GRADIENT DESCENT METHOD AND PARAMETER SELECTION FOR IMAGE RESTORATION PROBLEM

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ABSTRACT

Image enhancement is a meaningful problem in many practical applications. It plays an important role in preprocessing steps for recognition and information extraction. Image restoration is often considered one of the data processing steps before the training process for machine learning models is performed. Image restoration problems are often solved by iterative algorithms, where the choice of iteration parameters plays an important role in improving the algorithm's convergence rate. The problem is determining the parameters that ensure the algorithm has the fastest convergence rate. In this paper, we propose a parameter selection method for the Gradient descent algorithm to recover the original image data from the image obtained after performing morphological transformation on the original image. According to this method, we analyze the eigenvalues of the morphological transformation matrix to derive a formula to determine the optimal parameters for the Gradient descent algorithm. We have proven that the iterative process converges from the parameter determination formula. The experimental results also show that the proposed theory is consistent and confirms that the approximate solution converges to the solution of the original problem.

Keywords: Image restoration; Gradient descent; iterative method; convex optimization; optimal parameter.

1. Introduction

Advanced image enhancement is a significant issue in many practical applications, playing a crucial role in preprocessing steps for recognition and information extraction. Image restoration, in particular, is often considered one of the essential preprocessing stages for image data before training machine learning models, especially in applications that use image data [1], [2], [3]. These steps are critical for preparing and improving the quality of input data before feeding it into training processes. Key reasons for performing image preprocessing and restoration before training include: Removing noise from image data, helping machine learning models recognize and learn important patterns more accurately [4], [5], [6], [7]; Ensuring that input data is complete and free from loss of important information helps the model learn and predict more effectively with sufficient data [8]; Highlighting important features in images allows machine learning models to identify objects and features more efficiently [9]; Removing backgrounds and detecting objects focuses on critical areas of the image, minimizing distractions and allowing machine learning models to concentrate on essential features in the data [10]. Image data preprocessing and restoration are often crucial steps in data preparation before training machine learning models and are typically performed to improve model quality and performance. Image preprocessing is often done through morphological transformations, which modify images by altering orientation, dimensions, angles, edges, and more. These transformations can be linear or nonlinear operators, depending on the goals of the image-processing task.

In practice, for various reasons, each morphological transformation of an image may result in blurred images due to the impact of noise [11], [12]. This paper's problem is restoring the original image from these morphologically transformed images.

Let A denote the morphological transformation operator that transforms image x into a new morphology b, with e as the noise vector resulting from environmental factors. Thus, the image restoration problem is reduced to finding a solution $x^* \in R^N$ that satisfies the operator equation.

$$Ax + e = b \tag{1}$$

In cases where the noise e has a negligible impact on image quality, the image can be restored using the formula

$$x^* = A^{-1}b \tag{2}$$

From (1) and (2), we have

$$x^* = x + A^{-1}e \tag{3}$$

It is evident that, in some cases, $A^{-1}e$ can be pretty significant, even though e itself is negligible compared to b. This discrepancy introduces differences between the original image and the restored image. In this scenario, the problem is called an ill-posed problem, where solving (1) directly using (2) becomes infeasible. Instead, regularization methods must be used to approximate (1) as a well-posed problem and find a solution for this approximation.

For removing noise, numerous traditional methods in image processing, typically basic and widely used techniques, have been applied long before modern approaches like machine learning and artificial intelligence became prevalent. These traditional methods include Median Filtering [13], Gaussian Filtering [14], and Fourier Transform [15], among others. Such techniques are usually applied directly to image data and do not demand extensive computational resources, which makes them popular and straightforward for basic image processing applications. However, they may be less effective in handling complex images and substantial noise. In contrast, regularization methods offer advantages in flexibility, performance, and quality over traditional approaches. Notably, the adaptability and customizability of regularization methods allow users to produce more visually appealing and realistic images. For this reason, various regularization techniques and their applications in image restoration have been chosen as the research topic.

In essence, the problem addressed in this paper is: Given a blurred, noisy, or otherwise degraded input image, the goal of image restoration is to generate a reconstructed image that closely resembles the original image. This process may include noise removal, reconstruction of lost information, contrast enhancement, colour balance adjustment, and restoration of lost details.

In this paper, we focus on the iterative regularization technique to find a solution to the image restoration problem [16], [17], [18]. This approach is widely used in image restoration tasks, especially when dealing with ill-posed problems where direct solutions may be unstable or prone to noise amplification. This technique helps approximate the solution to the image restoration problem by applying a sequence of iterations that gradually improve the image quality.

2. Method

From problem (1), we transform it into the equivalent problem of finding $\tilde{x}$ such that

$$J(\tilde{x}) = min_{x \in R^N} \frac{1}{2} \|Ax - b\|^2 \tag{4}$$

Where $J(x)$ is the loss function describes the error between the left and right sides of (1). In this section, we will use an iterative approach to solve (4), which is often more efficient for large and complex problems, as it does not require direct computation of the matrix inverse. Specifically, we use the Gradient Descent method and propose optimal parameter choices that ensure the stability and convergence of the solution. This is necessary when the data is strongly noisy, or the problem is unstable.

The gradient of $J(x)$ is

$$\nabla \mathrm{J(x)} = \mathrm{A}^T (Ax - b), \tag{5}$$

Applying the Gradient Descent method to minimize (4), we get the iterator sequence

$$x^{(k)} = x^{(k-1)} - \mu A^T \left(Ax^{(k-1)} - b \right), k = 1,2, \dots \tag{6}$$

From the iteration sequence (6), we have

$$\begin{aligned} x^{(k)} - x^* &= x^{(k-1)} - x^* - \mu \left(A^T A x^{(k-1)} - A^* b \right) \\ &= x^{(k-1)} - x^* - \mu \left(A^T A x^{(k-1)} - A^T A x^* \right) \\ &= x^{(k-1)} - x^* - \mu A^T A \left(x^{(k-1)} - x^* \right). \end{aligned}$$

Let $e^{(k)} = x^{(k)} - x^*$, we have

$$e^{(k)} = e^{(k-1)} - \mu A^T A e^{(k-1)} = (I - \mu A^T A) e^{(k-1)} = (I - \mu A^T A)^k e^{(0)}$$

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If $x^{(0)}$ is selected, $z^{(0)}$ is the eigenvector of the matrix $A^T A$, we have

$$e^{(k)} = (1 - \mu\lambda)^k e^{(0)} \tag{7}$$

where, λ is the eigenvalue of the matrix $A^T A$ and satisfied $|\lambda| < \|A^T A\|$, so for the sequence to converge, $|1 - \mu\lambda| < 1$ or can be written equivalently $0 < \mu\lambda < 2$.

So, the iteration process (6) converges when the parameter μ is chosen so that

$$0 < \mu < \frac{2}{\|A^T A\|} \tag{8}$$

Now, we will choose the parameter μ for the fastest convergence iteration. First, we set $M = E - \mu A^T A$ as the iteration matrix, $R(M) = max|1 - \mu\lambda_i|$ as the spectral radius (largest eigenvalue of M), where λ_i are the eigenvalues of $A^T A$. Suppose the corresponding smallest and largest eigenvalues are λ_{min} and λ_{max}, then we have $R(M) = \max_i\{|1 - \mu\lambda_{min}|, |1 - \mu\lambda_{max}|\}$.

The iteration process (6) converges fastest if $R(M)$ is the smallest. The equilibrium conditions must be ensured, that is

$$|1 - \mu\lambda_{min}| = |1 - \mu\lambda_{max}| \tag{9}$$

We need to choose μ satisfy (9); that is, it is necessary to choose parameters that satisfy

$$\mu = \frac{2}{\lambda_{min} + \lambda_{max}}, \tag{10}$$

$R(M)$ reaches its minimum value

$$R(M) = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{min} + \lambda_{max}} \tag{11}$$

So, the following theorem shows how to choose optimal parameters and the convergence rate of the iterative sequence (6).

Theorem 1. *Let the smallest and largest eigenvalues of $A^T A$ be λ_{min} and λ_{max}, respectivel, if selecting calibration parameter $\mu = \frac{2}{\lambda_{min} + \lambda_{max}}$, then the iteration process (6) converges with a convergence rate of $(R(M))^k$.*

If with choice (10), then condition (8) can be replaced by condition

$$0 < \mu < \frac{2}{\lambda_{max}} \tag{12}$$

The above selection shows that the optimal parameters overcome the disadvantages if μ is small; the iterative sequence converges very slowly. Otherwise, μ is approaching $\frac{2}{\lambda_{max}}$ oscillating, unstable sequence. The convergence rate is a quantity that depends on $\lambda_{min}, \lambda_{max}$, if $\lambda_{min} \ll \lambda_{max}$, then the convergence process will be slow; conversely, the convergence rate will be increased if λ_{min} is closer to λ_{max}.

In case, matrix A is a large and irregular matrix, the eigenvalues of $A^T A$ non-uniform distribution with many very small and approximately zero values, then determining the correction parameter depending on the number of iterations becomes more efficient. To select parameters depending on the number of iterations, we choose a balanced strategy between fast convergence in the early stage, stability, and reduced oscillation when approaching the solution; the overall convergence rate is optimized. In addition, the choice of the tuning parameter needs to ensure that the iteration steps are enough to reach the exact solution gradually; this condition is important in ensuring two conditions: the first is that the iteration process does not stop early and approaches the solution of the problem; the second is that the total accumulated error does not cause divergence. So, the conditions set for the tuning parameter are

$$\lim_{k\to\infty} \mu_k = 0 \tag{13}$$

$$\sum_{k=1}^{\infty} \mu_k = \infty \tag{14}$$

$$\sum_{k=1}^{\infty} \mu_k^2 < \infty \tag{15}$$

To ensure these criteria, we choose a linear reduction parameter tuned to the spectrum of eigenvalues of $A^T A$, specifically

$$\mu_k = \frac{2}{\lambda_{min} + \lambda_{max} + ck} \tag{16}$$

where c is a reduction factor and is adjusted to control the convergence rate.

By this choice, we can see that in the first iterations, if μ_k is large, it can help the iteration sequence move faster to the solution. When close to the solution, if μ_k is smaller, it will help reduce oscillation and stabilize convergence. Suppose λ_{min} and λ_{max} are not known precisely. In that case, flexibly changing the calibration parameter over the number of iterations will help adapt to the spectrum of eigenvalues of $A^T A$. The following theorem allows us to choose a parameter depending on the number of iterations to obtain a convergent sequence.

Theorem 2. *If* $u_k = \frac{2}{\lambda_{min} + \lambda_{max} + ck}$ *, then we have (13), (14, (15) and* $\lim_{k\to\infty} \left\| x^{(k)} - x^* \right\| = 0$

First, we prove μ_k satisfies (13), (14), (15) in turn. Let $a = \lambda_{min} + \lambda_{max}$. The parameter can be rewritten as

$$\mu_k = \frac{2}{a + ck} \tag{17}$$

Then, $\lim_{k\to\infty} \mu_k = \lim_{k\to\infty} \frac{2}{a+ck} = 0$. So condition (13) is satisfied. Now, let us consider condition (14).

We have

$$\sum_{k=1}^{\infty} \mu_k = \sum_{k=1}^{\infty} \frac{2}{a+ck} \approx \int_1^{\infty} \frac{2}{a+cx} dx \tag{18}$$

$$\int_1^{\infty} \frac{2}{a+cx} dx = \frac{2}{c}\left(\lim_{x\to\infty} \ln(a+cx) - \ln(a+c)\right) = \infty \tag{19}$$

From (18) and (19), we deduce (14). Considering condition (15), we have

$$\sum_{k=1}^{\infty} (\mu_k)^2 = \sum_{k=1}^{\infty} \frac{4}{(a+ck)^2} \approx 4\int_1^{\infty} \frac{1}{(a+cx)^2} dx \tag{20}$$

$$\int_1^{\infty} \frac{1}{(a+cx)^2} dx = \frac{1}{c(a+c)} \tag{21}$$

From (20) and (21), we imply (15).

From (7), we have

$$\left\| e^{(k)} \right\| = (1 - \mu_k \lambda)^k \left\| e^{(0)} \right\| \tag{22}$$

Combined with the hypothesis of the theorem, we have

$$1 - \mu_k \lambda < 1 \tag{23}$$

From (22) and (23), we have $\left\| e^{(k)} \right\| \to 0$ when $k \to \infty$.

Based on Theorem 2, we have the following implementation algorithm:

Algorithm: Gradient descent method and parameter selection for image restoration problem

Input: A, b, c, *maximum number of iterations*

1: Initialize: $\mathbf{x}$, d , *k=1*

2: Compute $\lambda_{min} + \lambda_{max}$

Repeat

$\mu_k = \frac{2}{\lambda_{min}+\lambda_{max}+ck}$

$I = x - \mu_k A^T(Ax - b)$

$MSE = \frac{1}{m.n}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|x_{i,j} - I_{i,j}\right|^2$

$PSRN = 20\log_{10}\frac{Max_I}{\sqrt{MSE}}$

$x = I;$

$k{+}{+};$

Until ($PRSN \geq d$ or $k \geq maximum\ number\ of\ iterations$)

Output: $\boldsymbol{x = I}$

The following section presents some experimental results for the proposed algorithm.

3. Numerical results

The proposed algorithm has been applied to restore the original image from the blurred image data. We use the image blurring operator as matrix A, the input image has size 38x50 (Figure 1). The matrix code A has size *N*x*N* with *N*=1900 and is defined as a diagonal matrix $A = diag\left(0.5 + \frac{i}{5N}\right), i = 1, 2, \ldots, N$. The image vector will be a column vector of length 1900. Eigenvalues $\lambda_{min} = 0.25$; $\lambda_{max} = 1$. Therefore, $\mu_k = \frac{2}{1.25+ck}$.

After performing the algorithm, we get the blurred image and the restored image as in Figure 1, 2 and 3.

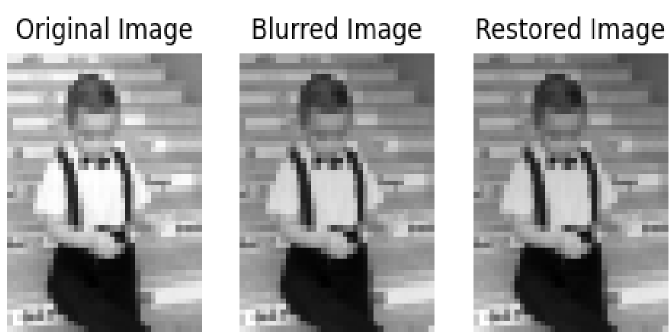

Figure 1: *Image size 38x50 and coefficient c=10*

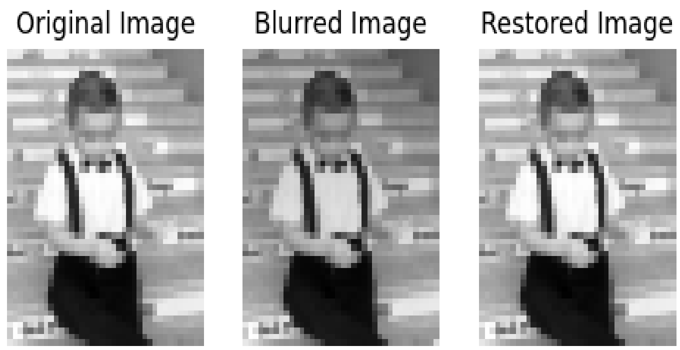

Figure 2: *Image size 38x50 and coefficient c=1*

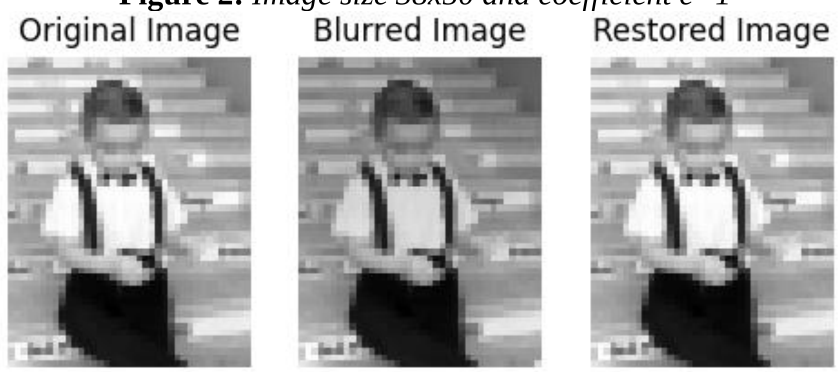

Figure 3: *Image size 38x50 and coefficient c=0.1*

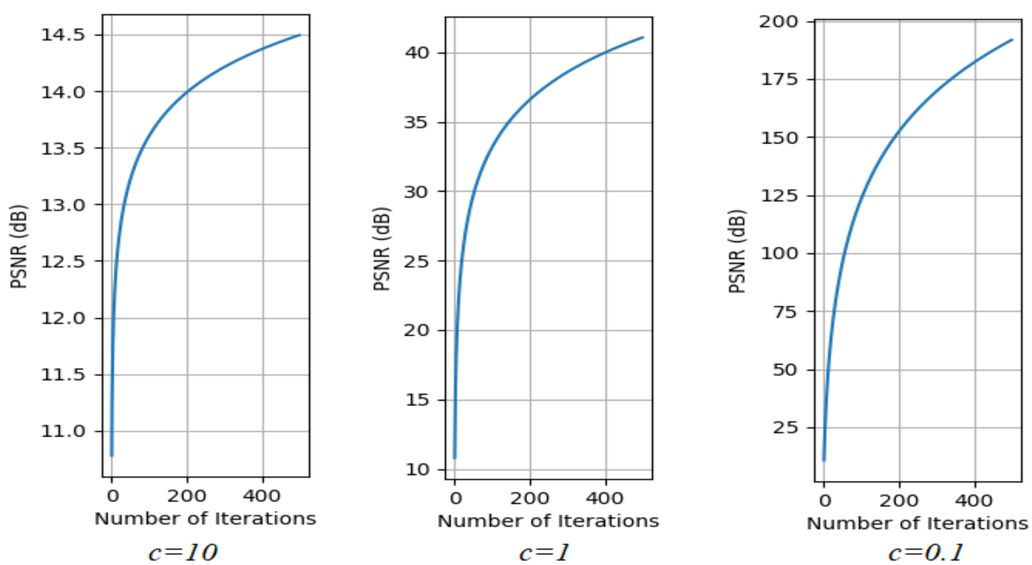

Figure 4: *Graph shows PSNR depending on the number of iterations corresponding to each case of choosing parameter c for the result of restoring a 38x50 image*

Mean Squared Error between the restored and original images was used to indicate the accuracy of the recovery process: $MSE = \frac{1}{m.n}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|x_{i,j}^{\alpha(\delta)} - x_{i,j}^{0}\right|^2$.

PSNR (Peak Signal-to-Noise Ratio) PSNR (Peak Signal-to-Noise Ratio) was used to measure the quality of the restored image compared to the original image. Usually, the higher the PSNR, the better the quality of the restored image. Although there is no specific threshold for all situations, below is a table of PSNR calculation according to the number of iterations for the case c=0.1 and some commonly used PSNR benchmarks, in which: Below 20 dB: Poor image quality, with much noise; 20-30 dB: Average image quality, acceptable but still noisy; 30-40 dB: Good image quality, and little noticeable noise; Above 40 dB: Excellent image quality with negligible noise. However, the specific PSNR level considered reasonable depends on the application and user requirements. For example, the quality requirements in medical applications or satellite imagery may be higher than in general applications.

Table 1: *Calculation results of the algorithm (c=0.1)*

Iteration	MSE	PSNR (dB)
50	1.718057964894565e-05	95.78042548631602
100	3.066169479777623e-08	123.26484204446757
150	6.362211074625241e-10	140.094722875734
200	3.867337828200004e-11	152.25668249690958
250	4.312061004389293e-12	161.78395464514313
300	7.10305837927509e-13	169.61634976761414
350	1.536221802444018e-13	176.26626436433355
400	4.0614173483435126e-14	182.04402741294243
450	1.2528264066459461e-14	187.1518946214721
500	4.3669278354988115e-15	191.72904345704896

The above results show that as the noise level decreases, PSRN increases, which means that MSE also decreases, which means that the restored image gradually approaches the original image. Here are some experimental results to illustrate the theory presented in the paper.

4. Conclusion

This paper proposes a method for selecting optimal parameters for the Gradient Descent algorithm. With the proposed method of selecting optimal parameters, we prove that the iteration sequence reduces oscillation when the approximate solution is close to the exact solution and ensures the iteration sequence is stable and convergent. We present the convergence rate and illustrate it with experimental results.

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TÓM TẮT

PHƯƠNG PHÁP GRADIENT DESCENT VÀ LỰA CHỌN THAM SỐ CHO BÀI TOÁN PHỤC HỒI ẢNH

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Ngày nhận bài 15/10/2024, ngày nhận đăng 24/12/2024

Xử lý nâng cao chất lượng hình ảnh là một bài toán có ý nghĩa trong nhiều ứng dụng thực tế. Nó đóng vai trò quan trọng trong các bước tiền xử lý cho việc nhận dạng và trích xuất thông tin. Trong đó, khôi phục ảnh thường được coi là một trong các công đoạn tiền xử lý dữ liệu trước khi thực hiện quá trình huấn luyện cho các mô hình học máy. Bài toán khôi phục ảnh thường được xử lý bằng các thuật toán lặp, trong đó việc lựa chọn tham số lặp đóng vai trò quan trọng trong việc nâng cao tốc độ hội tụ của thuật toán. Bài toán đặt ra ở đây là cần xác định được tham số đảm bảo tốc độ hội tụ của thuật toán nhanh nhất. Trong bài báo này, chúng tôi đề xuất một phương pháp lựa chọn tham số cho thuật toán Gradient descent nhằm khôi phục dữ liệu ảnh gốc từ ảnh thu được sau khi thực hiện phép biến đổi hình thái lên ảnh gốc ban đầu. Theo phương pháp này, chúng tôi phân tích giá trị riêng của ma trận biến đổi hình thái để từ đó đưa ra công thức xác định tham số tối ưu cho thuật toán Gradient descent. Từ công thức xác định tham số, chúng tôi đã chứng minh được quá trình lặp hội tụ. Các kết quả thực nghiệm cũng cho thấy lý thuyết đưa ra là phù hợp và khẳng định nghiệm xấp xỉ hội tụ về nghiệm của bài toán ban đầu.

Từ khóa: Khôi phục ảnh; Gradient descent; phương pháp lặp; tối ưu lồi; tham số tối ưu.