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A SHRINKING PROJECTION METHOD FOR SOLVING THE SPLIT COMMON FIXED POINT PROBLEM IN HILBERT SPACES

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ABSTRACT

We study the split common fixed point problem in two Hilbert spaes. Let H_1 and H_2 be two real Hilbert spaces. Let $S_1 : H_1 \rightarrow H_1$, and $S_2 : H_2 \rightarrow H_2$, be two nonexpansive mappings on H_1 and H_2, respectively. Consider the following problem: find an element $x^\dagger \in H_1$ such that

$$x^\dagger \in \Omega := \mathrm{Fix}(S1) \cap T-1(\mathrm{Fix}(S2)) \neq \emptyset,$$

where $T : H_1 \rightarrow H_2$ is a given bounded linear operator from H_1 to H_2.

Using the shrinking projection method, we propose a new algorithm for solving this problem and establish a strong convergence theorem for that algorithm.

Key words: *Hilbert space, metric projection, monotone operator, nonexpansive mapping, split common fixed point problem*

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PHƯƠNG PHÁP CHIẾU THU HẸP GIẢI BÀI TOÁN ĐIỂM BẤT ĐỘNG CHUNG TÁCH TRONG KHÔNG GIAN HILBERT

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TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu bài toán điểm bất động chung tách trong 2 không gian Hilbert. Cho H_1 và H_2 là hai không gian Hilbert thực. Cho S_1: $H_1 \rightarrow H_1$, và S_2: $H_2 \rightarrow H_2$, là hai ánh xạ không giãn trên không gian H_1 và H_2 tương ứng. Bài toán đặt ra là: tìm một phần tử $x^\dagger \in H_1$ sao cho:

$$x^\dagger \in \Omega := \mathrm{Fix}(S_1) \cap T-1(\mathrm{Fix}(S_2)) \neq \emptyset,$$

Khi $T : H_1 \rightarrow H_2$ là một ánh xạ tuyến tính bị chặn cho trước từ H_1 vào H_2. Sử dụng phương pháp chiếu thu hẹp, chúng tôi đề xuất một thuật toán mới (Thuật toán 3.1) để giải bài toán này và thiết lập một định lý hội thụ mạnh cho thuật toán (Định lý 3.3).

Từ khóa: *Không gian Hilbert, phép chiếu metric, toán tử đơn điệu, ánh xạ không giãn, bài toán điểm bất động chung tách*

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1. Introduction

Let K and Q be nonempty, closed and convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator and let $T^*: H_2 \longrightarrow H_1$ be its adjoint. The *split convex feasibility problem* (SCFP) is formulated as follows:

(1.1)

$$\text{Find an element } x^* \in K \text{ such that } Tx^* \in Q.$$

The SCFP was first introduced by Y. Censor and T. Elfving [1] for modeling certain inverse problems. It plays an important role in medical image reconstruction and in signal processing (see [2, 3]). Several iterative algorithms for solving (1.1) were presented and analyzed in [2–14], and in references therein.

It is known that the SCFP is a special case of the *split common fixed point problem* (SCFPP), which is formulated as follows. Let $S_1: H_1 \longrightarrow H_1$ and $S_2: H_2 \longrightarrow H_2$ be two nonexpansive mappings and let $T: H_1 \longrightarrow H_2$ be a bounded linear operator such that $\Omega = \mathrm{Fix}(S_1) \cap T^{-1}(\mathrm{Fix}(S_2)) \neq \emptyset$. The SCFPP is to find an element $x^* \in \Omega$.

In this paper, by combining the proximal point algorithm with the shrinking projection method, we introduce and analyze a new iterative method for solving the SCFPP in Hilbert spaces. Using these methods, we also remove the assumptions imposed on the norm $\|T\|$ (see Section 3 below).

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H. It is well known that for each $x \in H$, there is unique point $P_C^H x \in C$ such that

$$\|x - P_C^H x\| = \inf_{u \in C} \|x - u\|. \tag{2.1}$$

The mapping $P_C^H: H \longrightarrow C$ defined by (2.1) is called the *metric projection* of H onto C. Moreover, we have (see, for example, Section 3 in [15])

(2.2)

$$\langle x - P_C^H x, y - P_C^H x\rangle \leq 0 \quad \forall x \in H,\ y \in C.$$

Recall that a mapping $T: C \longrightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of fixed points of T by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$.

The following lemma is used in the sequel in the proofs of the main result of this paper.

From (2.2), we have the following Lemma.

Lemma 2.1. *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Then for all $x \in H$ and $y \in C$, we have*

$$\|x - P_C^H x\|^2 + \|y - P_C^H x\|^2 \leq \|x - y\|^2.$$

3. Main results

Let H_1 and H_2 be two real Hilbert spaces. Let $S_1: H_1 \longrightarrow H_1$, and $S_2: H_2 \longrightarrow H_2$, be two nonexpansive mappings on H_1 and H_2, respectively. Consider the following problem: find an element $x^\dagger \in H_1$ such that

(3.1)

$$x^\dagger \in \Omega := \mathrm{Fix}(S_1) \cap T^{-1}(\mathrm{Fix}(S_2)) \neq \emptyset,$$

where $T: H_1 \longrightarrow H_2$ is a given bounded linear operator from H_1 to H_2.

Using the shrinking projection method, we introduce in this section a new algorithm for solving Problem (3.1).

Algorithm 3.1. For any initial guess $x_0 = x \in H_1$, $C_0 = D_0 = H_1$, define the sequence $\{x_n\}$ by

$$y_n = S_1(x_n),$$

$$z_n = S_2(Ty_n),$$

$$C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\},$$

$D_{n+1} = \{z \in D_n : \ \|z_n - Tz\| \le \|Ty_n - Tz\|\}$,
$x_{n+1} = P^{H_1}_{C_{n+1}\cap D_{n+1}} x_0, \quad n \ge 0.$

The following theorem yields the strong convergence of the sequence generated by Algorithm 3.1.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $P^{H_1}_{\Omega} x_0$.*

Proof. We divide the proof of this theorem into four steps.

Step 1. The sequence $\{x_n\}$ is well defined.

First, we claim that C_n and D_n are closed and convex subsets of H_1 for all $n \ge 0$. To see this, we rewrite, for each integer $n \ge 0$, the subsets C_{n+1} and D_{n+1} in the following forms:

$C_{n+1} = C_n \cap \Big\{z \in H_1 : \ \langle x_n - y_n, z\rangle \le \frac{1}{2}(\|x_n\|^2 - \|y_n\|^2)\Big\}$,

$D_{n+1} = D_n \cap \Big\{z \in H_1 : \langle Ty_n - z_n, Tz\rangle \le \frac{1}{2}(\|Ty_n\|^2 - \|z_n\|^2)\Big\}$,

$= D_n \cap \Big\{z \in H_1 : \ \langle T^*(Ty_n - z_n), z\rangle \le \frac{1}{2}(\|Ty_n\|^2 - \|z_n\|^2)\Big\}$,

respectively. Now, using induction and the fact that $C_0 = D_0 = H_1$, we see that C_n and D_n are indeed closed and convex subsets of H_1 for all $n \ge 0$, as claimed.

Next, we show that $\Omega \subset C_n \cap D_n$ for all $n \ge 0$. It is clear that $\Omega \subset C_0 \cap D_0 = H_1$. Suppose that $\Omega \subset C_n \cap D_n$ for some $n \ge 0$. Taking any point $p \in \Omega$, we have $S_1(p) = p$ and $S_2(Tp) = Tp$. Therefore, the nonexpansivity of S_1 and S_2 implies that

$$\|y_n - p\| = \|S_1(x_n) - S_1(p)\| \le \|x_n - p\|$$
$$\|z_n - Tp\| = \|S_2(Ty_n) - S_2(Tp)\| \le \|Ty_n - Tp\|.$$

Hence the definitions of C_{n+1}, D_{n+1} and the fact that $\Omega \subset C_n \cap D_n$ imply that $\Omega \subset C_{n+1} \cap D_{n+1}$. Hence, by induction, we obtain that $\Omega \subset C_n \cap D_n$ for all $n \ge 0$ and hence that $C_n \cap D_n$ is a nonempty, closed and convex subset of H_1 for each integers $n \ge 0$. This implies that the sequence $\{x_n\}$ is indeed well defined, as asserted.

Step 2. $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

We first show that the sequence $\{x_n\}$ is bounded. Indeed, let $x^\dagger = P_\Omega x_0$. It follows from the fact that $\Omega \subset C_n \cap D_n$, $x^\dagger \in C_n \cap D_n$ for all $n \ge 0$. Thus, using $x_n = P_{C_n \cap D_n} x_0$, we obtain that

(3.2)
$$\|x_0 - x_n\| \le \|x_0 - x^\dagger\| \quad \text{for all} \ \ n \ge 0.$$

Hence the sequence $\{x_n\}$ is bounded.

Next, using $x_{n+1} = P_{C_{n+1}\cap D_{n+1}} x_0 \in C_n \cap D_n$, $x_n = P_{C_n \cap D_n} x_0$ and Lemma 2.1, we obtain that

$$\begin{aligned}\|x_n - x_0\|^2 &\le \|x_{n+1} - x_0\|^2 - \|x_{n+1} - x_n\|^2 \\ &\le \|x_{n+1} - x_0\|^2.\end{aligned}$$

This implies that the sequence $\{\|x_n - x_0\|\}$ is increasing. The boundedness of $\{x_n\}$ now implies that the limit of $\{\|x_n - x_0\|\}$ exists and is finite.

Next, we show that sequence $\{x_n\}$ converges strongly to some point $p \in H_1$. Indeed, for all $m \ge n$, we have $C_m \cap D_m \subset C_n \cap D_n$. Thus, $x_m \in C_n \cap D_n$. By Lemma 2.1, we have

$$\|x_m - x_n\|^2 \le \|x_m - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \quad \text{as} \quad m, n \to \infty.$$

So, $\{x_n\}$ is Cauchy sequence. Hence there exists the limit $\lim_{n\to\infty} x_n = q$. Thus we have

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - q\| + \|x_n - q\| \to 0,$$

which implies that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, as claimed.

Step 3. $\|x_n - y_n\| \to 0$ and $\|z_n - Ty_n\| \to 0$ as $n \to \infty$.

From $x_{n+1} = P^{H_1}_{C_n \cap D_n} x_0 \in C_n$ and the definition of C_n, we have

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\|.$$

So, from $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we obtain that

$$\|x_{n+1}-y_n\|\to 0. \tag{3.3}$$

Since

$$\|x_n-y_n\|\le\|x_{n+1}-y_n\|+\|x_{n+1}-x_n\|,$$

it follows that

$$\|x_n-y_n\|\to 0. \tag{3.4}$$

From $x_{n+1}=P_{C_n\cap D_n}x_0\in D_n$ and the definition of D_n we have

$$\begin{aligned}\|z_n-Tx_{n+1}\|&\le\|Ty_n-Tx_{n+1}\|\\&\le\|T\|\|x_{n+1}-y_n\|.\end{aligned}$$

It now follows from (3.3) that

$$\|z_n-Tx_{n+1}\|\to 0. \tag{3.5}$$

So, using (3.3) and the estimate

$$\begin{aligned}\|z_n-Ty_n\|&\le\|z_n-Tx_{n+1}\|+\|Tx_{n+1}-Ty_n\|\\&\le\|z_n-Tx_{n+1}\|+\|T\|\|x_{n+1}-y_n\|,\end{aligned}$$

we obtain

$$\|z_n-Ty_n\|\to 0. \tag{3.6}$$

Step 4. $x_n\to x^\dagger=P_\Omega x_0$ as $n\to\infty$.

Since $x_n\to q$ and T is bounded linear operator, $Tx_n\to Tq$. It follows from (3.4), (3.6), the continuity S_1 and S_2 that $q\in\Omega$.

Letting $n\to\infty$ in (3.2), we get that

$$\|x_0-p\|\le\|x_0-x^\dagger\|$$

and the uniqueness of $x^\dagger$ yields the equality $p=x^\dagger$.

This completes the proof. □

The following result which concerns finding a fixed point of a nonexpansive mapping in a real Hilbert space.

Corollary 3.2. *Let H be a real Hilbert space and let $S:\ H\longrightarrow H$ be a nonexpansive mapping such that $\Omega=\mathrm{Fix}(S)\neq\emptyset$. Then the sequence $\{x_n\}$ is generated by $x_0=x\in H$, $C_0=H_1$ and*

$$\begin{aligned}&y_n=S(x_n),\\&C_{n+1}=\big\{z\in C_n:\ \|y_n-z\|\le\|x_n-z\|\big\},\\&x_{n+1}=P_{C_{n+1}}x_0,\quad n\ge 0,\end{aligned}$$

converges strongly to $x^\dagger=P_\Omega^{H_1}x_0$.

Proof. We obtain this result by applying Theorem 3.1 with $H_1=H_2=H$, $S=S_1$, $S_2=I^H$ and $T=I^H$, the identity operator on H. □

We now have the following result for solving the SCFP in Hilbert sapces.

Let H_1 and H_2 be two real Hilbert spaces, and let K and Q, be two closed and convex subsets of H_1 and H_2, respectively. Letting $T:\ H_1\longrightarrow H_2$ be a bounded linear operator such that $\Omega=K\cap T^{-1}(Q)\neq\emptyset$, we now consider the following problem:

$$\text{Find an element } x^\dagger\in\Omega. \tag{3.7}$$

Using Theorem 3.1, we obtain the following result concerning Problem (3.7).

Theorem 3.3. *The sequence $\{x_n\}$ generated by $x_0\in H_1$, $C_0=D_0=H_1$ and*

$$\begin{aligned}&y_n=P_K^{H_1}x_n,\\&z_n=P_Q^{H_2}Ty_n,\\&C_{n+1}=\big\{z\in C_n:\ \|y_n-z\|\le\|x_n-z\|\big\},\\&D_{n+1}=\big\{z\in D_n:\ \|z_n-Tz\|\le\|Ty_n-Tz\|\big\},\\&x_{n+1}=P^{H_1}_{C_{n+1}\cap D_{n+1}}x_0,\quad n\ge 0,\end{aligned}$$

converges strongly to $x^\dagger=P_\Omega^{H_1}x_0$.

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