{R}^m_+,\forall y\in K(x,\gamma). \quad (1)$$

Given $S(\gamma)$ the solution set of $(MPVQEP)_\gamma$. we always assume that $S(\gamma)\neq\emptyset$ for all $\gamma\in\Gamma$. To illustrate motivations for this setting, we provide some special cases of the problem $(\mathrm{MPVQEP})_\gamma$:

(a) If $m=1$, $K(x,\gamma)\equiv A$, $H_1\equiv 0$, $T_1(\gamma,x)=T_1(x)$ $\kappa(y,x)=y-x,\forall\gamma\in\Gamma,x,y\in A,$ then $(\mathrm{MPVQEP})_\gamma$ reduces to the following variational inequality problem (briefly, (VIP)) studied in Yamashita and Fukushima (1997) of finding $x\in A$ such that

$$\langle T_1(x),y-x\rangle\geq 0,\forall y\in A.$$

(b) If $m=1,\kappa\equiv 0,K(x,\gamma)=K(x)$, $\forall\gamma\in\Gamma$, $x\in A$ then the problem $(\mathrm{MPVQEP})_\gamma$ reduces to the following abstract quasiequilibrium problem (briefly, (QEP)) studied in Bigi and Passacantando (2016) of finding $x\in K(x)$ such that

$$H_1(x,y)\geq 0,\forall y\in K(x).$$

In this paper, we study regularized gap functions and error bounds for the problem $(\mathrm{MPVQEP})_\gamma$ under suitable assumptions. We also provide some examples to support the results presented in this paper. Our main results extend and differ from those corresponding ones in the current literatures.

We recall some notations and definitions used in the sequel.

Definition 1. (See Rockafellar and Wets (1998)) A real valued function $f:A\to\mathbb{R}$ is said to be *convex* if

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for every $x,y\in A$ and $\lambda\in[0,1]$.

Definition 2. Let $T:\Gamma\times A\to\mathbb{R}^n$, $f:A\times A\to\mathbb{R}$ and $\kappa:A\times A\to\mathbb{R}^n$ be functions. Then

(i) (See Mastroeni (2003)) f is said to be *strongly monotone* with modulus $\alpha>0$ if, for each $(x,y)\in A\times A$,

$$f(x,y)+f(y,x)\leq-\alpha\|x-y\|^2;$$

(ii) T is said to be $\kappa-$*strongly monotone* with modulus $\mu>0$ if, for each $\gamma\in\Gamma$, $(x,y)\in A\times A$,

$$\langle T(\gamma,y)-T(\gamma,x),\kappa(y,x)\rangle\geq\mu\|x-y\|^2.$$

2. Regularized gap functions and error bounds for (MPVQEP)$_\gamma$

In the section, we propose the regularized gap function and error bound for $(MPVQEP)_\gamma$. Motivated by Auslender (1976), Bigi and Passacantando (2016), we consider the following definition of gap functions. Let

$$\mathrm{P}(\gamma)=\{x\in A: x\in K(x,\gamma)\},\forall\gamma\in\Gamma$$

and we assume that $\mathrm{P}(\gamma)\neq\emptyset,\forall\gamma\in\Gamma.$

Definition 3. A real valued function $p:\mathbb{R}^n\times A\to\mathbb{R}$ is said to be a *gap function* of (MPVQEP)$_\gamma$ if it satisfies the following conditions: for each $\gamma\in\Gamma$ fixed,

(a) $p(\gamma,x)\geq 0,$ for all $x\in\mathrm{P}(\gamma)$.

(b) for any $x_0\in\mathrm{P}(\gamma)$, $p(\gamma,x_0)=0$ if and only if x_0 is a solution of (MPVQEP)$_\gamma$.

Inspired by the approaches of Yamashita and Fukushima (1997), we construct a regularized gap function for (MPVQEP)$_\gamma$. Suppose that $K(x,\gamma)$ is a compact set for each $x\in A$ and $\gamma\in\Gamma$, then for each $\theta>0$ and $\gamma\in\Gamma$ fixed, we consider a function $\Omega_\theta:\Gamma\times A\to\mathbb{R}$ defined by

$$\Omega_\theta(\gamma,x)=\max_{y\in K(x,\gamma)}\left\{h(\gamma,x,y)-\theta\|x-y\|^2\right\},\quad(2)$$

where

$$h(\gamma,x,y)=\min_{1\leq i\leq m}\left\{-H_i(x,y)+\langle T_i(\gamma,x),\kappa(x,y)\rangle\right\}.$$

Remark 1. The function Ω_θ in (2) is well-defined. Indeed, as H_i,T_i and κ are continuous for any $i=1,2,...,m$, the function h is continuous. Combine the continuity of h, $\|\cdot\|$ and the compactness of $K(x,\gamma)$ for each $x\in A$ and $\gamma\in\Gamma$, we have Ω_θ is well-defined.

We show that Ω_θ is a gap function for (MPVQEP)$_\gamma$ under suitable conditions.

Theorem 1. *Assume that*

(i) K *has compact and convex values;*

(ii) H_i *is convex in the second component for all* $i=1,2,...,m$;

(iii) *for each* $t\in\mathbb{R}^n$ *and* $x\in A$, *the function* $y\mapsto\langle t,\kappa(y,x)\rangle$ *is convex.*

Then, for each $\gamma\in\Gamma$ *and* $\theta>0$, *the function* Ω_θ *defined by* (2) *is a gap function for (MPVQEP)*$_\gamma$.

Proof. (a) For each $\gamma\in\Gamma$ fixed, it is clear that for any $x\in\mathrm{P}(\gamma)$, i.e., $x\in K(x,\gamma)$ and so

$$\begin{aligned}\Omega_\theta(\gamma,x)&=\max_{y\in K(x,\gamma)}\left\{h(\gamma,x,y)-\theta\|x-y\|^2\right\}\\&\geq h(\gamma,x,x).\end{aligned}\quad(3)$$

We have

$$\begin{aligned}&h(\gamma,x,x)\\&=\min_{1\leq i\leq m}\left\{-H_i(x,x)+\langle T_i(\gamma,x),\kappa(x,x)\rangle\right\}=0.\end{aligned}$$

Then, from (3), we conclude that $\Omega_\theta(\gamma,x)\geq 0$ for any $x\in\mathrm{P}(\gamma)$.

(b) If there exists $x_0\in\mathrm{P}(\gamma)$, i.e., $x_0\in K(x_0,\gamma)$ such that $\Omega_\theta(\gamma,x_0)=0$, then

$$h(\gamma,x_0,y)-\theta\|x_0-y\|^2\leq 0,\forall y\in K(x_0,\gamma)$$

or

$$\begin{aligned}&\min_{1\leq i\leq m}\{-H_i(x_0,y)+\langle T_i(\gamma,x_0),\kappa(x_0,y)\rangle\}\\&\leq\theta\|x_0-y\|^2,\forall y\in K(x_0,\gamma).\end{aligned}$$

For arbitrary $x\in K(x_0,\gamma)$ and $\lambda\in(0,1)$, let $y_\lambda=x+\lambda(x_0-x)$. Since $K(x_0,\gamma)$ is a convex set, we get $y_\lambda\in K(x_0,\gamma)$ and so

$$\begin{aligned}&\min_{1\leq i\leq m}\left\{-H_i(x_0,y_\lambda)+\langle T_i(\gamma,x_0),\kappa(x_0,y_\lambda)\rangle\right\}\\&\quad\leq\theta\|x_0-y_\lambda\|^2.\end{aligned}\quad(4)$$

Since H_i is convex in the second component for all $i=1,2,...,m$, we have

$$\begin{aligned}-H_i(x_0,y_\lambda)&\geq-\lambda H_i(x_0,x_0)-(1-\lambda)H_i(x_0,x)\\&=-(1-\lambda)H_i(x_0,x).\end{aligned}\quad(5)$$

It follows from condition (iii) that

$$\begin{aligned}\langle T_i(\gamma,x_0),\kappa(y_\lambda,x_0)\rangle&\leq(1-\lambda)\langle T_i(\gamma,x_0),\kappa(x,x_0)\rangle\\&\quad+\lambda\langle T_i(\gamma,x_0),\kappa(x_0,x_0)\rangle.\end{aligned}$$

Since $\kappa(x,y)+\kappa(y,x)=0_{\mathbb{R}^n}$ for all $x,y\in A,$ we have $\kappa(x_0,x_0)=0_{\mathbb{R}^n}$ and so

$$\langle T_i(\gamma,x_0),\kappa(x_0,y_\lambda)\rangle \geq (1-\lambda)\langle T_i(\gamma,x_0),\kappa(x_0,x)\rangle. \quad (6)$$

We have

$$\| x_0-y_\lambda\|^2=\|x_0-x-\lambda(x_0-x)\|^2$$
$$=(1-\lambda)^2\| x_0-x\|^2. \quad (7)$$

From (4)-(7), we get that

$$\min_{1\le i\le m}\{-(1-\lambda)H_i(x_0,x)+(1-\lambda)\langle T_i(\gamma,x_0),\kappa(x_0,x)\rangle\}$$
$$\le (1-\lambda)^2\theta\| x_0-x\|^2.$$

Equivalently,

$$(1-\lambda)\min_{1\le i\le m}\{-H_i(x_0,x)+\langle T_i(\gamma,x_0),\kappa(x_0,x)\rangle\}$$
$$\le (1-\lambda)^2\theta\| x_0-x\|^2.$$

So,

$$\min_{1\le i\le m}\{-H_i(x_0,x)+\langle T_i(\gamma,x_0),\kappa(x_0,x)\rangle\}$$
$$\le (1-\lambda)\theta\| x_0-x\|^2. \quad (8)$$

Taking the limit in (8) as $\lambda\to 1^-$, we obtain

$$\min_{1\le i\le m}\{-H_i(x_0,x)+\langle T_i(\gamma,x_0),\kappa(x_0,x)\rangle\}\le 0.$$

Then, for any $x\in K(x_0,\gamma)$, there exits $1\le i_0\le m$ such that

$$H_{i_0}(x_0,x)+\langle T_{i_0}(\gamma,x_0),\kappa(x,x_0)\rangle\ge 0,$$

that is,

$$H(x_0,x)+\langle T(\gamma,x_0),\kappa(x,x_0)\rangle\notin -\text{int}\mathbb{R}^m_+,\forall x\in K(x_0,\gamma).$$

Hence, $x_0\in S(\gamma)$.

Conversely, if $x_0\in S(\gamma)$, then there exists $i_0\in\{1,...,m\}$ such that

$$H_{i_0}(x_0,y)+\langle T_{i_0}(\gamma,x_0),\kappa(y,x_0)\rangle\ge 0,\forall y\in K(x_0,\gamma).$$

This means that

$$\min_{1\le i\le m}\{-H_i(x_0,y)+\langle T_i(\gamma,x_0),\kappa(x_0,y)\rangle-\theta\| x_0-y\|^2\}\le 0,$$
$\forall y\in K(x_0,\gamma)$

or

$$\max_{y\in K(x_0,\gamma)}\min_{1\le i\le m}\{-H_i(x_0,y)+\langle T_i(\gamma,x_0),\kappa(x_0,y)\rangle-\theta\| x_0-y\|^2\}$$
$\le 0.$

Hence, $\Omega_\theta(\gamma,x_0)\le 0.$ Since $\Omega_\theta(\gamma,x)\ge 0$ for any $x\in K(x,\gamma)$, $\Omega_\theta(\gamma,x_0)=0$. □

Example 1. Let $n=1, m=2$, $A=[0,1]$, $\Gamma=\mathbb{R}_+$, $\theta=1$, $\gamma\in\mathbb{R}_+$, $K(x,\gamma)=[0,1+x]$, $T_1(\gamma,x)=\gamma x$, $T_2(\gamma,x)=2\gamma x$, $H_1(x,y)=y^2+2xy-3x^2$, $H_2(x)=\ y^2+7xy-8x^2$ and $\kappa(y,x)=y-x$ for all $x,y\in A$, $\gamma\in\Gamma$. Then, the problem (MPVQEP)$_\gamma$ is equivalent to finding $x\in[0,1+x]\cap[0,1]$ such that

$$H(x,y)+\langle T(\gamma,x),\kappa(y,x)\rangle$$
$$=((y-x)((3+\gamma)x+y),(y-x)((8+2\gamma)x+y))\notin\text{-int}\mathbb{R}^2_+$$

for all $y\in[0,1+x]$. It follows from some direct computations that $\mathcal{S}(\gamma)=\{0\}$.

It is clear that all assumptions imposed in Theorem 1 are satisfied. Hence, the function Ω_θ defined by (2) is the gap function for (MPVQEP)$_\gamma$. Indeed,

$$\Omega_\theta(\gamma,x)=\max_{y\in K(x,\gamma)}\{h(\gamma,x,y)-\theta\| x-y\|^2\}$$
$$=\max_{y\in[0,1+x]}\{\min\{(x-y)((3+\gamma)x+y),$$
$$(x-y)((8+2\gamma)x+y)\}-(x-y)^2\}$$
$$=(2+\gamma)x^2.$$

Next, we investigate the error bound for (MPVQEP)$_\gamma$ via the gap function Ω_θ.

For each $i\in\{1,...,m\}$, we now consider the following problem $(\text{GPVQEP})^{(i)}_\gamma$: find $x\in K(x,\gamma)$ for each parameter $\gamma\in\Gamma$ fixed such that

$$H_i(x,y)+\langle T_i(\gamma,x),\kappa(y,x)\rangle\ge 0,\forall y\in K(x,\gamma).$$

Given $S^{(i)}(\gamma)$ the solution set of $(\text{GPVQEP})^{(i)}_\gamma$.

Remark 2. If $x_0\in\cap_{i=1}^m S^{(i)}(\gamma)$, then x_0 is the same solution of $(\text{GPVQEP})^{(i)}_\gamma$ for all $i\in\{1,...,m\}$. Thus, it is clear that x_0 is a solution of the problem (MPVQEP)$_\gamma$.

Theorem 2. *For each $\gamma\in\Gamma$, let x_0 be a solution of the problem (MPVQEP)$_\gamma$. Suppose that all the conditions of Theorem 1 hold and for each $i=1,2,...,m$, H_i is strongly monotone with modulus $\alpha_i>0$ and T_i is κ-strongly monotone with modulus $\mu_i>0$. Let $\alpha=\min_{1\le i\le m}\alpha_i$ and*

$\mu = \min_{1\le i\le m} \mu_i$. *Assume further that* $\cap_{i=1}^{m} S^{(i)}(\gamma) \neq \emptyset$, $x_0 \in K(x,\gamma)$ *for any* $x \in K(x_0,\gamma)$ *and* $\theta > 0$ *satisfying* $\alpha + \mu > \theta$.

Then, for each $x \in K(x_0,\gamma)$, *we have*

$$\| x - x_0 \| \le \sqrt{\frac{\Omega_\theta(\gamma,x)}{\alpha+\mu-\theta}}. \quad (9)$$

Proof. Since $\cap_{i=1}^{m} S^{(i)}(\gamma) \neq \emptyset$, all the problems $\left(\text{GPVQEP}\right)_\gamma^{(i)}$ have the same solution. Without loss of generality, we assume that x_0 is the same solution. Since $x_0 \in K(x,\gamma)$ for any $x \in K(x_0,\gamma)$,

$$\Omega_\theta(\gamma,x) = \max_{y\in K(x,\gamma)} \left\{ \min_{1\le i\le m}\{-H_i(x,y) + \langle T_i(\gamma,x), \kappa(x,y)\rangle\} - \theta\| x-y\|^2 \right\}$$
$$\ge \min_{1\le i\le m}\{-H_i(x,x_0) + \langle T_i(\gamma,x),\kappa(x,x_0)\rangle\} - \theta\| x-x_0\|^2. \quad (10)$$

Then, we can assume that there exists $i_0 \in \{1,...,m\}$ such that

$$\min_{1\le i\le m}\{-H_i(x,x_0) + \langle T_i(\gamma,x),\kappa(x,x_0)\rangle\} = -H_{i_0}(x,x_0) + \langle T_{i_0}(\gamma,x),\kappa(x,x_0)\rangle$$

and so, (10) follows that

$$\Omega_\theta(\gamma,x) \ge -H_{i_0}(x,x_0) + \langle T_{i_0}(\gamma,x),\kappa(x,x_0)\rangle - \theta\| x-x_0\|^2. \quad (11)$$

As $x_0 \in S^{(i_0)}(\gamma)$, we obtain

$$H_{i_0}(x_0,x) + \langle T_{i_0}(\gamma,x_0),\kappa(x,x_0)\rangle \ge 0. \quad (12)$$

Since H_{i_0} is strongly monotone with modulus α_{i_0}, we conclude that

$$-H_{i_0}(x_0,x) - H_{i_0}(x,x_0) - \alpha_{i_0}\| x-x_0\|^2 \ge 0. \quad (13)$$

It follows from the κ-strong monotonicity of T_{i_0} with modulus μ_{i_0} that

$$\langle T_{i_0}(\gamma,x),\kappa(x,x_0)\rangle - \langle T_{i_0}(\gamma,x_0),\kappa(x,x_0)\rangle - \mu_{i_0}\| x-x_0\|^2 \ge 0. \quad (14)$$

Employing (12) – (14), we obtain

$$-H_{i_0}(x,x_0) + \langle T_{i_0}(\gamma,x),\kappa(x,x_0)\rangle \ge (\alpha_{i_0} + \mu_{i_0})\| x-x_0\|^2 \ge (\alpha+\mu)\| x-x_0\|^2. \quad (15)$$

From (11) and (15), we get

$$\Omega_\theta(\gamma,x) \ge (\alpha+\mu-\theta)\| x-x_0\|^2.$$

Therefore,

$$\| x-x_0\| \le \sqrt{\frac{\Omega_\theta(\gamma,x)}{\alpha+\mu-\theta}}$$

and hence the proof is completed. □

Example 2. Let $n, m, \Gamma, A, \theta, K, T_1, T_2, H_1, H_2, \kappa$ as be in Example 1. From Example 1, we have the solution of $(\text{GPVQEP})_\gamma$,

$$S(\gamma) = \{0\}$$

and the regularized gap function of $(\text{MPVQEP})_\gamma$ is defined by

$$\Omega_\theta(\gamma,x) = (2+\gamma)x^2.$$

It is easy to check that H_1 and H_2 are strongly monotone with moduli $\alpha_1 = 2$ and $\alpha_2 = 7$, respectively. Also T_1 and T_2 are κ-strongly monotone with the moduli $\mu_1 = \gamma$ and $\mu_2 = 2\gamma$, respectively. Then, $\alpha = 2$ and $\mu = \gamma$. Therefore, the assumptions of Theorem 2 are satisfied, and so Theorem 2 holds. Some numerical results of Theorem 2 are shown in Table 1.

Table 1. Illustrate the error bounds given by (9) with $\gamma = 0.15$, $\gamma = 0.3$ and $\gamma = 0.5$

x	$\lVert x-x_0 \rVert$	Error bounds		
		$\gamma = 0.15$	$\gamma = 0.3$	$\gamma = 0.5$
0.0	0.0	0.000	0.000	0.000
0.1	0.1	0.137	0.133	0.129
0.2	0.2	0.273	0.266	0.258
0.3	0.3	0.410	0.399	0.387
0.4	0.4	0.547	0.532	0.516
0.5	0.5	0.684	0.665	0.645
0.6	0.6	0.820	0.798	0.775
0.7	0.7	0.957	0.931	0.904
0.8	0.8	1.094	1.064	1.033
0.9	0.9	1.231	1.197	1.162
1.0	1.0	1.367	1.330	1.291

Remark 3. In special cases of (a), (b) mentioned in Sect. 1, the regularized gap function Ω_θ for $(\text{GPVQEP})_\gamma$ reduces to the regularized gap

function for (VIP) and (QEP) considered in Bigi and Passacantando (2016) and Yamashita and Fukushima (1997), respectively. Therefore, for these cases, Theorem 1 and Theorem 2 extend to the existing ones in Bigi and Passacantando (2016) and Yamashita and Fukushima (1997), and are different from the corresponding results in Anh *et al.* (2018), Hung *et al.* (2020a, 2020b, 2021) and Khan and Chen (2015b) in the form of the problem $(GPVQEP)_\gamma$ perturbed by parameters.

3. Conclusions

The class of mixed parametric vector quasi-equilibrium problems $(GPVQEP)_\gamma$ is introduced in this paper. Regularized gap functions and error bounds are stated for this kind of problems under suitable assumptions. Examples are given to support the results presented here.

It would be interesting to consider the study of Levitin-Polyak well-posedness by perturbations and Hölder continuity of solution mapping for the class of mixed parametric vector quasi-equilibrium problems $(GPVQEP)_\gamma$ based on regularized gap functions.

Acknowledgements

This work is supported by the Ministry of Education and Training of Vietnam under Grant No. B2021.SPD.03. The authors are grateful to the anonymous referees for their valuable remarks which improved the results and presentation of the paper.

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