# ON THE SYNCHRONIZATION OF HETEROGENEOUS PHASE OSCILLATOR NETWORKS

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## ABSTRACT

This paper presents a study on synchronization behaviors in phase oscillator networks where the oscillators are interconnected through a general nonlinear function and their interconnections are bidirectional. Consequently, we investigate two contexts namely symmetric and asymmetric couplings among oscillators. In both cases, we show that if the coupling strengths are greater than some specific levels then the frequencies of oscillators in the network are synchronized. Furthermore, the synchronization rate is pointed out. Lastly, several numerical examples are presented to illustrate the theoretical results.

*Keywords:* phase oscillator networks, collective synchronization, nonlinear coupled oscillators.

## 1. INTRODUCTION

Synchronization is a ubiquitous phenomenon that occurs in many real-world systems such as circadian rhythms [1, 2], central pattern generator networks [3], a group of crickets, a swarm of fire flights or arrays of lasers [4]. In many systems, especially in biology, synchronization is not only phenomenon but also a mechanism. Therefore, understanding the synchronization behaviors in oscillator networks is an important research topic for decades. Furthermore, throughout the synchronization analysis, we may derive useful results for designing oscillator networks in engineering applications.

Phase oscillator networks has been widely utilized to investigate the oscillations in many systems in different fields, for examples biology and neuroscience [5], electrical and electronics engineering [6], chemical engineering and physics [7]. Under some assumptions, the ordinary differential equations describing the processes can be reduced to obtain the phase oscillator models [5 - 7] which only concern about the phases and frequencies of oscillations. Hitherto, the phase oscillator models have been proved to be very useful in representing, explaining and further exploring the oscillating phenomena in many distinct systems. Particularly, the collective synchronization of a set of interconnected oscillators is intensively studied since it is found in many real applications.

Kuramoto model is a typical phase oscillator network that has been extensively investigated where the couplings among oscillators are represented by the sinusoidal function and each oscillator is connected to every other. This type of phase oscillator networks has been applied to explain and analyze a lot of oscillating networks [4, 7, 8, 9]. Nevertheless, there exist other classes of oscillator networks where the couplings among oscillators are not all-to-all. Moreover, the coupling function would be other nonlinear functions rather than the sinusoid. Thus, it is emergent to investigate the phase oscillator networks whose models are more general than Kuramoto model. An application of these general phase oscillator networks model would be found in the design of wireless sensor networks [10, 11].

Bearing those points in mind, we aim at studying the collective synchronization behaviors in phase oscillator networks whose coupling functions are more general than sinusoidal function and the interactions among oscillators are sparse. Accordingly, the contributions of this paper are twofold. First, we propose a sufficient condition for the couplings among oscillators such that their frequencies are synchronized as the interactions among oscillators are undirected. Second, a sufficient condition is proposed for frequency synchronization when the couplings among oscillators are bidirectional and asymmetric. In both cases, we figure out what are the synchronized frequencies and speeds of synchronization.

## 2. PROBLEM FORMULATION

Consider a network of $n$ heterogeneous phase oscillators, each oscillator is represented by its phase $\theta_k$ and natural frequency $\omega_k$. The heterogeneity of oscillators here is due to the difference on their natural frequencies. Suppose that each oscillator in the network interacts with some other oscillators then the network dynamics is described as follows,

$$\dot{\theta}_k = \omega_k - \sum_{j=1}^{n} a_{kj} f\left(\theta_k - \theta_j\right), k = 1,\dots,n, \tag{2.1}$$

where $a_{kj}, k, j = 1,\dots,n$ is the coupling weight between the $k$-th and $j$-th oscillators; $a_{kj} > 0$ if the $k$-th and $j$-th oscillators are connected, otherwise $a_{kj} = 0$; $f$ is a nonlinear function which represents how the oscillators are coupled.

Employing algebraic graph theory, we can describe our network of coupled phase oscillators as follows. Denote $\mathcal{G}\left(\mathcal{V},\mathcal{E}\right)$ a graph where each node in $\mathcal{G}$ represents a phase oscillator and each edge in $\mathcal{G}$ represents a coupling between two corresponding oscillators, $\mathcal{V}$ is the set of all nodes, $\mathcal{E}$ is the set of all edges in the graph. Furthermore, the weights on the edges of $\mathcal{G}$ are equal to $a_{kj}, k, j = 1,\dots,n$ which are the coupling weights among oscillators. Then denote $A \in \mathbb{R}^{n\times n}$ the adjacency matrix whose elements are $a_{kj}$, and $D \in \mathbb{R}^{n\times n}$ the degree matrix in which the $k$ th element on the diagonal is equal to $\sum_{j=1}^{n} a_{kj}, k = 1,\dots,n$ and all off-diagonal elements are 0. Accordingly, $L = D - A$ is call the Laplacian matrix associated with the graph $\mathcal{G}$.

The following assumptions are employed in our paper.

*A1.* $\mathcal{G}$ is undirected and connected.

*A2.* $f$ is continuous, bounded, odd.

*A3.* There exists a real constant $\gamma > 0$ such that $f(x) > 0 \; \forall \; x \in (0, \gamma]$ and $\frac{\partial f(x)}{\partial x} > 0 \; \forall \; x \in [-\gamma, \gamma]$.

The meaning of assumption A3 is to ensure that the network of heterogeneous phase oscillators can be synchronized since the synchronization does not occur if $\frac{\partial f(x)}{\partial x} < 0 \; \forall \; x \in [-\gamma, \gamma]$.

Denote $\lambda_2(L)$ the algebraic connectivity of the Laplacian matrix $L$ which is the smallest non-zero eigenvalue of $L$. Let the undirected graph $\mathcal{G}$ be assigned with an arbitrary direction, then denote $B \in \mathbb{R}^{n \times |\mathcal{E}|}$ the associated oriented incidence matrix in which $B_{kj} = 1$ if the node $k$ th is the sink node of the edge $j$ th, $B_{kj} = -1$ if the node $k$ th is the source node of the edge $j$ th, and $B_{kj} = 0$ if otherwise; $|\mathcal{E}|$ denotes the number of elements in $\mathcal{E}$. Consequently, $B^T\theta$ is a vector including all the phase differences among connected oscillators and $L = B\text{diag}\left(a_{kj}\right)_{k,j=1,\dots,n} B^T$. In the following, we introduce some properties of the matrices $L, B$.

*(i).* $L$ has a single eigenvalue $0$ with the associated eigenvector $\mathbf{1}_n$.

*(ii).* $B^T \mathbf{1}_n = 0$.

*(iii).* $\left\| B^T \right\|_\infty = 2$.

*(iv).* $L^\dagger L = I_n - \frac{1}{n}\mathbf{1}_{n\times n}$ in which $L^\dagger$ is the pseudo-inverse matrix of $L$, $\mathbf{1}_{n\times n}$ is the $n$ by $n$ matrix whose elements are all equal to $1$.

Our problem is to study the collective synchronization behaviors in the oscillator network (2.1). When the nonlinear function $f$ is sinusoidal, i.e., $f(x) = \sin(x)$ and each oscillator is connected to all other ones, (2.1) becomes the celebrated Kuramoto model and there is a rich collection of results for it. However, when $f$ is a general nonlinear function, very few results are available. Thus, in the next sections, we will contribute sufficient conditions for frequency synchronization of oscillators in two scenarios, one is symmetric couplings and the other is asymmetric couplings among oscillators, for a class of the nonlinear function $f$ in the assumptions A2-A3.

## 3. SYNCHRONIZATION IN SYMMETRICALLY COUPLED OSCILLATOR NETWORK

Let us denote $\Delta_p^{\gamma}(x) = \left\{x \in \mathbb{R}^m : \|x\|_p \le \gamma\right\}$ where $\|x\|_p$ is the $p$-norm of a vector $x$; $\omega = \left[\omega_1 \quad \ldots \quad \omega_n\right]^T$ Suppose that the frequencies of the oscillator network (2.1) are synchronized at $\omega_{sym}$ then

$$\omega_{sym} = \omega_k - \sum_{j=1}^{n} a_{kj} f\left(\theta_k - \theta_j\right), k = 1, \ldots, n, \tag{3.1}$$

By summing up both sides of (3.1) with $k = 1, \ldots, n$, we obtain

$$n\omega_{sym} = \sum_{k=1}^{n} \omega_k - \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} f\left(\theta_k - \theta_j\right). \tag{3.2}$$

Since the couplings among oscillators are symmetric, i.e., $a_{kj} = a_{jk}\ \forall\ k, j = 1, \ldots, n$, and $f$ is an odd function, $\sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} f\left(\theta_k - \theta_j\right) = 0$. As a result,

$$\omega_{sym} = \frac{1}{n} \sum_{k=1}^{n} \omega_k \tag{3.3}$$

This means if the synchronization occurs in the network (2.1) when the couplings are symmetric, the synchronized frequency will be the average of all natural frequencies of oscillators. Denote $\omega_{ave} = \frac{1}{n} \sum_{k=1}^{n} \omega_k$, then subtracting both sides of (2.1) by $\omega_{ave}$, we have

$$\dot{\theta}_k - \omega_{ave} = \omega_k - \omega_{ave} - \sum_{j=1}^{n} a_{kj} f\left(\theta_k - \theta_j\right), k = 1, \ldots, n. \tag{3.4}$$

Before showing a sufficient condition for synchronization, we introduce the following lemma.

**Lemma 1.** Consider the oscillator network (2.1) where the couplings among oscillators are symmetric and the graph representing the interconnection structure in the network is connected. The following statements hold.

(1) The Jacobian matrix $J\left(\theta\right)$ of the oscillator network (2.1) is given by

$$J\left(\theta\right) = -B \mathrm{diag}\left(a_{kj} \left.\frac{\partial f\left(x\right)}{\partial x}\right|_{x=\theta_k - \theta_j}\right)_{k,j=1,\ldots,n} B^T \tag{3.5}$$

(2) If there exists an equilibrium point $\theta^*$ such that $B^T\theta^* \in \Delta_{\infty}^{\pi/2}\left(B^T\theta\right)$ then

(i) $-J\left(\theta^*\right)$ is a Laplacian matrix.

(ii) The equilibrium point is unique and locally exponentially stable.

*Proof:*

This lemma can be considered as a generalization of Lemma 3.2 in [8] and its proof is similar to the proof in [8], so we ignore it here for brevity. ■

Now, we present one of the main contributions of this paper in the following theorem.

**Theorem 1.** Consider the oscillator network (2.1) as in Lemma 1. If the algebraic connectivity satisfies

$$\lambda_2\left(L\left(B^T\theta\right)\right) \geq \frac{2\left\|\omega - \omega_{ave}\mathbf{1}_n\right\|_2}{\rho}, \tag{3.6}$$

where $\rho$ is defined as follows,

$$\rho = \sup_{\gamma}\left(\gamma \min_{x\in[0,\gamma]}\frac{f(x)}{x}\right), \tag{3.7}$$

then the following statements hold.

(1) The set $\Delta_2^{\gamma}\left(B^T\theta\right)$ is positively invariant, i.e., if initial phases of oscillators belongs to $\Delta_2^{\gamma}\left(B^T\theta\right)$ then they will remain inside it.

(2) The frequencies of oscillators exponentially synchronize to the average frequency $\omega_{ave}$. Furthermore, the sync rate is lower bounded by $\lambda_2\left(L\right)\inf_{\gamma}\min_{|x|\leq\gamma}\frac{\partial f(x)}{\partial x}$.

*Proof:*

The equilibrium point of (3.4) is determined by the following equation.

$$\begin{aligned}\omega_k - \omega_{ave} &= \sum_{j=1}^{n} a_{kj} f\left(\theta_k^* - \theta_j^*\right), k = 1,\ldots,n,\\ \Leftrightarrow \omega - \omega_{ave}\mathbf{1}_n &= B\text{diag}\left(a_{kj} f\left(\theta_k^* - \theta_j^*\right)/\left(\theta_k^* - \theta_j^*\right)\right)B^T\theta^*,\end{aligned} \tag{3.8}$$

where $\theta^* = \left[\theta_1^* \quad \theta_2^* \quad \ldots \quad \theta_n^*\right]^T$. Denote $L\left(B^T\theta^*\right)$ the Laplacian matrix corresponding to $B\text{diag}\left(a_{kj} f\left(\theta_k^* - \theta_j^*\right)/\left(\theta_k^* - \theta_j^*\right)\right)B^T$ and $L\left(B^T\theta^*\right)^\dagger$ the pseudo-inverse of $L\left(B^T\theta^*\right)$. Next, multiplying to the left of both sides of (3.7) with $B^T L\left(B^T\theta^*\right)^\dagger$ gives us

$$\begin{aligned}B^T L\left(B^T\theta^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right) &= B^T L\left(B^T\theta^*\right)^\dagger L\left(B^T\theta^*\right)\theta^*,\\ &= B^T\left(I_n - \frac{1}{n}\mathbf{1}_{n\times n}\right)\theta^*,\\ &= B^T\theta^*,\end{aligned} \tag{3.9}$$

Note that (3.9) is a continuous and convex equation of $B^T\theta^*$. Employing Brouwer's fixed point theorem [12], there exits an equilibrium point

$$B^T\theta^* \in \Delta_2^\gamma\left(B^T\theta\right) = \left\{B^T\theta, \theta \in \mathbb{R}^n : \left\|B^T\theta\right\|_2 \leq \gamma\right\}$$

if and only if $\left\|B^T L\left(B^T\theta^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \leq \gamma$ for all $B^T\theta^* \in \Delta_2^\gamma\left(B^T\theta\right)$. Denote $\tilde{\theta}^* = B^T\theta^*$ then there exists an equilibrium point $\tilde{\theta}^* \in \Delta_2^\gamma\left(B^T\tilde{\theta}\right)$ if and only if

$$\max_{\tilde{\theta}^* \in \Delta_2^\gamma\left(B^T\theta\right)} \left\|B^T L\left(B^T\tilde{\theta}^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \leq \gamma. \tag{3.10}$$

Since

$$\left\|B^T L\left(B^T\tilde{\theta}^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \leq \left\|B^T\right\|_\infty \left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 = 2\left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|$$

inequality (3.10) is satisfied if

$$2\max_{\tilde{\theta}^* \in \Delta_2^\gamma\left(B^T\theta\right)} \left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \leq \gamma. \tag{3.11}$$

Moreover, $\left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \leq \left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\right\|_2 \left\|\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2$, hence (3.11) is true if

$$2\left\|\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \max_{\tilde{\theta}^* \in \Delta_2^\gamma\left(B^T\theta\right)} \left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\right\|_2 \leq \gamma. \tag{3.12}$$

On the other hand, it can be shown that $\left\|L\left(B^T\tilde{\theta}^*\right)^\dagger\right\|_2 = 1/\lambda_2\left(L\left(B^T\tilde{\theta}^*\right)\right)$ since all eigenvalues of $L\left(B^T\tilde{\theta}^*\right)^\dagger$ except 0 are the inverse of eigenvalues of $L\left(B^T\tilde{\theta}^*\right)$. Furthermore, $\Delta_2^\gamma\left(B^T\theta\right) \in \Delta_\infty^\gamma\left(B^T\theta\right)$, hence (3.12) is satisfied if

$$2\left\|\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \max_{\tilde{\theta}^* \in \Delta_2^\gamma\left(B^T\theta\right)} \frac{1}{\lambda_2\left(L\left(B^T\tilde{\theta}^*\right)\right)} \leq \gamma. \tag{3.13}$$

Since $f$ is an odd function, $\frac{f\left(\tilde{\theta}^*\right)}{\tilde{\theta}^*} \geq 0$ is an even function. Moreover, $\frac{f\left(\tilde{\theta}^*\right)}{\tilde{\theta}^*} \geq 0 \;\forall\; \tilde{\theta} \in \left|-\gamma, \gamma\right|$ due to assumption A3, therefore

$$L\left(B^T\theta^*\right) \geq \min_{x \in \Delta_\infty^\gamma(x)} \frac{f(x)}{x} B\mathrm{diag}\left(a_{k_l}\right)B^T = \min_{x \in \Delta_\infty^\gamma(x)} \frac{f(x)}{x} L. \tag{3.14}$$

Accordingly, (3.13) is satisfied if

$$2\left\|\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \left|\lambda_2(L) \min_{x \in \Delta_\infty^\gamma(x)} \frac{f(x)}{x}\right|^{-1} \leq \gamma,$$

$$\Leftrightarrow 2\left\|\left(\omega - \omega_{ave}\mathbf{1}_n\right)\right\|_2 \left|\gamma \min_{x \in \Delta_\infty^\gamma(x)} \frac{f(x)}{x}\right|^{-1} \leq \lambda_2(L), \tag{3.15}$$

Let us denote $\rho$ as in (3.7) then it can be seen that (3.15) is satisfied if (3.6) is satisfied. Thus, by employing Lemma 1, the equilibrium point $\theta^*$ such that $B^T\theta \in \Delta_2^\gamma\left(B^T\theta\right)$ is locally stable if the condition (3.6) is satisfied. Furthermore, the sync rate is lower bounded by

$$\lambda_2\left(-J\left(\theta^*\right)\right) \geq \lambda_2\left(B\mathrm{diag}\left(a_{kj}\right)_{k,j=1,\dots,n} B^T\right) \min_{x \in \Delta_2^\gamma(x)} \left.\frac{\partial f(x)}{\partial x}\right|_{x=\theta_i^*-\theta_j^*}$$

$$\geq \lambda_2(L) \min_{x \in \Delta_\infty^\gamma(x)} \frac{\partial f(x)}{\partial x}$$

$$\geq \lambda_2(L) \inf_\gamma \min_{|x| \leq \gamma} \frac{\partial f(x)}{\partial x} \tag{3.16}$$

■

Theorem 1 shows that the algebraic connectivity of the graph representing the interconnection structure in the network should be greater than a value specified by the natural frequencies of oscillators and the nonlinear function $f$ such that the frequency synchronization can occur. Since the algebraic connectivity is related to the coupling strengths among oscillators, this means the oscillators should be strongly interconnected enough to achieve the frequency synchronization.

**Example 1.** Consider a network of 50 phase oscillators representing the circadian oscillators whose autonomous frequencies of oscillators are assumed to be slightly different and around the frequency of daily light-dark cycle, i.e., $2\pi/24$. In particular,

$$\omega_k = \frac{2\pi}{24} + \varepsilon_k, k = 1,\dots,50, \tag{3.17}$$

where $\varepsilon_k$ is a random variable described by the standard normal distribution. The coupling function $f$ is hyperbolic tangent function $\tanh$. Then, we can easily find that $\rho = 1$. Therefore, the sufficient condition (3.6) becomes

$$\lambda_2(L) \geq 2\left\|\omega - \omega_{ave}\mathbf{1}_n\right\|_2^2. \tag{3.18}$$

where $L$ is the Laplacian matrix of the graph representing the symmetric interconnections among oscillators. We first randomly generate $\omega_k$ as in (3.17) then compute $2\left\|\omega - \omega_{ave}\mathbf{1}_n\right\|_2$. Next, we randomly generate the Laplacian matrix $L$ and verify the condition (3.18). The simulation results are displayed in Figure 1 where the upper and lower plots show the phase and frequency responses of oscillators, respectively. We can observe that the frequencies of oscillators are synchronized. Moreover, the synchronized frequency is approximately $2\pi/24$. In this simulation, $2\left\|\omega - \omega_{ave}1_n\right\|_2 = 1.2134$ and $\lambda_2(L) = 1.8162$, so we can see that the condition (3.18) is satisfied.

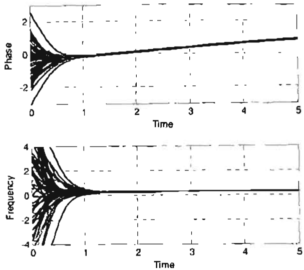


*Figure 1.* Time plot of a symmetrically coupled oscillator network.

## 4. OSCILLATOR NETWORKS WITH ASYMMETRIC COUPLINGS

In the previous section, we have studied the synchronization in the oscillator network (2.1) with bidirectional, symmetric couplings among oscillators. Nonetheless, the interactions among oscillators may be asymmetric due to the uncertainties, disturbances, or noises in the communication links. Therefore, we investigate in this section the scenario of asymmetrically coupled oscillator networks and accordingly propose a sufficient condition for synchronization as well as the value of synchronized frequency and the sync rate.

Suppose that the coupling weight from every other oscillator to the $k$-th oscillator is perturbed by a same quantity $\delta_k, k = 1,\ldots,n$. Consequently, the oscillator network model is

$$\dot{\theta}_k = \omega_k - \sum_{j=1}^{n} a_{kj}\delta_k f\left(\theta_k - \theta_j\right), k = 1,\ldots,n. \tag{4.1}$$

Denote $\Sigma = \operatorname{diag}\left(\delta_k\right)_{k=1,\ldots,n}$, we can rewrite (4.1) as follows,

$$\Sigma^{-1}\dot{\theta} = \Sigma^{-1}\omega - \Sigma^{-1}\tilde{L}\left(B_1^T\theta\right)\theta, \tag{4.2}$$

where $\theta = \begin{bmatrix}\theta_1 & \ldots & \theta_n\end{bmatrix}^T$ is the vector of oscillators' phases; $B_1$ is the incidence matrix of the graph representing the interconnection matrix in the network; and

$$\tilde{L}\left(B_1^T\theta\right) = B_1\operatorname{diag}\left(a_{ij}f\left(\theta_i - \theta_j\right)/\left(\theta_i - \theta_j\right)\right)B_1^T$$

Let $\omega_{\text{asym}}$ be the synchronized frequency in the oscillator network (4.1) with asymmetric couplings. Subsequently, we obtain the following equations from (4.2)

$$\begin{aligned}
\omega_{\text{asym}}\Sigma^{-1}\mathbf{1}_n &= \Sigma^{-1}\omega - \Sigma^{-1}\tilde{L}\left(B_1^T\theta\right)\theta, \\
\Rightarrow \omega_{\text{asym}}\mathbf{1}_n^T\Sigma^{-1}\mathbf{1}_n &= \mathbf{1}_n^T\Sigma^{-1}\omega - \mathbf{1}_n^T\Sigma^{-1}\tilde{L}\left(B_1^T\theta\right)\theta, \\
&= \mathbf{1}_n^T\Sigma^{-1}\omega, \\
\Rightarrow \omega_{\text{asym}} &= \left(\mathbf{1}_n^T\Sigma^{-1}\mathbf{1}_n\right)^{-1}\mathbf{1}_n^T\Sigma^{-1}\omega. \qquad (4.3)
\end{aligned}$$

Equation (4.3) shows how we can compute the synchronized frequency as the synchronization occurs in the oscillator network (4.1). Then, utilizing the same approach as in the previous section, we obtain the following result on a sufficient condition for synchronization in asymmetrically coupled oscillator networks.

**Theorem 2.** Consider the oscillator network (4.1) where the graph representing the interconnections in the network is connected. The frequencies of all oscillators synchronize to $\omega_{\text{asym}}$ if the algebraic connectivity of the Laplacian matrix associated with the graph satisfies

$$\lambda_2\left(\Sigma^{-1}\tilde{L}\left(B_1^T\theta\right)\right) \geq \frac{2\left\|\Sigma^{-1}\omega - \omega_{\text{asym}}\Sigma^{-1}\mathbf{1}_n\right\|_2}{\rho}. \qquad (4.4)$$

**Example 2.** Consider the same oscillator network as in Example 1, but the couplings among oscillators are perturbed to be asymmetric. Then, if the algebraic connectivity of the undirected graph associated with the Laplacian matrix $\Sigma^{-1}\tilde{L}\left(B_1^T\theta\right)$ is sufficiently large as shown in (4.4), the frequencies of oscillators are synchronized as illustrated in Figure 2.

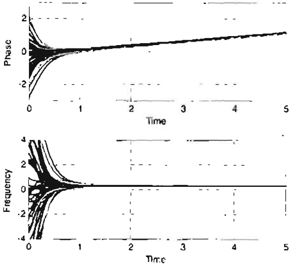


*Figure 2.* Time plot of an asymmetrically coupled oscillator network.

## 5. CONCLUSIONS

We have presented in this paper some new results on the synchronization of nonlinearly coupled phase oscillator networks which reveal that the coupling strengths among oscillators should be stronger than some determined values such that the frequency synchrony occurs. Several numerical examples were introduced to demonstrate the theoretical results. The next works would consider more complex situations where time delays exists in the couplings, or the models of phase oscillators are of higher orders.

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## TÓM TẮT

## NGHIÊN CỨU SỰ ĐỒNG BỘ CỦA MẠNG CÁC PHẦN TỬ DAO ĐỘNG PHA CÓ TẦN SỐ RIÊNG KHÁC NHAU

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Bài báo này trình bày một nghiên cứu về các đặc tính đồng bộ hóa trong mạng dao động gồm các phần tử dao động pha được kết nối qua một hàm phi tuyến tổng quát và sự kết nối giữa các phần tử dao động là hai chiều. Tiếp đó, tác giả nghiên cứu hai trường hợp bao gồm các kết nối đối xứng và phi đối xứng giữa các phần tử dao động. Trong cả hai trường hợp, tác giả chỉ ra rằng nếu độ lớn của các kết nối là lớn hơn một giá trị nhất định thì tần số của các phần tử dao động sẽ được đồng bộ. Hơn nữa, tốc độ đồng bộ hóa cũng được chỉ ra. Cuối cùng, một số ví dụ được giới thiệu để minh họa các kết quả lí thuyết.

*Từ khóa:* mạng dao động pha, sự đồng bộ hóa, các phần tử dao động phi tuyến được kết nối.