SOLVING SOME ELEMENTARY GEOMETRICAL PROBLEMS BY EUCLIDEAN GEOMETRY'S METHODS

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Article history

Received: 07/5/2022; Received in revised form:20/6/2022; Accepted: 14/7/2022

Abstract

The article introduces a method of exploiting the Euclidian geometry to find solutions for elementary geometry problems. By analyzing some special coordinates in the solution, we will detect some extra points or lines to help solve elementary problems.

Keywords: *Coordinate method, finding solutions, the Euclidian geometry.*

DOI: https://doi.org/10.52714/dthu.12.3.2023.1047

Cite: Nguyen Thi Thanh Loan, Tran Le Nam, and Phan Thi Hiep. (2023). Solving some elementary geometrical problems by Euclidean geometry's methods. *Dong Thap University Journal of Science*, *12*(3), 65-71.

SỬ DỤNG HÌNH HỌC Ơ-CLIT TÌM LỜI GIẢI HÌNH HỌC SƠ CẤP

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Lịch sử bài viết

Ngày nhận bài:07/5/2022; Ngày nhận chỉnh sửa: 20/6/2022; Ngày duyệt đăng: 14/7/2022

Tóm tắt

Bài báo giới thiệu phương pháp sử dụng hình học Ơ-clit để tìm lời giải các bài toán hình học sơ cấp. Bằng cách khảo sát các tọa độ đặc biệt, chúng ta phát hiện được các điểm hay đường phụ. Nhờ vào sự phát hiện đó, các bài toán hình học sơ cấp có thể tìm được lời giải.

Từ khóa: *Hình học Ơ-clit, phương pháp tọa độ, tìm lời giải.*

1. Introduction

Difficult elementary geometry problems often have some hidden properties, lines, or points (Lê Quốc Hán *et al.*, 2016). They are the keys to finding the solutions. Basing on accumulated experiences, teachers can use different advanced geometrical techniques to get the solution and easily get the keys to solving the primary solution for students.

This technique is covered in many advanced geometry books (Hsiung, 2005, Igor *et al.*, 2013, Ya *et al.*, 1982). However, books only give some suggestions for exploiting geometric properties with few illustrative examples. This article concretizes those orientations into a specific process which gives some illustrative examples and declares solutions to find many different solutions for elementary geometry problems. This technique can be combined with other techniques. From that, we can find many solutions to a geometry problem (Phạm Bảo, 2018, Top *et al., 2020*). Then, readers will understand the meaning of solving a problem through many methods, including the development of personal thinking.

The article also aims to guide a technique to exploit and find solutions for elementary problems. However, It is not a universal tool. Teachers need to flexibly, accumulate their experience, and combine many methods to find solutions and exploit problems. Hence, the quality of teaching has been improved.

2. Main contents

2.1. The process of finding solutions to elementary problems using the advanced geometry

After some experiments and discussions with high school teachers and students, we found the following process effective.

Step 1. Solving problems by using the coordinate method.

+ Choosing the coordinate system;

+ Transferring hypothesis and conclusion of the problem to the language of the coordinate geometry;

+ Using the facts of the Euclidean geometry to solve problems;

+ Conclusion of the problems.

Note: The conclusion by the coordinate geometry language is usually drafted, guiding the proof.

Step 2. Detecting hidden geometric properties based on calculation results.

Basing on the coordinates of the key points or vectors in solution by the coordinate method, we try to discover the parallel properties, collinear, perpendicular, ratio, and bisector angle. If the discovered properties are not enough to solve the problem, we need to find some extra points from the key points such as perpendicular projection, parallel projection, and midpoint.

Step 3. Proposed elementary solutions.

From discovered geometric properties in Step 2, we arrange the properties in the sketch to solve the problem.

Step 4. Solving the geometry problems using elementary properties.

Using the geometric properties at the high school to calculate and prove the properties outlined in Step 3, we solve the problems.

2.2. Example

2.2.1. The problem

Given the square $ABCD$. M is the midpoint of the side BC, N is the point on the side CD such that $CN = 2ND$. The segment AN and the segment BD intersect each other at H. Prove that triangle ΔAHM is an isosceles right triangle.

2.2.2. Solving the problem by the advanced geometry (Figure 1)

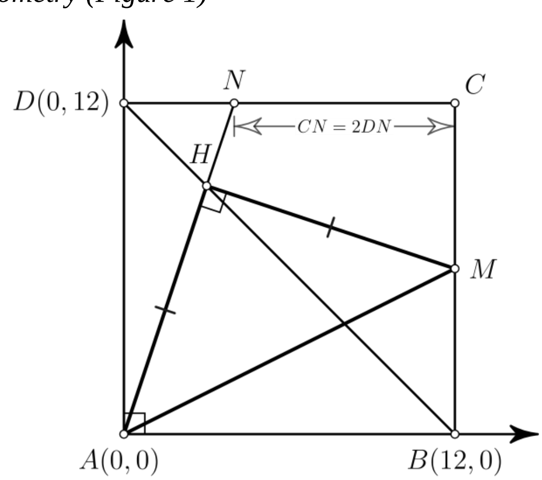

Figure 1

Using the basic knowledge of advanced geometry, teachers will solve the problem easily. This solution will guide some elementary solutions, suitable for high school students.

Let $A(0,0)$, $B(12,0)$ and $D(0.12)$.

Then, $M(12,6)$, $N(4,12)$.

The two lines (BD) and (AN) have the equations, respectively

$$(BD):\ x+y-12=0,$$
$$(AN):\ 3x-y=0.$$

It implies that, $(BD)\cap(AN)=H(3,9)$.

From that, $\overrightarrow{AH}=(3,9), \overrightarrow{MH}=(-9,3)$,

Therefore, $|\overrightarrow{AH}|=|\overrightarrow{MH}|=\sqrt{3^2+9^2}=3\sqrt{10}$.

Moreover, $\overrightarrow{AH}.\overrightarrow{MH}=3.(-9)+9.3=0$.

Therefore, $AH=MH$ and $AH\perp MH$.

So, $\triangle AHM$ is an isosceles right triangle.

2.2.3. Discovering the first elementary solution

Suppose points P and R are the orthographic projections of H to AD and AB, respectively.

From the coordinates of the point H, we get:

$$PH=AR=3=\frac{AB}{4},\ AP=9=\frac{3AB}{4}\cdot$$

Suppose line (PH) and line (BC) intersect at the point Q then its coordinates is (12,9).

Therefore, Q is the midpoint of CM.

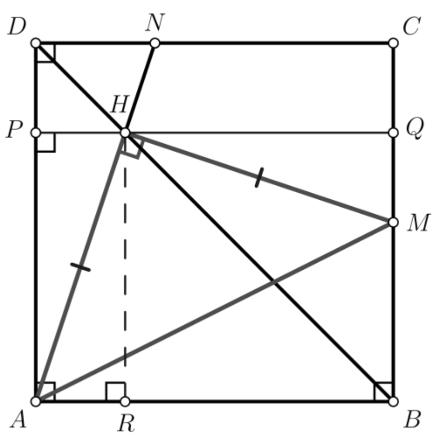

Figure 2

It then follows that $\triangle HAR=\triangle HMQ$, and the problem is proven.

From the above analysis, the key property to solve the problem that is $PH=\frac{AB}{4}\cdot$

Through the above analysis, the problem can be solved according to the following step.

- Prove that $\frac{PH}{AB}=\frac{1}{4}$;
- Prove that $PH=QM$ and $HQ=HR$;
- Prove that $AH=MH$ and $AH\perp MH$.

The elementary solution 1

We have

$$\frac{DH}{HB}=\frac{DN}{AB}=\frac{1}{3}\cdot$$

It implies that

$$\frac{PH}{AB}=\frac{DH}{DB}=\frac{DH}{DH+HB}=\frac{1}{4}\cdot$$

Since $\triangle PDH$ is an isosceles right triangle at P, it gets

$$CQ=DP=PH=\frac{1}{4}AB=\frac{1}{2}CM.$$

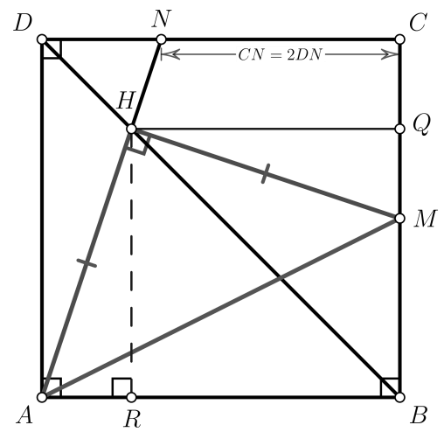

Figure 3

Therefore,

$$AM=AR=\frac{AB}{4},$$
$$HR=MB+QM=\frac{3AB}{4},$$
$$HQ=PQ-PH=\frac{3AB}{4}=HR.$$

It derives that $\triangle HAR=\triangle HMQ$ (s-a-s).

Considering $\triangle HAM$, we have:

$$HA=HM,$$
$$\widehat{AHM}=\widehat{PHQ}-\left(\widehat{PHA}+\widehat{MHQ}\right)$$
$$=\widehat{PHQ}-\left(\widehat{PHA}+\widehat{HAP}\right)$$
$$=180^\circ-90^\circ=90^\circ.$$

So, $\triangle AHM$ is an isosceles right triangle at H.

Moreover, *from the point coordinates H, the reader can discover a number of other congruent triangles. For example*

$$\triangle HAR = \triangle HAL = \triangle HMQ = \triangle HQC.$$

These results can lead to other elementary solutions.

2.2.4. Discovering the second elementary solution

We have $\overrightarrow{DH} = (3, -3)$ and $\overrightarrow{DB} = (12, -12)$. It derives that $BD = 4DH$.

Therefore, $AR = \frac{1}{4}AB = \frac{3}{4}RB = \frac{3}{4}BQ = QM$.

From the above analysis, teachers can directly solve the problem in the following way.

We have:

$$\frac{DH}{HB} = \frac{DN}{AB} = \frac{1}{3}.$$

It implies that

$$\frac{AR}{AB} = \frac{DH}{DB} = \frac{DH}{DH + HB} = \frac{1}{4}.$$

Since $\triangle QHB$ is an isosceles right triangle at Q, so

$$BQ = HQ = BR = \frac{3AB}{4} = \frac{3CB}{4}.$$

Therefore,

$$\begin{aligned} QM &= \frac{3CB}{4} - MB \\ &= \frac{3CB}{4} - \frac{CB}{2} = \frac{CB}{4} = \frac{3AB}{4}. \end{aligned}$$

It implies that $\triangle HAR = \triangle HMQ$ (s-a-s).

Otherwise, considering $\triangle HAM$, we have:

$$\begin{aligned} &HA = HM, \\ &\widehat{AHM} = \widehat{AHR} + \widehat{RHM} \\ &\quad = \widehat{MHQ} + \widehat{RHM} = \widehat{RHQ} = 90^\circ. \end{aligned}$$

So, $\triangle AHM$ is an isosceles right triangle at H.

2.2.5. Discovering the third elementary solution (Figure 4)

In exploiting extra points from existing points and lines, we can detect the intersection P of two lines (CD) and (HM) with coordinates $(-6,12)$. From there, the point H is the midpoint of PM. Moreover,

$\overrightarrow{AP} = (-6,12)$, $\overrightarrow{AM} = (12,6)$. Therefore, $\widehat{APM} = 90^\circ$ and AH is the perpendicular bisector of the segment PM.

On the other hand, from $\overrightarrow{PD} = (6,0)$ we get $PD = BM = 6$.

From there $\triangle ABM = \triangle ADP$.

These results are enough to prove the problem.

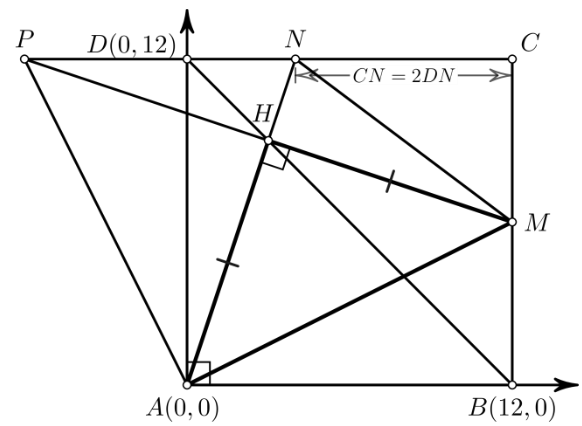

Figure 4

Third elementary solution orientation

Through the above analysis, the problem can be solved according to the following sketch.

- Let P be the intersection of two lines (CD) and (HM), prove that $PD = MB$ and $\triangle ABM = \triangle ADP$;
- Proving that $\triangle PAM$ square is balanced at A;
- Proving that (AH) is a bisector of $\widehat{PAM}$.

Third elementary solution

Let P be the intersection of two lines (CD) and (HM). By the Menelaus' theorem to $\triangle BCD$ for 3 collinear points P, H and M, we have

$$\frac{PD}{PC} \cdot \frac{MC}{MB} \cdot \frac{HB}{HD} = 1,$$
$$\Rightarrow \frac{PD}{PC} = \frac{HD}{HB} = \frac{DN}{AB} = \frac{1}{3}.$$

We get

$$PD = \frac{1}{3}PC = \frac{1}{3}(PD + DC).$$

It implies that $PD = \frac{1}{2}DC = \frac{1}{2}CB = BM$.

Therefore, $\triangle ABM = \triangle ADP$ (c.g.c).

We have

$$\begin{aligned} &AP = AM, \\ &\widehat{PAM} = \widehat{PAD} + \widehat{DAM} \\ &\quad = \widehat{MAB} + \widehat{DAM} \end{aligned}$$

$$= \widehat{DAB} = 90^{\circ}.$$

From that, we get $\triangle PAM$ is a right triangle at A.

On the other hand, assuming a square of the length $6a$, $a > 0$. Then,

$$PD = BM = CM = 3a,$$
$$CN = 2.DN = 4a.$$

Applying the Pythagorean theorem to $\triangle NMC$, we get:

$$MN = \sqrt{(3a)^2 + (4a)^2} = 5a = PD + DN = PN.$$

So, $\triangle ADN = \triangle AMN$.

Therefore, AH is the bisector of the angle $\widehat{PAM}$.

Since $\triangle APM$ is an isosceles right triangle at A, so

$$AH \perp PM,\ PH = \frac{1}{2}PM = HM.$$

Therefore, $\triangle HAM$ is an isosceles right triangle.

By experimenting and discussing with teachers on forums, some more elementary solutions were collected

2.2.6. Discovering the fourth elementary solution by using the Thales' theorem

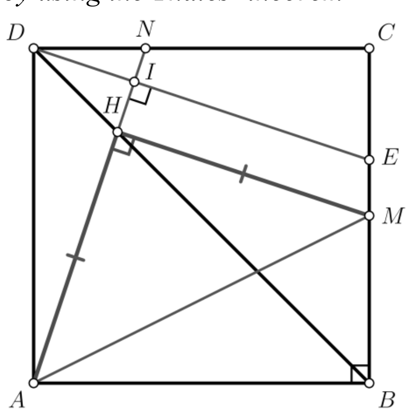

Figure 5

Let E be a point on the line segment CB such that $CE = DN$. Let I be the intersection point of line (AN) and line (DE).

We have:

$$\frac{NH}{HA} = \frac{DH}{HB} = \frac{DN}{AB} = \frac{1}{3}.$$

It is inferred that

$$\frac{AH}{AN} = \frac{AH}{AH + HN} = \frac{3}{4}.$$

On the other hand, we have

$$\frac{BM}{BE} = \frac{BC}{2} : \frac{2BC}{3} = \frac{3}{4}.$$

Therefore,

$$\frac{HM}{DE} = \frac{3}{4}.$$

Since $\triangle ADN = \triangle DCE$, we get $\triangle HAM$ is a right triangle.

So $\triangle HAM$ is an isosceles right triangle.

2.2.7. Discovering the fifth elementary solution using properties of orthocenter and parallelogram

Let O be the midpoint of BD and E the midpoint of AO, using the above results, we get EH the midsegment of $\triangle DAO$. It follows that E is the orthocenter of $\triangle HAB$.

From there $BE \perp AH$.

On the other hand, the quadrilateral $HEBM$ has sides HE and MB that are parallel and equal (parallel and equal to AD).

It implies that

$$HM \parallel EB \text{ and } HM = EB.$$

Therefore, $HM \perp AH$. and

It implies that

$$AH = EB = HM.$$

So $\triangle HAM$ is an isosceles right triangle.

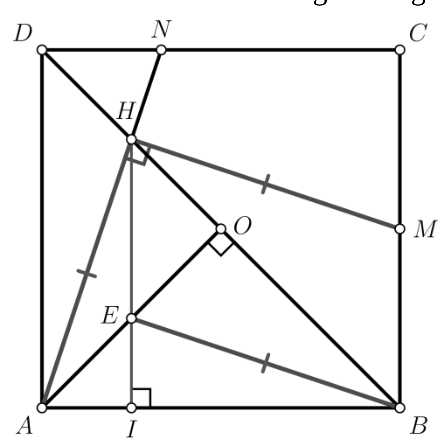

Figure 6

2.2.8. Discovering the sixth elementary solution using properties of the incircle

Let E be the midpoint of AD, using the above results, we get EH to be a midsegment of $\triangle DAO$. It infers that five points A, B, M, H and E are concyclic. They lie on a common circle with a diameter AM.

It derives that $\widehat{AHM} = 90^\circ$ and

$\widehat{HMA} = \widehat{HBA} = 45^\circ$.

Therefore, $\triangle HAM$ is right at H.

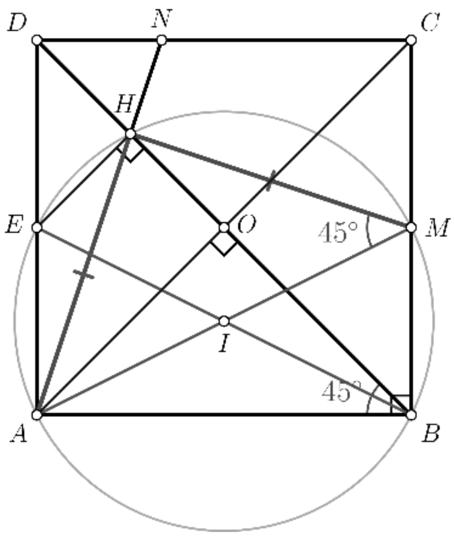

Figure 7

3. Conclusion

The article has proposed a technique for applying the Euclidan geometry in teaching. That is attaching the coordinate system to the geometry problem, solving the problem by the coordinate method, and exploiting the coordi-nates in the solution to find the key points and auxiliary lines. Thereby, the solution for the original problem can be discorved.

At the same time, the article also shows that this method can be combined with other methods. In particular, if we accumulate many elementary results, and apply the classical theorems, the solution can be profound.

This article mainly focuses on the technique of finding elementary solutions by using the Euclidean geometry results. In the process of teaching, teachers should combine the technique with mathematical softwares, encourage students to observe figures at multiple perspectives to detect hidden properties in the problem. Moreover, teachers should encourage students to draw accurate shapes, use rulers and compasses to predict geometric properties in the problem. Accordingly, students will find more solutions, better exploit the problem, and develop geometric thinking.

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