# FEM Limit Analysis of Beam Structures by Direct Method

**Trần Ngọc Trình, Trần Ngọc Thắng**


## Abstract

In this paper, we introduce a so-called direct method for the computation of limit loads of beam structures subjected to propotional loading. The forces acting on the beam structures are static. The material of beam is made of perfectly plastic elastic. There are two approaches based on fundamental theorems of limit analysis theory. This work presents a formulation which is based on upper bound approach combined with the von Mises yield criterion. After that an algorithm is given to find the limit load factor. The finite element method is used in this investigation. The method is illustrated by an example which is solved by both analytical and numerical methods.


Key words: Limit analysis, plastic analysis of structures, non-linear programming, finite element method

**Dr. Tran Ngoc Trinh**
Faculty of Civil Engineering,
Hanoi Architectural University
Email: trinhdhkt@gmail.com,
Tel: 0981111019
**Msc. Tran Ngoc Thang**
Faculty of Civil Engineering,
Hanoi University of Business and Technology (HUBT).
Email: Xaydunghubt@gmail.com,
Tel: 0973017918

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### 1. Introduction

Limit analysis plays an important role in safety assessment and structural design, especially in nuclear power plants, chemical industry, mechanical engineering and construction.

Based on the ideal elastic or ideal rigid material model, limit analysis theory allows calculating the load-bearing capacity that the structure is capable of withstanding under proportional action. When this limit is exceeded, the structure will be destroyed due to plastic flow. To calculate the ultimate load of the structure, we can use step-by-step analysis methods. In this method, we divide the loading process into small steps and investigate the entire process of developing stress and deformation of the structure until the structure is completely destroyed. This method is often used to calculate simple structures, which we can see in textbooks for university students [1]. This method has the disadvantage that besides the question of running time, we also have to know exactly the entire loading or unloading process, which is very difficult to do in real situations or technical problems. We can overcome the difficulties by another method called the direct method. The most convenient thing about this method is that we do not need to know the entire history of the loading process but are only interested in the maximum load value (corresponding to which is the deformation stress at the extreme state) at which the structure is collaped [2-10].

There are two basic methods in the group of direct methods: dynamic methods and static methods. Direct methods refer the problem of finding the limit load to the mathematical optimization problems. According to the static method, we are searching the largest value among the loads that makes the structure safe. According to the dynamic method, we find the smallest failure load in the admissible displacement field.

Limit analysis based on mathematical programming, the solution were not always easy to obtain. Therefore, researches were executed to find efficient algorithms for linear and nonlinear programming. Linear programming has been used widely in limit analysis because this approach permits the solving of large scale problems. Contributions for this approach can be seen in [5-10].

This article introduces the kinematic approach to find the value of the limit load acting on beam structures. The numerical method used to discretize the problem is the finite element method (FEM).

### 2. Approaches for Limit Analysis of beam structures

*2.1. Limit state of beam*

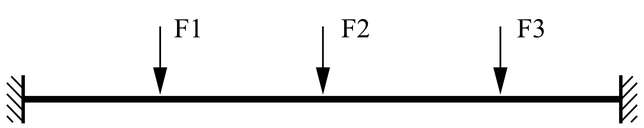
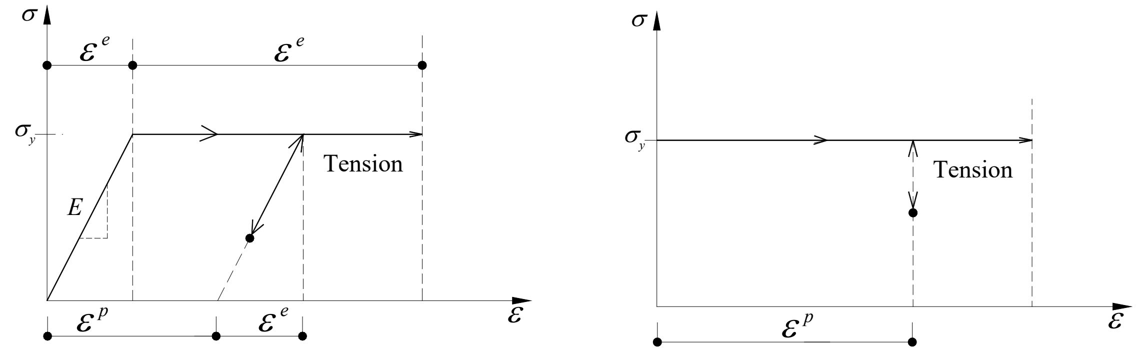



Figure 1. Beam subjected by a set of proportional forces

Consider a beam made of elastic-perfectly plastic or rigid-perfectly plastic material with material models are shown in figure 2. there is a set of forces $\boldsymbol{F}$ acting on it. A common assumption is that all the components of the set of forces change proportionally to a certain load parameter $\alpha$. This case is referred to as proportional loading. In matrix notation we can write:




Elastic plastic material    rigid-perfectly plastic material

Figure 2. material models

$$\mathbf{F} = \alpha \mathbf{F}_0$$

where $\mathbf{F}_0$ is some fixed reference load vector, for instant $\mathbf{F}_0 = \{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$

If the value of $\alpha$ remains sufficiently low, response of the structure is elastic. As $\alpha$ increases and reaches a special value, the first point in the body reaches the plastic state. This state of stress is called elastic limit. Further increase of $\alpha$ will lead to the expansion of plastic region in the structure. The structure gradually forms a collapse mechanism. At limit state, the beam is collapsed by applied forces. The value $\alpha = \alpha_{\lim}$ corresponding to the plastic collapse state is called the safety factor of beam or the limit load factor

*2.2. Statically and kinematically admissible states*

Let us specify the reference loading $\mathbf{F}_0 = (\bar{\mathbf{b}}, \bar{\mathbf{t}})$ by given body forces $\bar{\mathbf{b}}$ and given surface tractions $\bar{\mathbf{t}}$. The principle of virtual work states the static equilibrium of a stress field $\boldsymbol{\sigma}$ which is in equilibrium with $\mathbf{F}_0 = (\bar{\mathbf{b}}, \bar{\mathbf{t}})$

$$\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, dV = \int_V \alpha \bar{\mathbf{b}} \cdot \mathbf{u}\, dV + \int_{S_t} \alpha \bar{\mathbf{t}} \cdot \mathbf{u}\, dS \tag{1}$$

It is important to note that the kinematic quantities $\mathbf{u}, \boldsymbol{\varepsilon}$ are kinematically compatible by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[ (\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^T \right]$$

but otherwise arbitrary and have no causal relation with the static quantity $\boldsymbol{\sigma}$. The principle holds also for the time derivatives (virtual power):

$$\int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p\, dV = \int_V \alpha \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, dV + \int_{S_t} \alpha \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, dS \tag{2}$$

For continuous fields the principle is equivalent with the local form of the equlibirium conditions:

$$\begin{aligned} &\nabla \boldsymbol{\sigma}(\mathbf{x}, t) = -\mathbf{b} && \text{in } V \\ &\mathbf{n}\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{t} && \text{on } S_t \end{aligned} \tag{3}$$

A statically admissible state is described by a stress field $\boldsymbol{\sigma}$ and a load multiplier $\alpha^-$ such that (s.t.)

$$\begin{cases} -\boldsymbol{\sigma} \cdot \nabla = \alpha^- \bar{\mathbf{b}} & \text{in } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \alpha^- \bar{\mathbf{t}} & \text{on } S_t \\ f(\boldsymbol{\sigma}) \le 0 & \text{in } V \end{cases} \tag{4}$$

A kinematically admissible state is described by a displacement rate field $\dot{\mathbf{u}}$ and a plastic strain rate field $\dot{\boldsymbol{\varepsilon}}$ such that

$$\begin{cases} \dot{\boldsymbol{\varepsilon}} = (\nabla \dot{\mathbf{u}})_{sym} & \text{in } V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on } S_u \\ \int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, dV + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, dS > 0 \end{cases} \tag{5}$$

*2.3. Theorems of Limit Analysis*

Lower bound theorem states as follows:

If a stress field $\boldsymbol{\sigma}$ can be found which satisfies the statically admissible state (4) then the corresponding multiplier $\alpha^-$ cannot exceed the limit multiplier $\alpha_{\lim}$.

Upper bound theorem states:

Any multiplier $\alpha^+$ corresponding to a kinematically admissible state (5) is not less than the limit multiplier $\alpha_{\lim}$.

*2.4. Two basic approaches for limit analysis of beam structures*

There are two basic approaches to limit analysis corresponding to the two above theorems. The static approach is based on the lower bound theorem, according to which the safety factor can be obtained by looking for the maximum statically admissible load multiplier. This task lead to solving a maximum nonlinear optimization problem

$$\alpha_{\lim} = \max \alpha^-$$

$$\text{s.t.:} \begin{cases} -\boldsymbol{\sigma} \cdot \nabla = \alpha^- \bar{\mathbf{b}} & \text{in } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \alpha^- \bar{\mathbf{t}} & \text{on } S_t \\ f(\boldsymbol{\sigma}) \le 0 & \text{in } V \end{cases} \tag{6}$$

Constraints in (6) are the Cauchy equations of equilibrium, static boundary conditions and conditions of plastic admissibility, respectively.

The second one is the kinematic approach which is based on upper bound theorem, according to which the safety factor can be obtained by searching for the minimum kinematically admissible load multiplier. This can be achieved by solving the minimum optimization

$$\alpha_{\lim} = \min \alpha^+$$

$$\alpha^+ = \frac{\int_V D(\dot{\boldsymbol{\varepsilon}})\, dV}{\int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, dV + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, dS}$$

$$\text{s.t.:} \begin{cases} \dot{\boldsymbol{\varepsilon}} = (\nabla \dot{\mathbf{u}})_{sym} & \text{in } V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on } S_u \\ \int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, dV + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, dS > 0 \end{cases} \tag{7}$$

where $\int_V D(\dot{\boldsymbol{\varepsilon}})dV$ is the plastic dissipation power, the constraints (7) are the strain-displacement relations, kinematic boundary conditions, and the condition of positive external power.

We can restrict our attention to solutions with normalized external power by the condition

$$\int_V \overline{\mathbf{b}}\cdot\dot{\mathbf{u}}\,dV+\int_V \overline{\mathbf{t}}\cdot\dot{\mathbf{u}}\,dS=1 \tag{8}$$

Then problem can be rewritten as

$$\alpha_{\lim}=\min\int_V D(\dot{\boldsymbol{\varepsilon}})dV$$
$$\text{s.t.:}\begin{cases}\dot{\boldsymbol{\varepsilon}}=(\nabla\dot{\mathbf{u}})_{sym} & \text{in} V\\ \dot{\mathbf{u}}=\mathbf{0} & \text{on } S_u\\ \int_V \overline{\mathbf{b}}\cdot\dot{\mathbf{u}}\,dV+\int_V \overline{\mathbf{t}}\cdot\dot{\mathbf{u}}\,dS=1\end{cases} \tag{9}$$

**3. FEM formulation and solving algorithm for limit analysis of beam**

*3.1. FEM formulation*

The plastic disippation power of beam is as follows if we use the Von-Mises criteria:

$$D^p=\frac{2}{\sqrt{3}}\sigma_y\sqrt{\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{D}\dot{\boldsymbol{\varepsilon}}} \tag{10}$$

Where

- $\sigma_y$ is yield stress of material,
- vector of strain velocity $\dot{\boldsymbol{\varepsilon}}=\left[\dot{\varepsilon}_{11}\ \dot{\varepsilon}_{22}\ \dot{\gamma}_{12}\right]^T$

$$\mathbf{D}=\begin{bmatrix}1&1&0\\1&1&0\\0&0&1/2\end{bmatrix} \tag{11}$$

If the finite element method is used, the FEM discretized form of the deterministic formulation of the upper bound load factor $\alpha$-

$$\alpha^+=\min\sum_{i=1}^{NG}\sqrt{\frac{2}{3}}\sigma_y\sqrt{\dot{\boldsymbol{\varepsilon}}_i^T\mathbf{D}\dot{\boldsymbol{\varepsilon}}_i}$$
$$\text{s.t.:}\begin{cases}\dot{\boldsymbol{\varepsilon}}_i-\hat{\mathbf{B}}_i\dot{\mathbf{u}}=\mathbf{0} & i=1,..NG\\ \sum_{i=1}^{NG}\dot{\boldsymbol{\varepsilon}}_i^T\boldsymbol{\sigma}_i^E-1=0\end{cases} \tag{12}$$

In which $NG$ is the total number of Gauss points on the beam.

*3.2. Upper bound algorithm for problem of limit analysis of beams*

Problem (12) is a nonlinear optimization problem, we can use Lagrange multiplier method to convert problem (12) into an unconstrained programming problem. After that we solve Karush – Kuln – Tucker of unconstrained programming to obtain the optimal solution. The reader may found the detailed algorithm in [2].

**4. Example**

Consider the beam subjected to a concentrated force as shown in figure 3, Let us compute the limit load acting on the beam. The length of beam L=4m, plastic moment of beam $M_p$=4kNm.

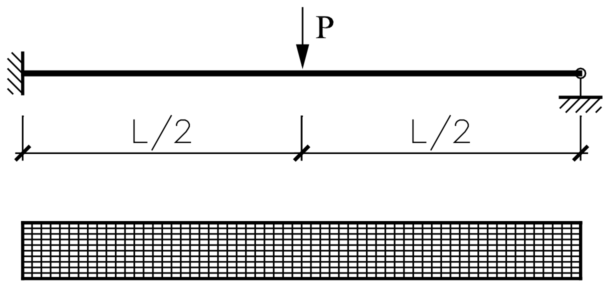



Figure 3. Beam subjected to a concentrate force and FE mesh

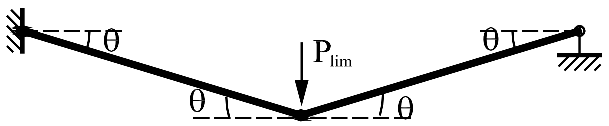



Figure 4. collapse mechanism of the beam

Analytical solution: At limit state as shown in figure 4, plastic dissipation energy is equal to external work, this lead to following equation:

$$\dot{W}_{in}=\dot{W}_{ext} \tag{13}$$

In which:

$$\dot{W}_{in}=M_p.\theta+M_p.2\theta=3M_p.\theta\,,\ \dot{W}_{\mathrm{ext}}=\frac{P.L}{2}.\theta$$

From equation (13) we have

$$P=\frac{6M_p}{l}$$

Substituting the input numerical datas:

$$M_p=4kNm,\ l=4m\ \ \rightarrow\ P_{\lim}=6kN$$

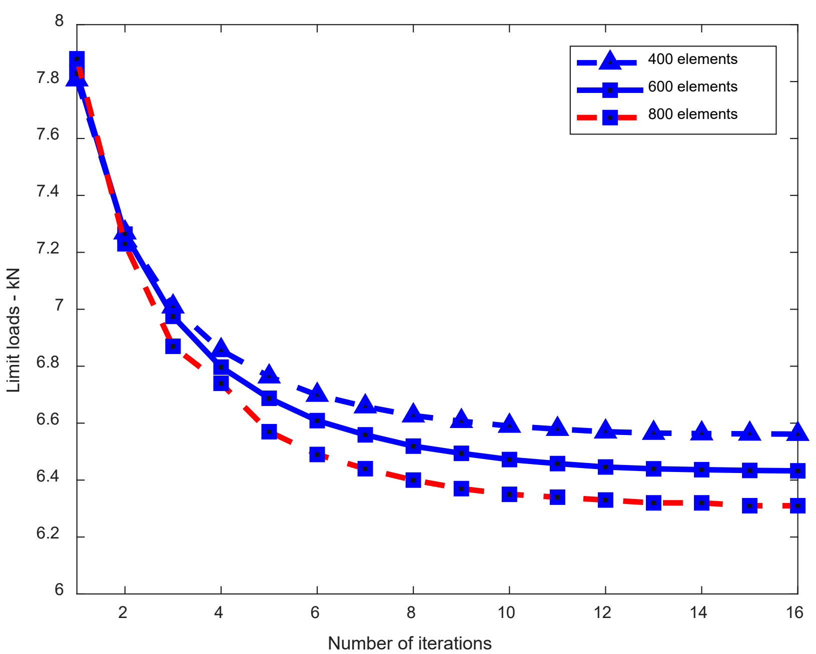



Figure 5. Convergence of limit loads

FEM solutions: Beam is modeled by 400 rectangular finite elements, the algorithm which mentioned above give us limit load P =6.45 kN. If we increase the number of elements to 600 and 800 then the results are 6.36 kN and 6.25 kN, respectively. The results are listed in table 1.

Table 1. Results of analytical and numerical solutions.

| Models | Limit load (kN) |
|---|---|
| Analytical | 6 |
| FEM - 400 elements | 6.45 |
| FEM - 600 elements | 6.36 |
| FEM - 800 elements | 6.25 |

**5. Conclusion**

The paper presents direct methods to solve problem of limit analysis of beam which is made of elasto-plastic material. Upper bound method is illustrated with a problem of computing the limit load of beam. The numerical solutions shown that the result will reach to analytical solution when the number of elements are increased.

This approach can be extended to problem of limit analysis of frame, plate structures./.

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## History and ways to carry out the process...

***(tiếp theo trang 34)***

the bathroom (wet the print), and does not shine too bright a light on the print ( avoid discoloration)..

- Cleaning the print after printing with clean water for water-based inks and with soap for oil-based inks.

**4. Conclusion**

Along with completing this research, the author had a wonderful time making prints using plastic and gel sheets. Beside using specialized ink for monotype printing, the author additionally use the acrylic paint with many different colors and materials available in nature, allowing artists to freely experiment to create unique images. Forms and patterns with high spontaneity give unexpected and unique beauty, it is truly interesting because the range of expression is very rich. Monoprinting provide flexible and entertain to approach to exploring and unleashing individual creativity. Whether experimenting with layered prints, with mixed media or monotype, the possibilities of expression are endless. Therefore, mastering the art of monoprinting helps to exploit our imagination deeply when creating beautiful paintings with fixed duplication./.

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